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Simultaneous Avoidance of a Vincular and a Covincular Pattern of Length 3

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Abstract

A pattern is said to be covincular if its inverse is vincular. In this paper we count the number of permutations simultaneously avoiding a vincular and a covincular pattern, both of length 3. We see familiar sequences, such as the Catalan and Motzkin numbers, but also some previously unknown sequences which have close links to other combinatorial objects such as ascent sequences, lattice paths and integer partitions. Where possible we include a generating function for the enumeration. We also give an alternative proof of the classic result that permutations avoiding 123 are counted by the Catalan numbers.

1 Introduction

A permutation π contains a *classical pattern* p, which is itself a permutation, if π contains a *subword* which is *order isomorphic* to p. Babson and Steingrímsson [5] introduced a generalisation of this that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. These are called *vincular patterns*. A further extension, called *bivincular patterns*, was provided by M. Bousquet-Mélou *et al.* [3]. Vincular patterns allow constraints on positions. For symmetry reasons it seems natural to also allow constraints on values and this is what bivincular patterns do. The special case when only constraints on values are allowed we shall call *covincular patterns*. The set of bivincular patterns is closed under the action of the symmetry group of the square and an alternative way of describing the covincular patterns is that they are inverses of vincular patterns.

Simultaneous avoidance of two vincular patterns was looked at by Claesson and Mansour [7]. Allowing one of the patterns to be covincular is a natural follow up question and leads to some interesting results. The overall goal of this paper is to count the number of permutations simultaneously avoiding a length 3 vincular and a length 3 covincular pattern. A summary of our results can be found in Table 1; these results are detailed in Sections 2 through 11.

In Section 12 we present a new, perhaps one of the simplest, proof that the permutations avoiding the classical pattern 123 are counted by the Catalan numbers. The Appendix contains all the results from the paper collected by their respective enumeration.

We shall now present the definitions and notation we use. An *alphabet*, X, is a non-empty set. An element of X is a *letter*. A finite sequence of letters from X is called a *word*. The word with no letters is called the *empty word* and is denoted ϵ. For a word w we say that the *length* of the word, denoted $|w|$, is the number of letters in it, that is if $w = x_1x_2 \dots x_n$ then $|w| = n$. A *subword* of w is a finite sequence $x_{i_1}x_{i_2} \dots x_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

As we are interested in permutations the alphabet we use is $[n] = \{1, 2, \dots, n\}$ for some $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. Any length n permutation is also a length n word $x = x_1x_2 \dots x_n$ of this alphabet where $x_i = \pi(i)$. When we refer to a permutation we are referring to this word. Let $\mathcal{S}_n$ denote the set of all length n permutations. Let $w = w_1w_2 \dots w_k$ and $v = v_1v_2 \dots v_k$ be words with distinct letters. We say that w is *order isomorphic* to v if, for all i and j, we have $w_i < w_j$ precisely when $v_i < v_j$. For example 53296 and 32154 are order isomorphic.

Definition 1.1 (Bousquet-Mélou *et al.* [3, page 4])**.** A *bivincular pattern* is a triple, $p = (\sigma, X, Y)$, where $\sigma \in \mathcal{S}_k$ is called the *underlying permutation* and X and Y are subsets of $\{0, 1, \dots, k\}$. An *occurrence* of p in $\pi \in \mathcal{S}_n$ is a subsequence $w = \pi(i_1) \dots \pi(i_k)$ order isomorphic to σ such that

$$\forall x \in X,\ i_{x+1} = i_x + 1 \quad \text{and} \quad \forall y \in Y,\ j_{y+1} = j_y + 1,$$

where $\{\pi(i_1), \dots, \pi(i_k)\} = \{j_1, \dots, j_k\}$ and $j_1 < j_2 < \cdots < j_k$. By convention, $i_0 = j_0 = 0$ and $i_{k+1} = j_{k+1} = n + 1$. If such an occurrence exists we say that π *contains* σ.

We define the *length* of a bivincular pattern $p = (\sigma, X, Y)$, denoted $|p|$, to be $|\sigma|$. Further, a permutation *avoids* p if it does not contain p. If $Y = \emptyset$ then p is a *vincular* pattern. If $X = \emptyset$ then p is a *covincular* pattern. If $X = Y = \emptyset$ then p is a *classical* pattern. For example, the permutation 15423 contains an occurrence of $(123, \{2\}, \emptyset)$, namely the subword 123, but avoids $(123, \{1\}, \emptyset)$. It contains an occurrence of $(312, \emptyset, \{1\})$, namely the subword 523, but avoids $(312, \emptyset, \{2\})$. The sets of all length n permutations avoiding the pattern p is denoted

$$\mathrm{Av}_n(p) = \{\pi \in \mathcal{S}_n : \pi \text{ avoids } p\},$$

and, for P a set of patterns, $\mathrm{Av}_n(P) = \cap_{p \in P} \mathrm{Av}_n(p)$ and $\mathrm{Av}(P) = \cup_{n \geq 0} \mathrm{Av}_n(P)$.

Below we shall use a pictorial representation of vincular and bivincular patterns that comes from viewing them as mesh patterns [4]: First draw the underlying permutation in a Cartesian coordinate system. Then, for each $i \in X$, shade the ith column and, for each $j \in Y$, shade the jth row; see Figure 1. The shading is used to denote the empty regions in the permutation if we were to overlay the grid onto an occurrence of the pattern in a permutation.

Remark 1.2. If $p = (\sigma, X, Y)$ and $p' = (\sigma, X', Y')$, where $X' \subseteq X$ and $Y' \subseteq Y$, then we immediately have that $\mathrm{Av}_n(p') \subseteq \mathrm{Av}_n(p)$.

We are interested in $|\mathrm{Av}_n(p, q)|$ where $p = (\sigma, X, \emptyset)$ is a length 3 vincular pattern with $X \in \{\emptyset, \{1\}, \{2\}\}$, and $q = (\tau, \emptyset, Y)$ is a length 3 covincular pattern with $Y \in \{\emptyset, \{1\}, \{2\}\}$, in a sense completing the work by Claesson and

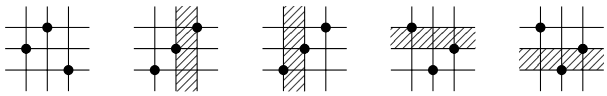


Figure 1: $(231, \emptyset, \emptyset)$, $(123, \{2\}, \emptyset)$, $(123, \{1\}, \emptyset)$, $(312, \emptyset, \{2\})$, and $(312, \emptyset, \{1\})$

Mansour [7]. We skip the case where $X = Y = \emptyset$ since this is classical avoidance of two length 3 patterns, which was done by Simion and Schmidt [15].

An important property of the set of bivincular patterns, as noted by Bousquet-Mélou et al. [3], is that it is closed under the symmetries of the square. The set of patterns we are interested in, i.e. the union of the vincular and the covincular patterns, is also closed under these symmetries. We recall the following observation from Claesson and Mansour [7, pages 3-4].

Lemma 1.3. *Let π be a permutation and p be a pattern, and let π^* and p^* be the permutation and pattern with the same symmetry applied. Then π avoids p if and only if π^* avoids p^*.*

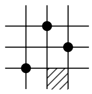
From this we immediately see that if we can find the enumeration of $\mathrm{Av}(p, q)$, for a single pair of patterns p and q, then we automatically have the enumeration for up to 8 other symmetric cases. This reduces the amount of work to be done considerably. In particular, we need only consider when $Y \neq \emptyset$ as otherwise we could take a symmetry to a case where instead $X = \emptyset$. Let

$$\mathcal{P} = \Big\{ (\sigma, X, \emptyset) \,:\, \sigma \in \mathcal{S}_3,\, X \in \big\{\emptyset, \{1\}, \{2\}\big\} \Big\};$$
$$\mathcal{Q} = \Big\{ (\sigma, \emptyset, Y) \,:\, \sigma \in \mathcal{S}_3,\, Y \in \{\{1\}, \{2\}\} \Big\}$$

In total we have $|\mathcal{P} \times \mathcal{Q}| = (3! \cdot 3) \cdot (3! \cdot 2) = 216$ pairs of patterns to consider. In Table 1 we summarize our results on permutations avoiding a pair of patterns from $\mathcal{P} \times \mathcal{Q}$.

