

THE MODULAR GROUP AND WORDS IN ITS TWO GENERATORS

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ABSTRACT. Consider the full modular group $\mathrm{PSL}_2(\mathbb{Z})$ with presentation $\langle U, S | U^3, S^2 \rangle$. Motivated by our investigations on quasi-modular forms and the Minkowski question mark function (so that this paper might be considered as a necessary appendix), we are lead to the following natural question. Some words in the alphabet $\{U, S\}$ are equal to the unity; for example, USU^3SU^2 is such a word of length 8, and $USU^3SUSU^3S^3U$ is such a word of length 15. We consider the following integer sequence. For each $n \in \mathbb{N}_0$, let $t(n)$ be the number of words in alphabet $\{U, S\}$ that equal the identity in the group. This is the new entry A265434 into the Online Encyclopedia of Integer Sequences. We investigate the generating function of this sequence and prove that it is an algebraic function over $\mathbb{Q}(x)$ of degree 3. As an aside, we formulate the problem of describing all algebraic functions with a Fermat property.

1. INTRODUCTION

1.1. Motivation and results. The topic of this paper is the following two new sequences. The first one $t(n)$, $n \geq 0$, starts from

$$1, 0, 1, 1, 1, 5, 2, 14, 13, 31, 66, 77, 240, 286, 722, 1226, 2141, 4760, 7268, 16473, \dots \quad (1)$$

This is the sequence A265434 in [13]. The second sequence $\mathfrak{t}(n)$, $n \geq 0$, starts from

$$1, 0, 1, 1, 0, 3, 0, 5, 3, 7, 16, 12, 50, 44, 123, 195, 301, 718, 928, 2244, \dots$$

They are defined as follows. Let $U^3 = I$ and $S^2 = I$ be two elements of order 3 and 2, respectively, I being the unity. Consider the *the full modular group* $\mathrm{PSL}_2(\mathbb{Z})$. It is known that it is freely generated by $U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$, element of order 3, and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, element of order 2. Let $n \in \mathbb{N}_0$, and A be any word in the alphabet U, S of total length n :

$$A = \prod_{j=1}^n (U^{\epsilon_j} S^{\delta_j}), \quad \epsilon_j, \delta_j \in \{0, 1\}, \quad \epsilon_j + \delta_j = 1.$$

Let $t(n)$ be the number of such words that in the group Γ are equal to the unity. Our method to calculate the first 20, $0 \leq n \leq 19$, terms of the sequence (1) was via a brute force calculation. As the model, let U and S be given by 2×2 matrices as above. Let us construct a binary tree, starting from the node $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Each node A in this binary tree generates two offspring - the right one AU , and the left one AS . For a given n , we then calculate which of the 2^n matrices in the n th generation are equal to $\pm I$. Via this method, a standard home computer

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can give few more terms of this sequence.

Further, we call such a word *primitive*, if

$$\prod_{j=1}^s (U^{\epsilon_j} S^{\delta_j}) = I \text{ only if } s = n.$$

Let $t(n)$ be the number of primitive (and nonempty) words of length n . Let us introduce the generating functions

$$\sum_{n=0}^{\infty} t(n)x^n = T(x), \quad \sum_{n=1}^{\infty} t(n)x^n = \mathfrak{T}(x).$$

Obviously,

$$\frac{1}{1 - \mathfrak{T}(x)} = T(x).$$

Indeed, we are just considering words broken up into primitive loops, so

$$T(x) = 1 + (t(1)x + t(2)x^2 + \dots) + (t(1)x + t(2)x^2 + \dots)^2 + \dots.$$

The number of total words of length n is equal to 2^n . So, $t(n) \leq 2^n$, and thus both series converge at least for $|x| < \frac{1}{2}$. The Table gives values for this function (and all words) for $0 \leq n \leq 8$.

Table. Sequences $t(n)$ and $\mathfrak{t}(n)$. Primitive elements are in bold			
n	$t(n)$	$\mathfrak{t}(n)$	Products
0	1	0	I
1	0	0	–
2	1	1	S^2
3	1	1	U^3
4	1	0	S^4
5	5	3	$U^3S^2, \mathbf{U^2S^2U}, \mathbf{US^2U^2}, S^2U^3, \mathbf{SU^3S}$
6	2	0	S^6, U^6
7	14	5	$U^3S^4, \mathbf{U^2S^4U}, \mathbf{US^4U^2}, S^4U^3, SU^3S^3, S^2U^2S^2U, \mathbf{SU^2S^2US}, U^2S^2US^2$ $S^2US^2U^2, \mathbf{SUS^2U^2S}, US^2U^2S^2, S^3U^3S, S^2U^3S^2, \mathbf{US^2US^2U}$
8	13	3	$S^8, U^6S^2, U^5S^2U, U^4S^2U^2, U^3S^2U^3, U^2S^2U^4, US^2U^5, S^2U^6$ $SU^6S, U^3SU^3S, U^2SU^3SU, USU^3SU^2, SU^3SU^3$

The integer $q(n, m)$ counts the number of words that are equal to unity in the group Γ , and which contain n copies of U and m copies of S . Note that

$$q(n, m) = 0 \text{ unless } n \equiv 0 \pmod{3} \text{ and } m \equiv 0 \pmod{2}.$$

We also introduce

$$\sum_{A \text{ is unity}} x^{\sum \epsilon_j} y^{\sum \delta_j} = \sum_{n,m=0}^{\infty} q(n, m)x^n y^m = Q(x, y).$$

Obviously, $T(x) = Q(x, x)$. The series $T(x)$ can be interpreted as a return generating function for a certain directed graph (see the Subsection 1.2 and Figure 1), and thus this function belongs to a hugely diverse and abundant family of functions with the following two features. All of them share the property that their Taylor coefficients grow exponentially, and all are related

to the graph theory and enumeration ([3], 5.6).

The main result of this paper is the following

Theorem. *The function $Q(x, y)$ is an algebraic 3rd degree function over $\mathbb{Q}(x, y)$, satisfying*

$$(y^6 - x^6 + 6y^2x^3 - 3y^4 + 2x^3 + 3y^2 - 1)Q^3 + (x^3y^2 - y^4 + x^3 + 2y^2 - 1)Q^2 + (x^3 - y^2 + 1)Q + 1 = 0.$$

In particular, the function $T(x)$ is an algebraic 3rd degree function over $\mathbb{Q}(x)$, satisfying

$$(6x^5 - 3x^4 + 2x^3 + 3x^2 - 1)T^3 + (x^5 - x^4 + x^3 + 2x^2 - 1)T^2 + (x^3 - x^2 + 1)T + 1 = 0.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{t(n)}{2^n} = \frac{14}{13} + \frac{6}{13}\sqrt{17}, \quad \sum_{n=0}^{\infty} \frac{t(n)}{2^n} \cdot \frac{1}{2^{n+1}} = 0.5443390725_+.$$

The last number can be interpreted as probability that a randomly chosen word is a unity, with a convention that each length n is given a weight $2^{-(n+1)}$, and then probability is equally distributed among all words of length n .

Two proofs of this result are given. The first one is longer, but uses only elementary combinatorics and considerations from the scratch. The second one is shorter, is included due to a suggestion and very clear guidance by the referee, and is a standard proof in the area of geometric and combinatoric group theory.

We finish this Subsection with the following

Proposition 1. *For any prime $p > 3$, $t(p) \equiv 0 \pmod{p}$.*

Proof. Indeed, if $AB = I$ in the group Γ , then $BA = I$ as well. So, any cyclic permutation of the word which is a unity is a unity again. So, words of length $p > 3$ which are equal to the unity split into groups each containing exactly p words. Indeed, otherwise all these permutations are equal. But $U^p \neq I$ and $S^p \neq I$ - a contradiction. \square

1.2. Context, previous results. In [1] we investigate the relation of modular forms to the Minkowski question mark function, introducing the notion of *mean-modular forms*. In particular, the analytic continuation of a certain bivariate analytic function $G(\varkappa, z)$ (an extension of the Stieltjes transform of the Minkowski question mark function) requires us to know the analytic formula for $Q(x, y)$ as a bivariate function; whence the principal motivation for the current paper. In fact, the result of Theorem is new, but it is an exercise for people in the field. The much more ambitious program of integrating the world of Minkowski question mark function and the world of modular forms for $\mathrm{PSL}_2(\mathbb{Z})$ justifies the current paper as a necessary appendix to [1].

As just mentioned, this particular question is new, though many intricately related problems were investigated and solved before, and the topic itself is of big importance in the theory of groups (growth and cogrowth rates), graphs (return and first-return paths), and non-commutative probability.

As a particular example of his more general results, Kuksov [6] considers the cogrowth rate of the product $\mathbb{Z}/(2) \star \mathbb{Z}/(3)$, which is the same modular group. The question of investigating

cogrowth rates amounts to the following. Count the number of *reduced* words in the alphabet $\{U, U^{-1}, S, S^{-1}\}$ that are equal to the unity. *Reduced* means that S and S^{-1} , and also U and U^{-1} never follow immediately one after another. It is obvious then that the total number of reduced words of length n is $4 \cdot 3^{n-1}$: the first letter can be anything, after that the choice is restricted. The generating function of this sequence (cogrowth series) turns out to be

$$v(x) = \frac{(x+1)\left(9x^5 - 3x^4 + 8x^3 - x^2 + x - (6x^2 - x + 2)\sqrt{R(x)}\right)}{2(3x-1)(3x^2+1)(3x^2+3x+1)(3x^2-x+1)},$$

where $R(x) = 81x^8 - 54x^7 + 9x^6 - 18x^5 - 8x^4 - 6x^3 + x^2 - 2x + 1$.