It is sometimes possible to show that avoiding a given pattern p is equivalent to avoiding a simpler pattern p'. The following lemma states three instances of this that are used here. This lemma is part of a more general result called The Shading Lemma, due to Hilmarsson *et al.* [10, Lemma 3.11].

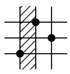
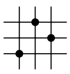
We first need to introduce the idea of a mesh pattern. In our previous pictures we were shading full rows or columns. In a mesh pattern we can shade zero or more individual squares in the diagram. As an example, below is a mesh pattern with a single square shaded:

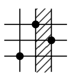
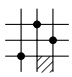
 .

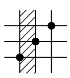
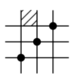
A subsequence $\pi(i)\pi(j)\pi(k)$ of $\pi \in \mathcal{S}_n$, that is order isomorphic to 132, is an occurrence of this particular pattern if there does not exist an m such that $j < m < k$ and $\pi(m) < \pi(i)$. Mesh patterns satisfy a property analogous to Remark 1.2: Given a pattern $p = (\sigma, B)$, where B is the set of squares shaded, and $p' = (\sigma, B')$, where $B' \subseteq B$, then $\mathrm{Av}_n(p') \subseteq \mathrm{Av}_n(p)$. The original definition of mesh patterns can be found in Brändén and Claesson [4].

$\lvert \mathrm{Av}_n(p,q) \rvert$	# pairs	OEIS
C_n	24	A000108
$\binom{n}{2}+1$	16	A000124
2^{n-1}	104	A000079
$\sum_{i=0}^{n} \binom{\binom{i+1}{2}}{n-i}$	8	A121690
$\sum_{k=0}^{n} \binom{\binom{k+1}{2}+n-k-1}{n-k}$	8	A098569
M_n	16	A001006
OGF: $1+\sum_{n\geq 0} x^{n+1} L_n(1+x)$	8	A249560
OGF: $1+\frac{x}{1-x}\sum_{n\geq 0}\sum_{k=0}^{n+1} x^{i+k} L_{n+1,k}\left(\frac{1}{1-x}\right)$	8	A249561
a complicated recurrence relation	8	A249563
a complicated recurrence relation	4	A249562
finite	12	-

Table 1: Permutations avoiding a pair of patterns in $\mathcal{P}\times\mathcal{Q}$. Here, C_n and M_n are the Catalan and Motzkin numbers, respectively. The sequences A249560–A249563 were added to the OEIS [16] by the authors. In A249560 and A249561, $L_n(q)=\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}_q$ and $L_{n,k}(q)=q^{n+\binom{k}{2}}\begin{bmatrix} n-1 \\ k-1\end{bmatrix}_q$

Lemma 1.4. *(i)* Av_n () $=$ Av_n ()

(ii) Av_n () $=$ Av_n ().

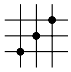
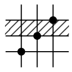
(iii) Av_n () $=$ Av_n ().

Proof. For (i) see [5, Lemma 2] and for (ii) and (iii) see [10, Lemma 3.11]. □

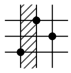
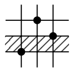
Remark 1.5. It is important to note that $\mathrm{Av}_n(132,\{2\},\emptyset)\neq\mathrm{Av}_n(132,\emptyset,\emptyset)$, for example $2413\notin\mathrm{Av}_4(132,\emptyset,\emptyset)$ but $2413\in\mathrm{Av}_4(132,\{2\},\emptyset)$.

2 Classical avoidance (A000108)

There are 24 pairs $(p,q) \in \mathcal{P} \times \mathcal{Q}$ such that $|\mathrm{Av}_n(p,q)| = C_n$. These break down to just five cases once symmetries are considered; see Table 2 in the Appendix for a full list. They can all be simplified to the avoidance of a single classical pattern. We will look at a case for each argument. By Remark 1.2 we have:

Proposition 2.1. $\mathrm{Av}_n\Bigg($  ,  $\Bigg) = \mathrm{Av}_n(123).$

Similarly, by first using Lemma 1.4i on the first pattern and then using Remark 1.2 on the resulting pair we have:

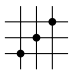
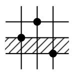
Proposition 2.2. $\mathrm{Av}_n\Bigg($  ,  $\Bigg) = \mathrm{Av}_n(132).$

It is well known that the enumeration for avoiding any classical pattern of length 3 is given by the Catalan numbers; see Knuth [11, Section 2.2.1, Exercises 4 and 5]. Thus the sets in the propositions above all have cardinality C_n. In Section 12 we present a new easy proof that $|\mathrm{Av}_n(123)| = C_n$.

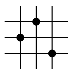
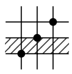
The rest of the pairs that are counted by the Catalan numbers all follow a very similar argument so we move on to the next case.

3 Central polygonal numbers (A000124)

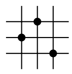
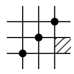
After considering symmetries there are three pairs $(p,q) \in \mathcal{P} \times \mathcal{Q}$ such that $|\mathrm{Av}_n(p,q)| = \binom{n}{2} + 1$; see Table 2 in the Appendix. They all reduce to the already known case $|\mathrm{Av}_n(123,231)| = \binom{n}{2} + 1$ done by Simion and Schmidt [15].

Proposition 3.1. $\mathrm{Av}_n\Bigg($  ,  $\Bigg) = \mathrm{Av}_n(123,231).$

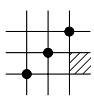
Proof. Use symmetry and Lemma 1.4i. □

Proposition 3.2. $\mathrm{Av}_n\Bigg($  ,  $\Bigg) = \mathrm{Av}_n(123,231).$

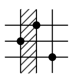
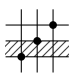
Proof. After applying Lemma 1.4 we see that the set we are interested in is

$\mathcal{B} = \mathrm{Av}\Bigg($  ,  $\Bigg).$

We want to show that $\mathcal{B}$ is equal to $\mathcal{A} = \mathrm{Av}(123,231)$. It is clear than $\mathcal{A} \subseteq \mathcal{B}$. We will show $\mathcal{B} \subseteq \mathcal{A}$ by contraposition. Assume that $\pi \notin \mathcal{A}$. If π contains 231 then it immediately follows that $\pi \notin \mathcal{B}$. If π contains 123 then either π contains



in which case $\pi \notin \mathcal{B}$, or else it contains the pattern where there is a point in the shaded square. This would create an occurrence of 1342 which contains an occurrence of 231 and hence we would have $\pi \notin \mathcal{B}$. Therefore $\mathcal{A} = \mathcal{B}$. □

Proposition 3.3. $\mathrm{Av}_n\Bigg($  ,  $\Bigg) = \mathrm{Av}_n(123,231).$

Proof. Apply Lemma 1.4i and then Proposition 3.2. □

4 Powers of 2 (A000079)

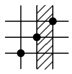
After considering symmetries there are 19 different pairs $(p, q) \in \mathcal{P} \times \mathcal{Q}$ such that $\mathrm{Av}_n(p, q) = 2^{n-1}$. Unsurprisingly a lot of the cases are very similar so we shall not show them all. For a complete list see Table 2 of the Appendix. Most of the cases rely on

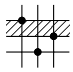
$$|\mathrm{Av}_n(123, 132)| = |\mathrm{Av}_n(132, 312)| = |\mathrm{Av}_n(231, 312)| = 2^{n-1},$$

as shown by Simion and Schmidt [15]. The cases that use similar proof methods to those already seen have not been included in this section. There were, however, two interesting cases which should help the reader get an idea of some of the methods used in the proofs in the sections coming up. In this section we will use generating functions, and a good introduction to generating functions can be found in Wilf [18].

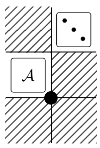
Proposition 4.1. *For $n \geq 1$, the number of permutations in*

$$\mathrm{Av}_n(p, q) = \mathrm{Av}_n\Bigg(\quad ,\quad \Bigg) \quad \textit{is} \quad 2^{n-1}.$$





Proof. Let $\mathcal{A}$ be the set of avoiders in question. Let $\pi \in \mathcal{A}$. As we are avoiding p, the points after the minimum of π form a decreasing sequence. Moreover, if the minimum point is not at the end, in order to ensure that the permutation avoids q, every point to the right of the minimum must be greater than every point on the left of the minimum. Therefore, all non-empty permutations of $\mathcal{A}$ have the form



where $\mathcal{A}$ symbolizes a possibly empty smaller permutation which avoids the patterns, and $\ddots$ symbolizes a decreasing permutation. As the structure is so rigid we can find the ordinary generating function of the avoiders by multiplying together the ordinary generating functions of the component parts. There is one decreasing permutation of length n and so the ordinary generating function is $1/(1 - x)$. The ordinary generating function of a single point will of course be x. Let A be the ordinary generating function for $\mathcal{A}$, then it follows that

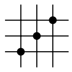
$$A = A \cdot x \cdot \frac{1}{1 - x} + 1$$

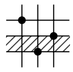
where we add 1 for the empty permutation which trivially avoids both patterns. Rearranging we get

$$A = \frac{1 - x}{1 - 2x} = 1 + \sum_{n \geq 1} 2^{n-1} x^n. \qquad \square$$

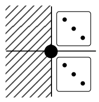
Proposition 4.2. *For $n \geq 1$, the number of permutations in*

$$\mathrm{Av}_n(p, q) = \mathrm{Av}_n\Bigg(\quad ,\quad \Bigg) \quad \textit{is} \quad 2^{n-1}.$$





Proof. Let $\mathcal{A}$ be the set of avoiders in question. Consider the leftmost point ℓ of a permutation in $\mathcal{A}$. To avoid p the points greater than ℓ must form a decreasing sequence and similarly to avoid q the points less than ℓ must form decreasing sequence:



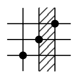
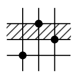
A permutation matching this picture cannot contain an occurrence of $p = 123$, and every occurrence of 312 will have the point ℓ preventing it from being an occurrence of q. Hence if we let A be the exponential generating function for $\mathcal{A}$ then

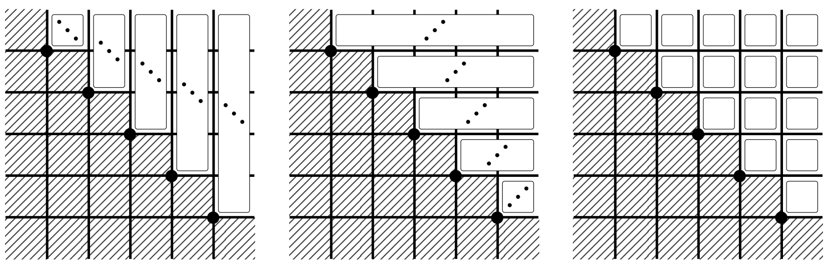
$$A = 1 + \int e^{2x} dx = 1 + \frac{e^{2x}}{2} = 1 + \sum_{n \geq 1} \frac{2^{n-1} x^n}{n!}. \qquad \square$$

5 Left-to-right minima boundaries (A121690)

After symmetries there is exactly one pair $(p, q) \in \mathcal{P} \times \mathcal{Q}$ enumerated by the formula in the following proposition.

Proposition 5.1. *The number of permutations in*

$\mathrm{Av}_n(p, q) = \mathrm{Av}_n \Bigg($  ,  $\Bigg)$ *is* $\displaystyle\sum_{k=0}^{n} \binom{\binom{k+1}{2}}{n-k}.$



(a) (b) (c)

Figure 2: The structure of $\mathrm{Av}_n(p, q)$

Proof. Consider the minimum point, 1, of a permutation in $\mathrm{Av}_n(p, q)$. From the pattern p we see that the points to the right of 1 form a decreasing sequence. Moreover, the points between any two adjacent left-to-right minima must also form a decreasing sequence giving the structure in Figure 2a, where the permutation we have drawn has five left-to-right minima.

Now, consider the leftmost point, say ℓ, of a permutation in $\mathrm{Av}_n(p, q)$. From the pattern q we see that the points greater than ℓ must form an increasing sequence. Moreover, considering the points above and below any two adjacent left-to-right minima we get the structure in Figure 2b, where, again, our permutation has five left-to-right minima. When we overlay the given conditions above we get the structure in Figure 2c, where each of the squares in the diagram must be both increasing and decreasing. Therefore each square must be empty or contain a single point. Also, the structure of the rows and columns will be determined as increasing and decreasing, respectively, no matter which squares have points. Therefore, placing any number of points into the squares (at most one in each) will create a unique permutation (see Figure 3).

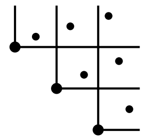


Figure 3: The permutation $673841952 \in \mathrm{Av}_9(p, q)$

Consider creating a permutation $\pi \in \mathrm{Av}_n(p, q)$ with k left-to-right minima. We need to know how to place the remaining $n - k$ points. There will be $\binom{k+1}{2}$ squares available to choose from (see the diagram) and placing the $n - k$ points into any subset of those squares will create a unique permutation. Thus, summing over the number of left-to-right minima, we get

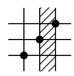
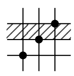
$$|\mathrm{Av}_n(p, q)| = \sum_{k=0}^{n} \binom{\binom{k+1}{2}}{n-k}. \qquad \square$$

It is interesting to note from the above proof and the formula that a permutation with k left-to-right minima will be of length at most $k + \binom{k+1}{2}$. Also, a length n avoider will have at least $\left\lceil \frac{-3+\sqrt{9+8n}}{2} \right\rceil$ left-to-right minima.

6 Barred patterns (A098569)

After considering symmetries there are two pairs $(p, q) \in \mathcal{P} \times \mathcal{Q}$ enumerated by the formula in the following proposition.

Proposition 6.1. *The number of permutations in*

$\mathrm{Av}_n(p, q) = \mathrm{Av}_n\Bigg($  ,  $\Bigg)$ *is* $\displaystyle\sum_{k=0}^{n} \binom{\binom{k+1}{2} + n - k - 1}{n-k}.$

Proof. Consider the left-to-right minima of a permutation $\pi \in \mathrm{Av}_n(p, q)$ as we did in Proposition 5.1. The points between any two adjacent left-to-right minima must form a decreasing sequence and the points above and below any two adjacent left-to-right minima must also form a decreasing sequence. If we overlay these two conditions we get a structure like that in Figure 2c, where, in this case, each of the squares in the diagram must be decreasing. Also, the structure of the rows and columns will be determined as decreasing no matter which squares have points. Therefore, placing any number of points into the squares will create a unique permutation, and so the ordinary generating function for $|\mathrm{Av}_n(p, q)|$ is

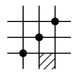
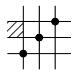
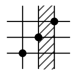
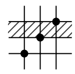
$$\sum_{k \geq 0} x^k \left(\frac{1}{1-x}\right)^{\binom{k+1}{2}}.$$

The coefficient of x^k in $1/(1-x)^n$ is $\binom{n+k-1}{k}$ which concludes the proof. $\square$

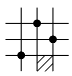
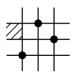
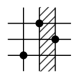
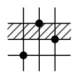
It is possible to show that avoiding the two patterns p and q, above, is equivalent to avoiding a single *barred* pattern. For a more detailed account of barred patterns see Pudwell [13]. The following is the definition which can be found in that reference.

Definition 6.2. (Pudwell [13, p. 1]) Let p be a barred pattern. Let q be the pattern with the bars removed and let p' be the reduced version of the pattern in which the barred numbers are removed. We say that a permutation contains p if every occurrence of p' can be extended to an occurrence of q.