The cogrowth rate (the inverse of the radius of convergence of the Taylor series for this function at the origin) turns out to be $2.9249_+ < 3$. The function $v(x)$ is quadratic algebraic function, as opposed to cubic algebraic function $T(x)$ in our case. The Taylor coefficients of $v(x)$ are

$$1, 0, 2, 2, 6, 24, 44, 136, 298, 914, 2462, 6464, \dots$$

For example, there are 6 reduced words of length 4:

$$SSSS, S^{-1}S^{-1}S^{-1}S^{-1}, U^{-1}SSU, USSU^{-1}, U^{-1}S^{-1}S^{-1}U, US^{-1}S^{-1}U^{-1}.$$

Also, Proposition 1 does not have an analogue in this case.

In a related direction, Quenell in [9] investigates the *return generating function* of Cayley graphs for the free products of finite groups. This is related to the thesis of McLaughlin [8] (see also [2]). In particular, in case of the modular group the return generating function counts words in $\{U, U^{-1}, S, S^{-1}\}$ of total length n that are equal to the unity (words are not necessarily reduced). The total number of all words of spell length n is 4^n . In this context, the function $T(x)$ can be interpreted as a return generated function for a *directed* Cayley graph for $\mathbb{Z}/(2) \star \mathbb{Z}/(3)$, where each 2-cycle and 3-cycle has a particular direction chosen in advance; see Figure 1.

Franz Lehner has pointed out that the question in consideration is a special case of a “free convolution” and can be obtained via Voiculescu-Woess transform [7, 10, 11, 12]. This technique is implemented as a package and it is part of the library of FriCAS. So, the current paper can be thought as a purely combinatoric demonstration of the result, with emphasis on a bivariate function $Q(x, y)$ rather than a univariate $T(x) = Q(x, y)$.

1.3. Algebraic functions with a Fermat property. Before passing to the proof of main result, we will formulate one interesting problem, which is motivated by Proposition 1.

Consider the following rational function

$$P(x) = \frac{x^2}{(1-2x)(1-x)} = \sum_{n=2}^{\infty} s(n)x^n = \sum_{n=2}^{\infty} (2^{n-1} - 1)x^n.$$

Thus, we have $s(p) \equiv 0 \pmod{p}$ for $p > 2$ prime.

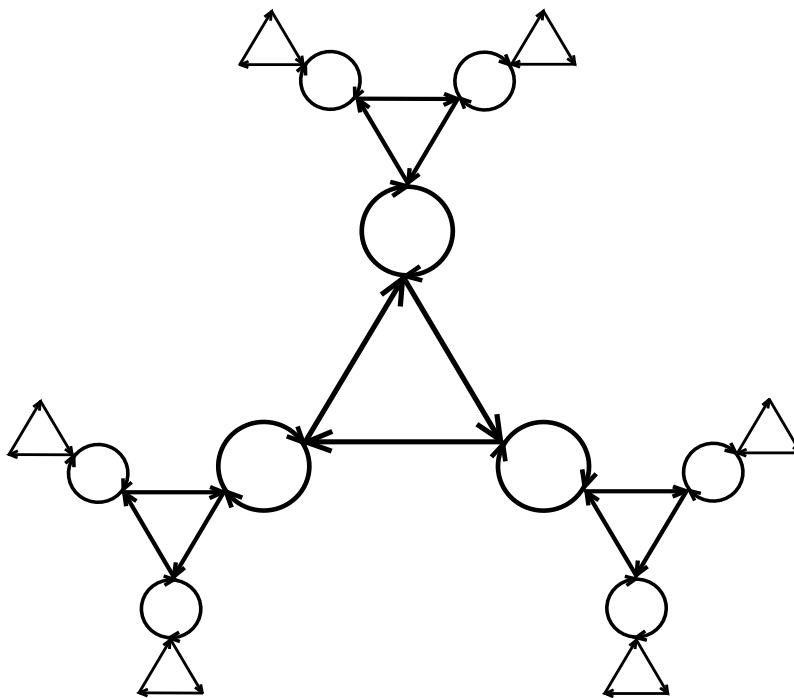


FIGURE 1. First few generations of a directed Cayley graph for $\mathbb{Z}/(2) \star \mathbb{Z}/(3)$.

Next, let

$$J(x) = \frac{1}{\sqrt{1-4x}} - \frac{2}{1-x} = \sum_{n=0}^{\infty} s(n)x^n = \sum_{n=0}^{\infty} \left(\binom{2n}{n} - 2 \right) x^n.$$

This also gives $s(p) \equiv 0 \pmod{p}$ for $p \geq 2$ prime.

Finally, as is implied by Proposition 1, we have the similar conclusion for a degree 3 algebraic function $T(x)$. Thus, we formulate

Problem. Let $R(x) = \sum_{n=0}^{\infty} s(n)x^n \in \mathbb{Z}[[x]]$ be an algebraic function over $\mathbb{Q}(x)$. Suppose, $s(p) \equiv 0 \pmod{p}$ for all sufficiently large prime numbers p . We call such a function $R(x)$ an algebraic function with a Fermat property. Characterize all such functions.

Of course, we can multiply the function $R(x)$ by an integer to get the congruence valid for all primes. All functions with a Fermat property form a \mathbb{Z} -module \mathcal{F} . If $U(x) = \sum_{n=0}^{\infty} a(n)x^n \in \mathbb{Z}[[x]]$ is an algebraic function, then $U^\partial(x) := xU'(x) = \sum_{n=0}^{\infty} na(n)x^n \in \mathcal{F}$. Let D be the union of all such possible U^∂ . Then D is a sub-module of \mathcal{F} , and so is $\mathbb{Z}[x]$. We may wonder, for example, about the structure of the \mathbb{Z} -module

$$\mathcal{F}/(D \oplus \mathbb{Z}[x]).$$

2. THE FIRST PROOF

Let us introduce the function $\hat{q}(n, m)$. This is defined similarly as $q(n, m)$. Namely, we count the number of words in U, S which are equal to unity in Γ , which have n copies of U and m

copies of S , but which do not contain S^2 (two S 's in a row). Let

$$\sum_{n,m \geq 0}^{\infty} \hat{q}(3n, 2m)x^{3n}y^{2m} = \widehat{Q}(x, y).$$

Proposition 2. *We have an identity*

$$q(3n, 2m) = \sum_{k \geq 0} \hat{q}(3n, 2m - 2k) \cdot \binom{3n + k}{k}.$$

This implies

$$Q(x, y) = \widehat{Q}\left(\frac{x}{1 - y^2}, y\right) \cdot \frac{1}{1 - y^2}.$$

Proof. Indeed, consider any word which has $3n$ copies of U and $2m$ copies of S . Now, replace each occurring segment $S^{2\ell+j}$, $\ell \geq 0$, $j \in \{0, 1\}$, with S^j . We get a word which lies in the set which defines $\hat{q}(3n, 2m - 2k)$ for some $k \in \mathbb{N}_0$. In the other direction, consider the latter word. Suppose, we have k spare copies of S^2 . We can plug k copies of S^2 into such a word, to get a word which defines $q(3n, 2m)$. As can be seen, we can confine in plugging to the left of each occurrence of U , plus to the right of the rightmost U . This gives $3n + 1$ possible places to plug in. We want to distribute k copies of S^2 . This gives the formula in Proposition. \square

Let us divide the words which define the quantity $\hat{q}(n, m)$ into 7 disjoint subsets. B stands for any non-empty word. Let $n, m \geq 0$. Here everywhere “—” stands for the phrase “be the number of such words that are of the form...”.

- a) $a(n, m)$ — $SUBUS$.
- b) $b(n, m)$ — either $U^\alpha SBSU^\beta$, $\alpha, \beta > 0$ and $\alpha + \beta \equiv 0 \pmod{3}$, or U^γ , $\gamma \equiv 0 \pmod{3}$.
- c) $c(n, m)$ — $U^\alpha SBS$, or $SBSU^\alpha$, $\alpha > 0$, $\alpha \equiv 0 \pmod{3}$.
- d) $d(n, m)$ — $U^\alpha SBSU^\beta$, $\alpha, \beta > 0$, and $\alpha + \beta \equiv 2 \pmod{3}$.
- e) $e(n, m)$ — $U^\alpha SBSU^\beta$, $\alpha, \beta > 0$, and $\alpha + \beta \equiv 1 \pmod{3}$.
- f) $f(n, m)$ — either $SBSU^\alpha$, or $U^\alpha SBS$, $\alpha \equiv 1 \pmod{3}$.
- g) $g(n, m)$ — either $SBSU^\alpha$, or $U^\alpha SBS$, $\alpha \equiv 2 \pmod{3}$.

Let us also introduce two subsets which define d and f , respectively, as follows:

- ḍ) $\mathfrak{d}(n, m)$ — $USBSU$,
- ḟ) $\mathfrak{f}(n, m)$ — $SBSU$.