For example, a permutation contains $\bar{4}25\bar{1}3$ if every occurrence of 132 is contained as 253 in a 42513 pattern. Barred patterns can often be thought of as mesh patterns (see Ulfarsson [17, p. 5]). For instance,

$\mathrm{Av}_n(\bar{4}23\bar{1}5) = \mathrm{Av}_n\Big($  ,  $\Big) = \mathrm{Av}_n\Big($  ,  $\Big)$

and

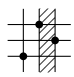
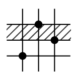
$\mathrm{Av}_n(\bar{4}25\bar{1}3) = \mathrm{Av}_n\Big($  ,  $\Big) = \mathrm{Av}_n\Big($  ,  $\Big)$,

where the second equalities in both equations come from Lemma 1.4

Corollary 6.3. *The number of permutations in* $\mathrm{Av}_n\left(\bar{4}23\bar{1}5\right)$ *is*

$$\sum_{k=0}^{n} \binom{\binom{k+1}{2} + n - k - 1}{n-k}.$$

This confirms the conjecture from Pudwell [13, page 8] that $\bar{4}25\bar{1}3$ and $\bar{4}23\bar{1}5$ are Wilf-equivalent. If we were to apply the same method as in the proof of Proposition 6.1 to

$\mathrm{Av}_n\Big($  ,  $\Big) = \mathrm{Av}_n(\bar{4}25\bar{1}3)$

then we would have a similar structure with the left-to-right minima, where, however, we get increasing sequences in the squares and along the rows and columns.

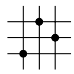
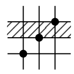
7 Motzkin numbers (A001006)

The Motzkin numbers, M_n, form a well known sequence which can be defined by a functional equation their ordinary generating function satisfies:

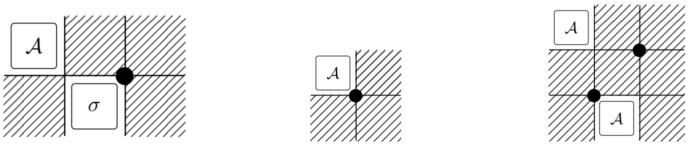
$$M = 1 + xM + x^2M^2 \quad \text{where} \quad M = \sum_{n\geq 0} M_n x^n.$$

For more information on Motzkin numbers see e.g. [16, A001006]. After considering symmetries and Lemma 1.4 we have two cases such that $|\mathrm{Av}_n(p,q)| = M_n$ and $(p,q) \in \mathcal{P} \times \mathcal{Q}$. For a full list see Table 2 of the Appendix.

Proposition 7.1. *The number of permations in*

$$\mathrm{Av}_n(p,q) = \mathrm{Av}_n\Bigg($$  ,  $$\Bigg) \quad \textit{is} \quad M_n.$$

Proof. Consider a permutation π in $\mathcal{A} = \mathrm{Av}(p,q)$. Further, consider the rightmost point of π. For π to avoid p the structure of π must be like Figure 4a. With regard to σ, let us consider two cases. Either σ is empty or it has at least



(a) (b) (c)

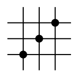
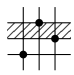
Figure 4: The structure of $\mathrm{Av}(p,q)$

one point. If σ is empty the structure looks like Figure 4b. If σ is non-empty then consider the maximum point, m, of σ. If there was a point to the left of m in σ then this point together with m and the rightmost point would create an occurrence of q. Therefore there must be no points to the left of m in σ. Thus we can place any possibly empty smaller permutation in $\mathcal{A}$ to the right of the maximum without creating an occurrence of p or q, and so we have the structure in Figure 4c.

In conclusion, any non-empty permutation in $\mathcal{A}$ either has a structure described by Figure 4b or a structure described by Figure 4c. Letting A denote the ordinary generating function for $\mathcal{A}$ we thus have $A = 1 + xA + x^2A^2$, from which the claim follows. $\square$

We now go on to the second case. We will consider the structure of the avoiders in terms of the left-to-right minima, as in Proposition 5.1.

Proposition 7.2. *The number of permutations in*

$$\mathrm{Av}_n(p,q) = \mathrm{Av}_n\Bigg($$  ,  $$\Bigg) \quad \textit{is} \quad M_n.$$

Proof. Let $\pi \in \mathrm{Av}_n(p,q)$ and consider the boundary of (the diagram of) π given by the left-to-right minima. As in Proposition 5.1, any cell in the diagram must be both increasing and decreasing and so the cell is empty or contains exactly one point. Because the rows are increasing and π avoids 123 there can be at most one point in each row. Moreover, if there is a point in a cell then we cannot place a point in a cell further to the right and above.

Pick the leftmost point in the leading diagonal of this grid. The points above this cell will then form a subword of π which is of shorter length and also avoids both patterns. The points below this cell will similarly form a subword which avoids both patterns; see Figure 5. Notice that this process is reversible: we can take a pair of avoiders of lengths k and $n-k-2$, respectively, and glue them together by adding two points in this way.

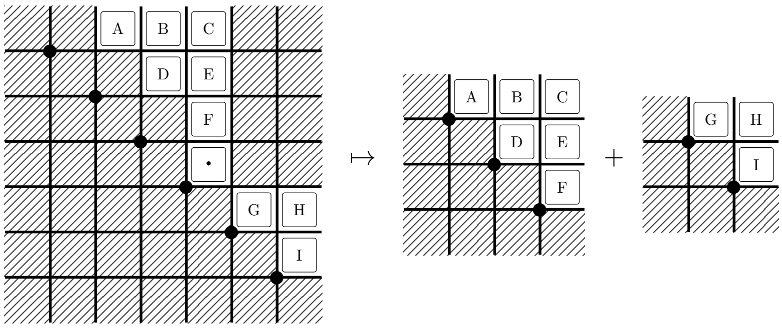


Figure 5: Decomposing a permutation in $\mathrm{Av}_n(p, q)$

There is also the case when there are no points directly to the right of a left-to-right minima. In this case we remove the minimum point and tuck the points in the same way we did for the top half of the previous case, producing an avoider which is shorter in length; see Figure 6. Again this is reversible, so

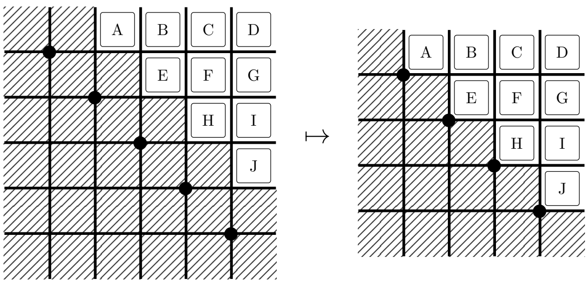


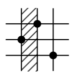
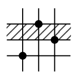
Figure 6: "Shortening" a permutation in $\mathrm{Av}_n(p, q)$

we can take any length $n-1$ avoider and append a new minimum in this way to create a length n avoider. Thus, letting A be the ordinary generating function for $\mathcal{A}$ we get that $A = 1 + xA + x^2A^2$. $\square$

We will use a similar method to give a new proof of the enumeration of $\mathrm{Av}_n(123)$ in Section 12.

8 Lattice paths and their area (A249560)

Up to symmetries there is a single pair (p, q) in $\mathcal{P} \times \mathcal{Q}$ with the enumeration given in Proposition 8.2, namely

$\mathrm{Av}_n\Big($  ,  $\Big)$.

To find the enumeration of this set we shall consider a different boundary than those seen in previous sections. Our boundary here will be left-to-right minima and right-to-left minima. We will first find a bijection between lattice paths and the boundaries of these permutations. Then we extend this bijection by considering the area under these paths.

For our purposes a *lattice path* of length n is a path that starts at $(0,0)$ and has n steps each of which is

$$N \;:\; (x,y) \mapsto (x, y+1) \text{ or}$$

$$E \;:\; (x,y) \mapsto (x+1, y).$$

Clearly there are 2^n paths of length n. The following result is due to Simion and Schmidt [15], but we give a proof that is different from theirs.

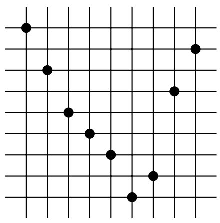
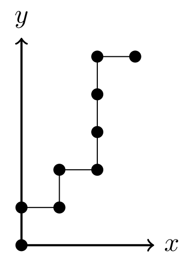
Proposition 8.1 (Simion and Schmidt [15]). *There is a bijection between the length $n-1$ lattice paths and the permutations in* $\mathrm{Av}_n(231,132)$.

Proof. For $\pi \in \mathrm{Av}_n(231,132)$ define the path $w = x_n x_{n-1} \dots x_2$ by

$$x_k = \begin{cases} N & \text{if } \pi^{-1}(k) < \pi^{-1}(1), \\ E & \text{if } \pi^{-1}(k) > \pi^{-1}(1). \end{cases}$$

To see that $\pi \mapsto w$ is invertible note that the points to the left of the minimum of π form a decreasing sequence, and, similarly, the points to the right of the minimum form an increasing sequence. Thus, any permutation $\pi \in \mathrm{Av}_n(231,132)$ is uniquely specified by the set $\{i : \pi^{-1}(i) < \pi^{-1}(1)\}$ which coincides with the set $\{i : x_i = N\}$. $\square$

For example

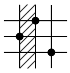
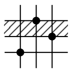
$\pi = 975431268 =$  $\mapsto$ 

Every lattice path has a certain area enclosed by the path and the x-axis. We use q-binomials to capture this, see for example Azose [2] for a detailed look into this. In terms of the q-binomial coefficients the number of length n paths, with m steps that are E, is given by $\begin{bmatrix} n \\ m \end{bmatrix}_q$ where the coefficient of q^k is the number of paths with area k. Let

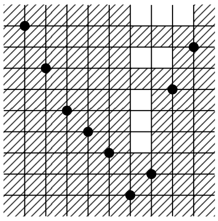
$$L_n(q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q,$$

which is the distribution of area over all length n paths. We will now link this to pattern avoidance. We use r for our second pattern to avoid confusion with q-binomials.

Proposition 8.2. *The ordinary generating function for*

$$\mathrm{Av}_n(p,r) = \mathrm{Av}_n\Big($$  ,  $$\Big) \quad \textit{is} \quad 1 + \sum_{n \geq 0} x^{n+1} L_n(1+x).$$

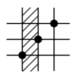
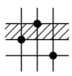
Proof. Every permutation has a left-to-right minima and right-to-left minima boundary that is in the set $\mathrm{Av}_n(231, 132)$. Consider a particular boundary of right-to-left minima and left-to-right minima of a permutation avoiding p and r. To avoid p the points to the left of the minimum form a decreasing sequence and hence there are no other points in this region; also, the columns in between the right-to-left minima are forced to be increasing. To avoid r, the rows in between the right-to-left minima must be decreasing, and the rows directly above a left-to-right minimum must be empty. As an example, for the boundary given by $\pi = 975431268$ we get the following restrictions.



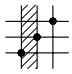
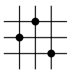
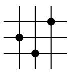
In each of the unshaded squares we can place either a single point or leave it empty, and each such choice will create a unique permutation. The number of unshaded squares is given by the area under the lattice path corresponding to the boundary as in Proposition 8.1. Hence, to count permutations in $\mathrm{Av}(p, q)$ we first fix the size of boundary, say $n + 1$, giving the factor x^{n+1}. Then we substitute $q = 1 + x$ into $L_n(q)$, since this is the generating function for all length $n + 1$ boundaries with q marking the squares that we can place a point in or leave empty. $\square$

9 Partitions into distinct parts (A249561)

Up to symmetries there is only a single pair (p, q) in $\mathcal{P} \times \mathcal{Q}$ with the enumeration given in Proposition 9.5, namely

$$\mathcal{A} = \mathrm{Av}(p,r) = \mathrm{Av}\Big($$  ,  $$\Big).$$

To find the enumeration of this set we consider the boundary of a permutation $\pi \in \mathcal{A}$ given by its right-to-left maxima and right-to-left minima. Taking this boundary of any permutation will result in some permutation that avoids 231 and 213. Since avoiding 231 implies avoiding r by Remark 1.2 it is clear that any such boundary for a permutation in $\mathcal{A}$ is in the set

$$\mathrm{Av}\Big($$  ,  ,  $$\Big).$$

If we take one of these boundaries and consider shading the restrictions of p and r we see that the number of right-to-left maxima between the two

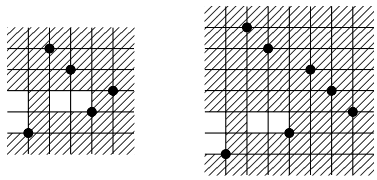
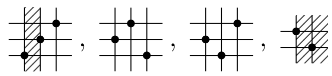


Figure 7: The boundaries given by 15423 and 1762543

rightmost right-to-left minima does not change the number of unshaded squares (see Figure 7).

We therefore start by considering π, where $\pi(n-1) = \pi(n) - 1$, i.e. with a single right-to-left maximum after the rightmost left-to-right maximum. In terms of pattern avoidance these boundaries are given by the set

$\mathcal{B}_n = \mathrm{Av}_n$ ().

Here the last pattern ensures that our condition of $\pi(n-1) = \pi(n) - 1$ is enforced.

We will now show that the permutations in $\mathcal{B}_n$ are in bijection with a subset of lattice paths.

Definition 9.1. Let $w = x_1 x_2 \dots x_n$ be a lattice path. We say w is a *restricted lattice path* if

(i) $x_0 = N$,

(ii) $x_n = E$ and

(iii) for all $i \in \{1, \dots, n-1\}$ we have $x_i x_{i+1} \neq EE$.

We define $\mathcal{R}_n$ to be the set of all restricted lattice paths of length n.

Remark 9.2. A restricted lattice path, w, represents a unique integer partition since w starts with an N step and ends with an E step. Furthermore, if an integer partition can be represented by some restricted lattice path then it must be an integer partition with distinct parts since we can never have two columns of the same height as there are no two consecutive E steps.

Proposition 9.3. *There is a bijection between the restricted lattice paths in $\mathcal{R}_n$ and the permutations in $\mathcal{B}_n$.*

Proof. Let $\pi \in \mathcal{B}_n$. Define the path $w = N x_1 x_2 \dots x_{n-1}$ by

$$x_k = \begin{cases} N & \text{if } \pi^{-1}(k) > \pi^{-1}(n), \\ E & \text{if } \pi^{-1}(k) < \pi^{-1}(n). \end{cases}$$

By definition the path w starts with an N step. Also, π ends with an ascent, and so $x_{n-1} = E$. That w doesn't contain EE can be seen by contraposition: Assume that $x_i x_{i+1} = EE$, then $\pi^{-1}(i) < \pi^{-1}(n)$ and $\pi^{-1}(i+1) < \pi^{-1}(n)$ which means that π either contains the subsequence $(i, i+1, n)$ or it contains the subsequence $(i+1, i, n)$. In the latter case we have an occurrence of 213 and we are done, so assume the former. If i and $i+1$ are adjacent in π then we have an occurrence of p. If not, then there must be a point in one of the lower three squares of the shading of p. But either of these three options leads to an occurrence of p or 213. This shows that the range of the mapping $\pi \mapsto w$ is contained in $\mathcal{R}_n$. To see that $\pi \mapsto w$ is invertible we can reason in a way that is similar to the proof of Proposition 8.1. □

Remark 9.4. Let λ be the integer partition obtained from applying the above bijection to the permutation π. By Remark 9.2 it is clear that λ has distinct parts. The number of points greater than $\pi(n)$ is one less than the maximum part of λ and the number of points less than $\pi(n)$ is the number of parts of λ. See Figure 8 for an example.