Example 1. We have: $\hat{q}(6, 2) = 5$. Words which correspond to a, b, c are, respectively, $\{SU^6S\}$, $\{U^2SU^3SU, USU^3SU^2\}$, $\{SU^3SU^3, U^3SU^3S\}$. All sets beyond “c” are empty. \square

Example 2. We have: $\hat{q}(9, 4) = 20$. Words which correspond to $a(9, 4) = 7$, $b(9, 4) = 5$, $c = 3$, $d(9, 4) = 1$, $e(9, 4) = 0$, $f(9, 4) = 2$, $g(9, 4) = 2$, are, respectively:

- a : $\{SUSU^3SU^5S, SUSU^6SU^2S, SU^2SU^3SU^4S, SU^2SU^6SUS, SU^3SU^3SU^3S, SU^4SU^3SU^2S, SU^5SU^3SUS\}$,
- b : $\{USUSU^3SU^2SU^2, USU^2SU^3SUSU^2, U^2SUSU^3SU^2SU, U^2SU^2SU^3SUSU, SUSU^3SU^2SU^3\}$,
- c : $\{SU^2SU^3SUSU^3, U^3SUSU^3SU^2S, U^3SU^2SU^3SUS\}$,
- d : $\{USU^3SUSU^3SU\}$,
- e : \emptyset ,
- f : $\{SU^3SU^2SU^3SU, USU^3SU^2SU^3S\}$,
- g : $\{SU^3SUSU^3SU^2, U^2SU^3SUSU^3S\}$. \square

Note that $a(n, m)$ and other 8 functions are potentially non-zero only for $n = 3k$, $m = 2l$ for $k \geq 0$, $l \geq 0$.

The first step to derive our main result is the following

Proposition 3. *We have the following recurrences:*

$$\begin{aligned}
a(3n, 2m) &= b(3n, 2m - 2) + d(3n, 2m - 2) + e(3n, 2m - 2), \\
b(3n, 2m) &= \sum_{k \geq 1} (3k - 1)a(3n - 3k, 2m) \text{ for } m \geq 1, \quad b(3n, 0) = 1 \text{ for } n > 0, \\
c(3n, 2m) &= 2 \sum_{k \geq 1} a(3n - 3k, 2m), \\
d(3n, 2m) &= \sum_{k \geq 0} (3k + 1)\mathfrak{d}(n - 3k, 2m), \\
e(3n, 2m) &= \sum_{k \geq 1} 3kf(n - 3k, 2m), \\
f(3n, 2m) &= 2 \sum_{k \geq 0} \mathfrak{f}(n - 3k, 2m), \\
g(3n, 2m) &= 2 \sum_{k \geq 0} \mathfrak{d}(n - 3k, 2m).
\end{aligned}$$

These hold for $n, m \geq 0$ assuming that all these functions vanish if one of the arguments is negative.

Proof. All these equalities are straightforward. Only the formula for e needs an explanation. Indeed, we note that (as already used in the proof of Proposition 1) if $AB = I$ in the group Γ , then $BA = I$. So, if $U^2BU^2 = I$, this gives $BU = I$. \square

Let

$$A(x, y) = \sum_{n, m \geq 0} a(3n, 2m)x^{3n}y^{2m}$$

be the generating function of the coefficients $a(3n, 2m)$, and similarly we define other bivariate functions $B, C, D, E, F, G, \mathfrak{D}$ and \mathfrak{F} . The identities of Proposition 3 now read as

$$\begin{aligned}
A &= y^2B + y^2D + y^2E, & B &= \frac{x^3}{1-x^3} + \frac{x^3(x^3+2)}{(1-x^3)^2}A, \\
C &= \frac{2x^3}{1-x^3}A, & D &= \frac{2x^3+1}{(1-x^3)^2}\mathfrak{D}, & E &= \frac{3x^3}{(1-x^3)^2}\mathfrak{F}, & F &= \frac{2}{1-x^3}\mathfrak{F}, & G &= \frac{2}{1-x^3}\mathfrak{D}.
\end{aligned}$$

This gives

$$A = \frac{y^2x^3}{1-x^3} + \frac{y^2x^3(x^3+2)}{(1-x^3)^2}A + \frac{y^2(2x^3+1)}{(1-x^3)^2}\mathfrak{D} + \frac{3y^2x^3}{(1-x^3)^2}\mathfrak{F}. \quad (2)$$

Our function \widehat{Q} is then

$$\widehat{Q} = 1 + A + B + C + D + E + G.$$

That is,

$$\widehat{Q}(x, y) = \frac{1}{1-x^3} + \frac{2x^3+1}{(1-x^3)^2}A + \frac{3}{(1-x^3)^2}\mathfrak{D} + \frac{x^3+2}{(1-x^3)^2}\mathfrak{F}. \quad (3)$$