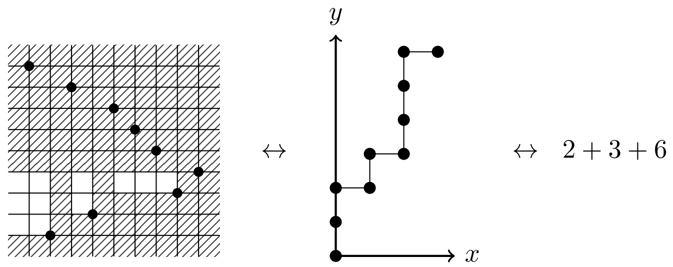


Figure 8: The boundary given by the permutation 918276534 with the corresponding lattice path and integer partition with distinct parts

Partitions are well studied objects (see for example Andrews [1]) and it can be shown that if q keeps track of the sum of the parts of our partition then the number of partitions with maximum part n into k distinct parts is given by

$$L_{n,k}(q) = q^{n+\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

Proposition 9.5. *The ordinary generating function for $\mathcal{A}$ is given by*

$$1 + \frac{x}{1-x} \sum_{n \geq 0} \sum_{k=0}^{n+1} x^{i+k} L_{n+1,k}\left(\frac{1}{1-x}\right).$$

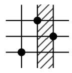
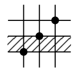
Proof. Let $\pi \in \mathcal{A}$. Consider the boundary given by right-to-left maxima and right-to-left minima. As before we shall assume that $\pi(n-1) = \pi(n)-1$ and thus the boundary is in $\mathcal{B}_n$. To avoid r the points above $\pi(n)$ must form a decreasing sequence. There must also be no points between a right-to-left minimum and right-to-left minimum in order to avoid p. In the remaining unshaded regions, columns are decreasing (to avoid p) and rows are decreasing (to avoid r). Thus, an unshaded square can contain a decreasing sequence of any length. The bijection in Proposition 9.3 gives a bijection that defines the available squares, and, considering Remark 9.4, it follows that the ordinary generating function for $\mathcal{A}$ is as claimed. □

10 Recurrence relations (A249562 & A249563)

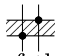
The two remaining pairs, $(p, q) \in \mathcal{P} \times \mathcal{Q}$, which are unique up to symmetries, we enumerate with recurrence relations.

10.1 A recurrence for A249563

The set we are enumerating is

$$\mathrm{Av}_n(p, q) = \mathrm{Av}_n\Bigg($$  ,  $$\Bigg).$$

Let $\pi \in \mathrm{Av}_n(p, q)$. Write $\pi = m_1\Pi_1 m_2\Pi_2 \dots m_k\Pi_k$ where $m_1, m_2, \dots, m_k$ are the left-to-right minima of π, and $\Pi_1, \Pi_2, \dots, \Pi_k$ are the remaining points in between the minima. To avoid p each Π_i must be increasing. We shall call $m_i\Pi_i$ the ith *block* of π.

Assume that π has an occurrence of the pattern . Because π avoids q there cannot be any points above and to the right of this occurrence. This motivates the following definition.

Definition 10.1. For a permutation $\pi \in \mathrm{Av}_n(q)$, if there exists i and j such that $j > i$ and $\pi(i) = \pi(j) - 1$ then we call $\pi(j)$ a *ceiling point*.

Going back to analyzing the structure of $\pi \in \mathrm{Av}_n(p, q)$, notice that if we remove the maximum, n, from π then the resulting permutation will be in $\mathrm{Av}_{n-1}(p, q)$. This gives us ground for a recursion. Consider inserting n into a permutation in $\mathrm{Av}_{n-1}(p, q)$. Where we can place n depends on several factors. Let $a_{n,k,i,\ell}$ be the number of avoiders where n is the length of the permutation; k is the number of blocks; i is the block that the maximum is in and ℓ is the block containing the leftmost ceiling point; if there is no ceiling point then we let $\ell = 0$.

It is clear that we can have at most n blocks and that n cannot occur after the leftmost ceiling point. Therefore if $n < k$ or $i > \ell$ (while $\ell > 0$) then $a_{n,k,i,\ell} = 0$. There is a unique length n permutation with n blocks namely the decreasing one. The maximum is in the first block (except when $k = 0$, in which case there is no maximum so we say $i = 0$), hence we have

$$a_{n,n,1,0} = 1 = a_{0,0,0,0}.$$

We have three cases to consider. The new maximum, n, is inserted to become a ceiling point (this is when $i = \ell$); n is inserted to create a new block (when $i = 1$) or n is inserted into an existing block but is not a ceiling point.

We first consider inserting n into a length $n - 1$ permutation so as to ensure it is a ceiling point. It must be placed after the current maximum but before the leftmost ceiling point. If the smaller permutation has no ceiling point then we can freely insert n. Hence

$$a_{n,k,\ell,\ell} = a_{n-1,k,1,0} + \sum_{j=1}^{i} \sum_{m=i+1}^{k} a_{n-1,k,j,m}.$$

Now we consider inserting n so as it is not a ceiling point. We may either create a new block (when $i = 1$) or place it into an already existing block.

Consider inserting it into an existing block, then it cannot be placed after the current maximum or else it will become a ceiling point. The leftmost ceiling point will carry over to the larger permutation. Therefore, if $i < \ell$,

$$a_{n,k,i,\ell} = \sum_{j=i+1}^{k} a_{n-1,k,i,\ell}.$$

To create a new block we can add n to any length $n-1$ avoider but there will of course be a shift of indices. If $i = 1$ we get

$$a_{n,k,i,\ell} = \sum_{j=i+1}^{k} a_{n-1,k,j,\ell} + \sum_{j=0}^{k-1} a_{n-1,k-1,j,\ell-1}.$$

This, with the initial conditions, gives a recursion for $a_{n,k,i,\ell}$.

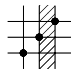
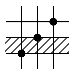
Proposition 10.2. *The number of permutations in* $\mathrm{Av}_n(p,q)$ *is given by*

$$\left\{ \sum_{k=0}^{n} \sum_{i=0}^{k} \sum_{\ell=0}^{k} a_{n,k,i,\ell} \right\}_{n \geq 0} = \{1, 1, 2, 4, 9, 22, 57, 156, 447, 1335, 4140, \dots\}.$$

This sequence was added to the OEIS by the authors [16, A249563].

10.2 A recurrence for A249562

Here we enumerate the set

$$\mathrm{Av}_n(p,q) = \mathrm{Av}_n\Bigg($$  ,  $$\Bigg).$$

Let $\pi \in \mathrm{Av}_n(p,q)$. Write $\pi = m_1\Pi_1 m_2\Pi_2 \dots m_k\Pi_k$ where $m_1, m_2, \dots, m_k$ are the left-to-right minima of π, and $\Pi_1, \Pi_2, \dots, \Pi_k$ are the remaining points in between the minima. To avoid p each Π_i must be decreasing. We shall call $m_i\Pi_i$ the ith *block* of π. Notice that removing n from π will result in a permutation in $\mathrm{Av}_{n-1}(p,q)$. Therefore we will build these recursively by adding in a new maximum.