Example 3. The sets which define “ d ” through “ g ” are non-empty starting only from the length 13 (that is, where $3n + 2m = 13$). Thus, if we use the above recurrences only minding the sets “ a ”, “ b ” and “ c ” (that is, assuming that $\mathfrak{D} = 0$ and $\mathfrak{F} = 0$), we obtain

$$A(x, y) \gg \frac{y^2x^3(1-x^3)}{(1-x^3)^2 - y^2x^3(x^3+2)}.$$

By the sign \gg we mean that the inequality \geq holds for Taylor coefficients of corresponding functions on the left and on the right. This gives

$$\widehat{Q}(x, y) \gg 1 + A + B + C \gg \frac{1-x^3-y^2x^3}{(1-x^3)^2 - y^2x^3(x^3+2)},$$

and consequently

$$T(x) = \widehat{Q}\left(\frac{x}{1-x^2}, x\right) \cdot \frac{1}{1-x^2} \gg \frac{(x-1)(x+1)(x^6+x^5-3x^4+x^3+3x^2-1)}{1-5x^2-2x^3+10x^4+2x^5-9x^6+2x^7+5x^8-2x^9-x^{10}}.$$

And indeed, minding the values given by (1), MAPLE confirms that the first discrepancy occurs only for $n = 13$. Namely, $286 > 281$, and the five missing words are precisely those given by Example 2 in the sets “ d ” through “ g ”. For $n = 14$ there are no words in the sets beyond “ c ”, so the above is in fact the equality $722 = 722$. The Galois group of the splitting field of the polynomial in the denominator is equal to S_{10} . This polynomial has the unique root θ of the smallest absolute value, it is real and positive: $\theta = 0.5394737936_+$, $\theta^{-1} = 1.853658161_+$. So, this single observation gives the lower bound

$$t(n) > C(1.853658161)^n$$

for a certain $C > 0$. In fact, our main Theorem implies that for the function $t(n)$ we have the sharper bound

$$t(n) > C(1.971480194)^n,$$

where $\phi^{-1} = 1.971480194_+$, ϕ being the smallest (in absolute value) root of $x^7 - 20x^5 + 12x^4 - 8x^3 - 12x^2 + 4 = 0$, the factor of the discriminant of the cubic polynomial in the Theorem. \square

To prove the formula for A , and hence for \widehat{Q} , we need to express \mathfrak{D} and \mathfrak{F} in terms of A . In order to accomplish this, we introduce the notion of a *primitive a-word*. This, by definition, is the word which belongs to the subset which defines the function “ a ” (an *a-word*), but which cannot be written as

$$A_1U^{\alpha_1}A_2U^{\alpha_2}\dots U^{\alpha_{s-1}}A_s, \quad s \geq 2, \quad \alpha_i \geq 1, \quad (4)$$

where each A_i is an *a-word*.

Thus, let us continue our classification given by *a)* through *f)*, and introduce

a) $\mathbf{a}(n, m)$ — number of primitive *a-words*, which have n copies of U and m copies of S .

Example 4. In the Example 2 above, six of the *a-words* are primitive, only $SU^3SU^3SU^3S$ is not. \square

Proposition 4. *We have the recurrence*

$$\begin{aligned}
a(3n, 2m) &= \mathbf{a}(3n, 2m) + \sum_{k=1}^{\infty} \sum_{\substack{3n_1+3n_2=3n-3k \\ 2m_1+2m_2=2m}} \mathbf{a}(3n_1, 2m_1)\mathbf{a}(3n_2, 2m_2) \\
&+ \sum_{k=1}^{\infty} \binom{3k-1}{1} \sum_{\substack{3n_1+3n_2+3n_3=3n-3k \\ 2m_1+2m_2+2m_3=2m}} \mathbf{a}(3n_1, 2m_1)\mathbf{a}(3n_2, 2m_2)\mathbf{a}(3n_3, 2m_3) \\
&+ \sum_{k=1}^{\infty} \binom{3k-1}{2} \sum_{\substack{3n_1+3n_2+3n_3+3n_4=3n-3k \\ 2m_1+2m_2+2m_3+2m_4=2m}} \mathbf{a}(3n_1, 2m_1)\mathbf{a}(3n_2, 2m_2)\mathbf{a}(3n_3, 2m_3)\mathbf{a}(3n_4, 2m_4) \\
&+ \dots
\end{aligned}$$

Proof. We note that each \mathbf{a} -word is either primitive, or can be written in the form (4), where each A_i is a primitive \mathbf{a} -word. This claim follows by induction. Now we are left to count, which gives the above formula. Here $\binom{3k-1}{s}$ stands for a number of ways the number $3k$ can be written as a sum of $s+1$ positive integers. \square

Let

$$W(x, y) = \sum_{n, m \geq 0} \mathbf{a}(3n, 2m) x^{3n} y^{2m}.$$

The identity in Proposition 4 can be written as

$$A = W + \sum_{k=1}^{\infty} \sum_{s \geq 0} \binom{3k-1}{s} x^{3k} W^{s+2} = W + \sum_{k=1}^{\infty} x^{3k} W^2 (1+W)^{3k-1} = W + \frac{x^3 W^2 (1+W)^2}{1-x^3(1+W)^3}.$$