We set up as follows: let n be the length of the permutation; let k be the number of blocks; let i be the position of the maximum; and let ℓ be the position of the leftmost ceiling point (if no ceiling points we set $\ell = 0$). Let $\hat{a}_{n,k,i,\ell}$ be the number of avoiders where the maximum is a ceiling point; let $\bar{a}_{n,k,i,\ell}$ be the number of avoiders where the maximum is a left-to-right minimum; and let $\check{a}_{n,k,i,\ell}$ be the number of avoiders where the maximum is neither a ceiling point nor a left-to-right minimum. Then we are interested in

$$a_{n,k,i,\ell} = \hat{a}_{n,k,i,\ell} + \bar{a}_{n,k,i,\ell} + \check{a}_{n,k,i,\ell}.$$

First consider adding the maximum as a ceiling point. If we want to add n to the first block then we must have $m_1 = n-1$ and the leftmost ceiling point can be anywhere. Therefore,

$$\hat{a}_{n,k,1,\ell} = \sum_{m=0}^{k} \bar{a}_{n-1,k,1,m}.$$

Otherwise we want to add n to any of the other blocks. We can do this to a permutation starting with a maximum as long as it is before the leftmost ceiling point. If the previous maximum is not a ceiling point then we must add it after the maximum but before the leftmost ceiling point. We cannot create a new maximum ceiling point if the previous one is already a ceiling point. Hence, if $i > 1$,

$$\hat{a}_{n,k,i,\ell} = \sum_{m=\ell}^{k} \bar{a}_{n-1,k,1,m} + \sum_{j=1}^{i-1} \sum_{m=i}^{k} \check{a}_{n-1,k,j,m}.$$

We can add a maximum to the far left to any length $n-1$ avoider to create a length n avoider, so we get

$$\bar{a}_{n,k,i,\ell} = \sum_{j=1}^{k-1} a_{n-1,k-1,j,\ell-1}.$$

We can add a new maximum to an existing block so that it is not a ceiling point as long as it comes before the current maximum, so

$$\check{a}_{n,k,i,\ell} = \sum_{j=i}^{k} \hat{a}_{n-1,k,j,\ell} + \check{a}_{n-1,k,j,\ell}.$$

This together with $\bar{a}_{n,n,1,0} = 1$, $\hat{a}_{n,n-1,i,\ell} = 1$, and the conditions that $n > k > i$ and $i < \ell$ is enough to enumerate these permutations recursively.

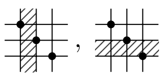
Proposition 10.3. *The number of permutations in* $\mathrm{Av}_n(p, q)$ *is given by*

$$\left\{ \sum_{k=0}^{n} \sum_{j=0}^{k} \sum_{\ell=0}^{k} a_{n,k,i,\ell} \right\}_{n \geq 0} = \{1, 1, 2, 5, 14, 43, 143, 509, 1921, 7631, 31725, \ldots\}.$$

This sequence was added to the OEIS by the authors [16, A249562].

11 A closer look at A249562

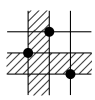
In this section we consider the set

$\mathrm{Av}(p, q) = \mathrm{Av}\Big($  $\Big)$

which is the reverse symmetry of the set enumerated in Section 10.2. We establish a bijection between this set and certain ascent sequences.

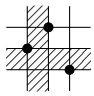
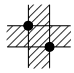
11.1 Modified ascent sequences

For the sake of brevity, the proofs in this section are omitted but instead included in the Appendix. All of our definitions and notation are from Bousquet-Mélou et al. [3]; they give a bijection ϕ from length n modified ascent sequences to length n permutations that avoid the bivincular pattern

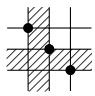
 .

Given a modified ascent sequence $\hat{x}$ their bijection is as follows: First take $\hat{x}$ and write the numbers 1 through to n below it. Sort the pairs $\binom{\hat{x}_i}{i}$ in ascending order with respect to the top entry and break ties in descending order with respect to the bottom entry. The resulting bottom row is $\phi(\hat{x})$. For example, if $x = (0,1,0,2,1,3,0,1)$ then $\hat{x} = (0,2,0,3,2,4,0,1)$ and if we sort $\substack{02032401\\12345678}$ as described we get $\substack{00012234\\73185246}$ and hence $\phi(\hat{x}) = 73185246$.

As pointed out in Parviainen [12, Section 5], if we restrict this bijection to modified ascent sequences of length n which have no two adjacent elements equal, then ϕ becomes a bijection into the set of length n permutations avoiding

 and  .

Parviainen further gives a bijection, that we shall call θ, from these modified ascent sequences into the set of permutations of length $n-1$ avoiding

 .

This bijection is defined as follows: Let $\hat{x}$ be a modified ascent sequence with no two adjacent elements equal. Write the numbers 1 through $n-1$ below it, skipping x_1. Sort the pairs $\binom{x_i}{i-1}$ with respect to ascending order in the top and break ties by sorting in ascending order with respect to the bottom entry; the bottom row in $\theta(\hat{x})$. Let us use the example sequence $\hat{x} = (0,2,0,3,2,4,0,1)$ (same as above). Sorting $\substack{2032401\\1234567}$ we get $\substack{0012234\\2671435}$ and thus $\theta(\hat{x}) = 2671435$. In summary we have the bijections shown in Figure 9.

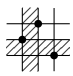
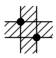
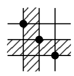
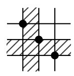
$\mathrm{Av}_n\Big($  ,  $\Big) \overset{\phi}{\leftrightarrow} \left\{ \begin{array}{c} \text{Length } n \text{ modified ascent} \\ \text{sequences with no two} \\ \text{adjacent elements equal} \end{array} \right\} \overset{\theta}{\leftrightarrow} \mathrm{Av}_{n-1}\Big($  $\Big)$

Figure 9: A summary of the bijections

We now return to the problem at hand. It is easy to see that

$\mathrm{Av}(p,q) \subseteq \mathrm{Av}_n(r) = \mathrm{Av}_n\Big($  $\Big) = B$

Pattern avoidance on modified ascent sequences can be defined in a natural way, see Duncan and Steingrimsson [8]. Let θ be the bijection defined above. Then we have the following result.

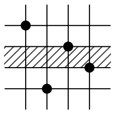
Lemma 11.1. *Let $\hat{x}$ be a modified ascent sequence with no two adjacent elements equal. Then*

(i) $\theta(\hat{x})$ *avoids* 321 *if and only if* $\hat{x}$ *avoids* 321*,*

(ii) $\theta(\hat{x})$ *avoids* q *if and only if* $\hat{x}$ *avoids* p *and*

(iii) $\theta(\hat{x})$ *avoids* p *if and only if* $\hat{x}$ *avoids* q *and the letter playing the role of the* 2 *is the leftmost of that value in* $\hat{x}$*.*

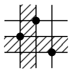
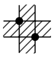
We now consider the bijection ϕ described above. If we let

$s =$ 

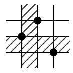
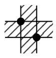
then we have the following lemma.

Lemma 11.2. *Let* $\hat{x}$ *be a modified ascent sequence with no two adjacent elements equal. Then* $\hat{x}$ *avoids* p *if and only if* $\phi(\hat{x})$ *avoids* s*.*

Let $\overline{\mathrm{Av}}_n(p)$ denote the set of length n modified ascent sequences with no two adjacent elements equal such that they avoid the pattern p. Combining the two previous lemmas we get the following correspondence:

$\mathrm{Av}_n\Big($  ,  $, s\Big) \xleftarrow{\phi \text{ injective}} \overline{\mathrm{Av}}_n(p, \bar{q}) \overset{\theta}{\longleftrightarrow} \mathrm{Av}_n(p, q)$

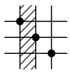
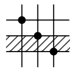
where $\bar{q}$ is the same as q except the 2 in $\bar{q}$ is the leftmost 2. This leads to the question: does avoidance of $\bar{q}$ correspond to something transparently describable (perhaps in terms of pattern avoidance) in

$\mathrm{Av}_n\Big($  ,  $\Big)$

under the bijection ϕ?

11.2 A249562 in terms of 321-avoiding permutations

For each permutation avoiding

$p =$  and $q =$ 

we consider the boundary given by the left-to-right maxima and the right-to-left minima. Since any avoider of 321 avoids p and q each boundary is in $\mathrm{Av}(p, q)$. Given such a boundary, the columns to the right of a left-to-right maximum must be empty to avoid p and the rows immediately above a left-to-right minimum must be empty to avoid q. There are however some unshaded squares. Within any such square the points must form a, possibly empty, increasing sequence. In fact, the positions of the points are determined as the rows must be increasing to avoid q, and the columns must be increasing to avoid p.

Let $m_1 < m_2 < \cdots < m_k$ be the right-to-left minima of π, and let

$$\mathrm{sq}(\pi) = \sum_{i=1}^{k-1} \max(c_i - 1, 0), \text{ where } c_i = \left|\left\{j \in \{1, 2, \ldots, m_i\} : \pi(j) > m_{i+1}\right\}\right|.$$

Then the number of unshaded squares in the boundary diagram of π is $\mathrm{sq}(\pi)$ and the ordinary generating function for $\mathrm{Av}(p, q)$ is

$$\sum_{n \geq 0} \sum_{\pi \in \mathrm{Av}_n(321)} x^n \left(\frac{1}{1-x}\right)^{\mathrm{sq}(\pi)}.$$

12 Avoiding 123

It is well known that $|\mathrm{Av}_n(123)| = C_n = \binom{2n}{n}/(n+1)$, the nth Catalan number. Inspired by Sections 6 and 7 we shall derive this fact in a alternative way.