Analogously we derive recurrences for \mathfrak{d} and \mathfrak{f} in terms of \mathbf{a} , which lead to the identities

$$\mathfrak{D} = \frac{x^3 W^2}{1-x^3(1+W)^3}, \quad \mathfrak{F} = \frac{x^3 W^2 (1+W)}{1-x^3(1+W)^3}.$$

Plugging all these three identities for A , \mathfrak{D} and \mathfrak{F} into (2), we readily obtain that W is an algebraic function:

$$W = \frac{y^2 x^3 (1-x^3)}{(1-x^3)^2 - y^2 x^3 (x^3+2)} - \frac{W^2 x^3 (1+W)^2}{1-x^3(1+W)^3} + \frac{W^2 x^3 y^2 (5x^3+1+3x^3 W)}{[1-x^3(1+W)^3][(1-x^3)^2 - y^2 x^3 (x^3+2)]}.$$

MAPLE simplifies this to a very elegant form

$$W = x^3 (W+1)^2 (W+y^2). \quad (5)$$

So, W is a third degree algebraic function over $\mathbb{Q}(x, y)$, and thus so is Q . The exact form of the cubic equation can be easily calculated with MAPLE, but we rather concentrate on $T(x)$, since, first, the equation for Q with the help of method in Section 3 can be easily calculated by hand, and second, the cubic equation for W is very convenient to calculate fast the Taylor coefficients of T recurrently. Let

$$Z(x) = W\left(\frac{x}{1-x^2}, x\right).$$

Then the equation for Z reads as

$$Z = \frac{x^3(Z+1)^2(Z+x^2)}{(1-x^2)^3}. \tag{6}$$

We are left to verify the cubic equation given in the formulation of the Theorem. Plugging known values into (3) and using Proposition 2, we obtain

$$T(x) = \frac{(1-x^2)^2(1+Z)}{1-3x^2-x^3+3x^4-x^6-3x^3Z-3x^3Z^2-x^3Z^3}. \tag{7}$$

Now, plug this value of $T(x)$ into the cubic equation given by the Theorem. Then factor the numerator. MAPLE confirms that one of the two multipliers is indeed $Z(1-x^2)^3 - x^3(Z+1)^2(Z+x^2)$, so the equation for $T(x)$ is verified. In fact, this equation was discovered by Robert Israel by finding a 3rd degree algebraic function whose first 42 Taylor coefficients coincide with $t(n)$, $0 \leq n \leq 41$. Our theoretical result thus double-checks this fact.

The equalities (6) and (7) give a polynomial-time method to calculate coefficients $t(n)$. Indeed, let us start from $Z_1(x) = x^5$ (the first primitive \mathbf{a} -word is SU^3S), and let us define polynomials $Z_N(x) \in \mathbb{Z}[x]$ recurrently by

$$Z_{N+1} = \frac{x^3(Z_N+1)^2(Z_N+x^2)}{(1-x^2)^3} \pmod{x^{3N+6}}.$$

Thus, Z_N is of degree $3N+2$, $Z_{N+1} \equiv Z_N \pmod{x^{3N+3}}$. After reaching enough terms, plug this into (7). This agrees perfectly with (1), which was calculated by a direct count. Robert Israel calculated 2000 terms of the sequence $t(n)$.

3. THE SHORT PROOF USING PDA

The following alternative proof was proposed by the referee. We reproduce it almost *verbatim*, since the ideas are clear and self-explanatory. The *PDA* stands for a *pushdown automaton*. Both proofs give exactly the same algebraic equation for the function Q .

The group $\Gamma = \text{PSL}_2(\mathbb{Z})$ is *virtually free*, so in the framework of geometric and combinatoric group theory it is known that the *word problem* (the set of all words in generators and inverses that are equal 1) is an unambiguous context-free language. However, for our purposes in [1] we need the regular language of all *positive words* $\{U, S\}^*$ to obtain the language of words counted by the generating functions T and Q . It follows immediately from the Chomsky-Schützenberger enumeration theorem that the functions T and Q are algebraic.

To obtain the explicit formulae, we can explicitly construct a PDA accepting the language, then follow standard methods to obtain generating functions from the PDA.

The language is very simple: a word equals 1 if some sequence of applications of the rules $S^2 \rightarrow 1$, $U^3 \rightarrow 1$ to factors of the input word reduces it to the empty word. This can be done using a PDA which has just one state, and a pushdown stack which uses the alphabet $\{0, 1, 2, 3\}$ where 0 is the bottom-of-stack marker, 1 means a single U , 2 means U^2 , and 3 means S . So that the model of a PDA can be used that accepts on empty stack. Let us introduce a symbol $\$$, and consider the language $L\$ = \{w\$ \mid w \in \{U, S\}^*, w =_{\Gamma} 1\}$.

Here are the transitions:

- 1) $U, 0 \rightarrow 10$
- 2) $U, 1 \rightarrow 2$
- 3) $U, 2 \rightarrow \epsilon$
- 4) $U, 3 \rightarrow 13$
- 5) $S, 0 \rightarrow 30$
- 6) $S, 1 \rightarrow 31$
- 7) $S, 2 \rightarrow 32$
- 8) $S, 3 \rightarrow \epsilon$
- 9) $\$, 0 \rightarrow \epsilon$

This is indeed self-explanatory. For example, take the item 4). This means U is the next letter, and 3 is on the top of the stack (that is, S). Since US does not reduce, replace 3 with 13. On the other hand, consider the item 3). It stands for a move that now we get a factor U^3 , which is removed.

The idea is that one reads a word in U, S and puts it into *normal form* on the fly, reducing U^3 , S^2 whenever they appear as one moves right. The normal form of the prefix of the input word is written (in code) on the stack. Accept if at the end the stack is empty (the normal form is 1).

Now we follow [4] to convert the deterministic PDA into an unambiguous context free grammar. Let N_i stand for the nonterminal $N_{q,i,q}$ (since there exists only one state). Start symbol is N_0 . We get

- 1) $N_0 \rightarrow UN_1N_0$
- 2) $N_1 \rightarrow UN_2$
- 3) $N_2 \rightarrow U$
- 4) $N_3 \rightarrow UN_1N_3$
- 5) $N_0 \rightarrow SN_3N_0$
- 6) $N_1 \rightarrow SN_3N_1$
- 7) $N_2 \rightarrow SN_3N_2$
- 8) $N_3 \rightarrow S$
- 9) $N_0 \rightarrow \$$

Next, following the method of Chomsky-Schützenberger we obtain the following system of equations for the generating function directly from the grammar:

$$\begin{cases} f_0 = xf_1f_0 + yf_3f_0 + z, \\ f_1 = xf_2 + yf_3f_1, \\ f_2 = x + yf_3f_2, \\ f_3 = xf_1f_3 + y. \end{cases} \quad (8)$$

That is, we replace U and S with x and y , respectively, and add the corresponding terms. If (f_0, f_1, f_2, f_3) is the solution, then $Q(x, y) = \frac{f_0}{z}$. Note that the variable z appears only in the expression for f_0 only as a linear factor z .

Solving easily by hand the last three equations of (8) gives the cubic equation for $f_3 = K$:

$$y^2K^3 - (2y + y^3)K^2 + (1 + 2y^2 - x^3)K - y = 0. \quad (9)$$

Plugging everything into the first equation of (8), we get

$$Q(x, y) = \frac{K}{y(1 - K^2)}. \quad (10)$$

Let K_1, K_2 and K_3 be three distinct roots of (9), and let $\sigma_1 = K_1 + K_2 + K_3$, $\sigma_2 = K_1K_2 + K_1K_3 + K_2K_3$, $\sigma_3 = K_1K_2K_3$ be the standard symmetric polynomials. Then

$$\sigma_1 = \frac{2 + y^2}{y}, \quad \sigma_2 = \frac{1 + 2y^2 - x^3}{y^2}, \quad \sigma_3 = \frac{1}{y}.$$

Let Q_i , $1 \leq i \leq 3$, are obtained from (10) by plugging K_i instead of K . We get

$$\begin{aligned} \frac{1}{Q_1} + \frac{1}{Q_2} + \frac{1}{Q_3} &= y \sum \frac{1}{K_i} - y \sum K_i = \frac{y\sigma_2}{\sigma_3} - y\sigma_1 = y^2 - x^3 - 1, \\ \frac{1}{Q_1Q_2} + \frac{1}{Q_2Q_3} + \frac{1}{Q_1Q_3} &= \frac{y^2\sigma_1}{\sigma_3} + y^2\sigma_2 - y^2 \frac{(\sigma_1\sigma_2 - 3\sigma_3)}{\sigma_3} = x^3y^2 - y^4 + x^3 + 2y^2 - 1, \\ \frac{1}{Q_1Q_2Q_3} &= \frac{y^3}{\sigma_3} - y^3\sigma_3 - y^3 \frac{\sigma_1^2 - 2\sigma_2}{\sigma_3} + y^3 \frac{\sigma_2^2 - 2\sigma_1\sigma_3}{\sigma_3} \\ &= x^6 - y^6 - 6y^2x^3 + 3y^4 - 2x^3 - 3y^2 + 1. \end{aligned}$$

We thus get the cubic equation for $\frac{1}{Q}$, and the reciprocal of it is the equation for Q , exactly as formulated in the Theorem.

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