Proposition (Hammersley [9], Rogers [14])**.** $|\mathrm{Av}_n(123)| = C_n$.

Proof. Given a permutation avoiding 123 we can use its left-to-right minima to partition the remaining points into cells. Each cell must be decreasing and the same is true for each row and each column, as noted by Claesson [6]. Therefore the permutation is uniquely determined by the number of points in each cell. If a cell is non-empty then all the cells strictly above and strictly to the right of it will be empty. See for example Figure 10 where we have five left-to-right minima and are assuming that $A \neq \epsilon$. This property allows us to construct

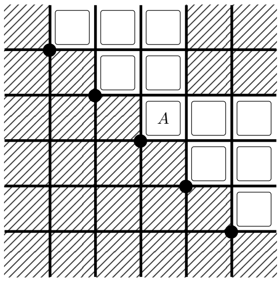


Figure 10: An avoider of 123 with five left-to-right minima where $A \neq \epsilon$

a larger avoider from two smaller ones. See Figure 11 where F' has one more

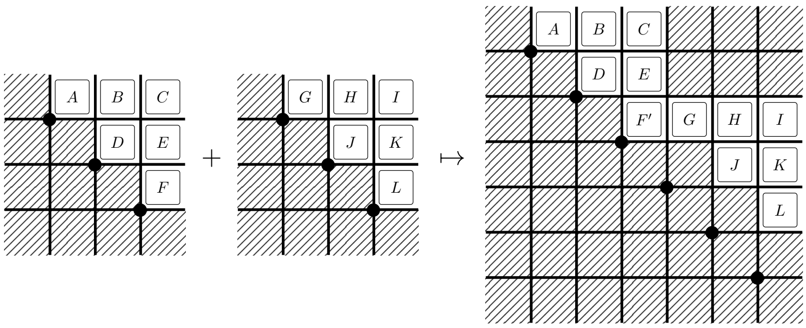


Figure 11: The sum of two 123-avoiding permutations

point than F. If we are adding the empty permutation, on the left, we instead add a left-to-right minimum. See Figure 12. This construction is reversible. Therefore, if we let A be the generating function then it is clear that it will satisfy $A = 1 + x(A-1)A + xA = 1 + xA^2$, which can be seen as the defining functional equation for the ordinary generating function of the Catalan numbers. $\square$

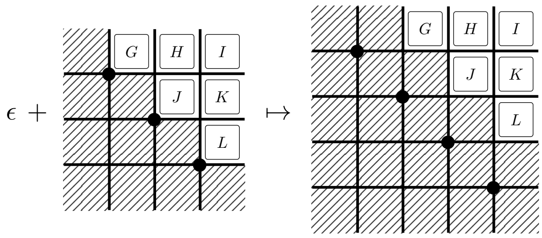


Figure 12: The sum of the empty permutation and a 123-avoiding permutation

Appendix

In the tables below the column titled "Method" will point the reader in the direction of an argument to confirm the enumeration. In some cases these links will be to the proposition or lemma with the patterns, but often just to a similar case where the same or a similar argument is used.

Method	p	q	Enumeration
Prop. 2.1	$(123, \emptyset, \emptyset)$	$(123, \emptyset, \{1\})$	C_n
	$(132, \emptyset, \emptyset)$	$(132, \emptyset, \{1\})$	
	$(132, \emptyset, \emptyset)$	$(132, \emptyset, \{2\})$	
Prop. 2.2	$(132, \{1\}, \emptyset)$	$(132, \emptyset, \{1\})$	C_n
	$(132, \{1\}, \emptyset)$	$(132, \emptyset, \{2\})$	
Prop. 2.1	$(123, \emptyset, \emptyset)$	$(231, \emptyset, \{1\})$	$\binom{n}{2} + 1$
Prop. 3.2	$(132, \emptyset, \emptyset)$	$(321, \emptyset, \{1\})$	$\binom{n}{2} + 1$
Lemma 1.4 and Prop. 3.2	$(123, \{2\}, \emptyset)$	$(231, \emptyset, \{1\})$	$\binom{n}{2} + 1$
Prop. 2.1	$(123, \emptyset, \emptyset)$	$(132, \emptyset, \{1\})$	2^{n-1}
	$(132, \emptyset, \emptyset)$	$(213, \emptyset, \{2\})$	
	$(132, \emptyset, \emptyset)$	$(231, \emptyset, \{1\})$	
	$(132, \emptyset, \emptyset)$	$(312, \emptyset, \{2\})$	
Prop. 2.2	$(132, \{1\}, \emptyset)$	$(213, \emptyset, \{2\})$	2^{n-1}
	$(132, \{1\}, \emptyset)$	$(231, \emptyset, \{1\})$	
Prop. 3.2	$(132, \emptyset, \emptyset)$	$(123, \emptyset, \{1\})$	2^{n-1}
	$(132, \emptyset, \emptyset)$	$(213, \emptyset, \{1\})$	
	$(132, \emptyset, \emptyset)$	$(231, \emptyset, \{2\})$	
	$(132, \emptyset, \emptyset)$	$(312, \emptyset, \{1\})$	
Lemma 1.4 and Prop. 3.2	$(123, \{1\}, \emptyset)$	$(132, \emptyset, \{1\})$	2^{n-1}
	$(132, \{1\}, \emptyset)$	$(213, \emptyset, \{1\})$	
	$(132, \{1\}, \emptyset)$	$(231, \emptyset, \{2\})$	

	$(132, \{1\}, \emptyset)$	$(312, \emptyset, \{1\})$	
Prop. 4.1	$(123, \{2\}, \emptyset)$ $(132, \emptyset, \emptyset)$ $(132, \{2\}, \emptyset)$	$(312, \emptyset, \{2\})$ $(321, \emptyset, \{2\})$ $(213, \emptyset, \{1\})$	2^{n-1}
Prop. 4.2	$(123, \emptyset, \emptyset)$ $(123, \{1\}, \emptyset)$	$(231, \emptyset, \{2\})$ $(312, \emptyset, \{1\})$	2^{n-1}
Prop. 5.1	$(123, \{2\}, \emptyset)$	$(132, \emptyset, \{2\})$	A121690
§ 6	$(123, \{1\}, \emptyset)$ $(132, \{2\}, \emptyset)$	$(123, \emptyset, \{1\})$ $(132, \emptyset, \{2\})$	A098569
Prop. 7.1	$(132, \emptyset, \emptyset)$	$(123, \emptyset, \{2\})$	M_n
Lemma 1.4 and Prop. 7.1	$(123, \{1\}, \emptyset)$	$(213, \emptyset, \{2\})$	M_n
Prop. 7.2	$(123, \emptyset, \emptyset)$	$(132, \emptyset, \{2\})$	M_n
Prop. 8.2	$(132, \{2\}, \emptyset)$	$(231, \emptyset, \{2\})$	A249563
Prop. 9.5	$(123, \{1\}, \emptyset)$	$(231, \emptyset, \{2\})$	A249561
§ 10	$(123, \{1\}, \emptyset)$	$(132, \emptyset, \{2\})$	A249560
§ 11	$(123, \{1\}, \emptyset)$	$(123, \emptyset, \{2\})$	A249562
	$(123, \emptyset, \emptyset)$ $(123, \{1\}, \emptyset)$ $(123, \{1\}, \emptyset)$	$(321, \emptyset, \{1\})$ $(321, \emptyset, \{1\})$ $(321, \emptyset, \{2\})$	finite

Table 2: Enumeration of $\mathrm{Av}_n(p, q)$

Ascent Sequences

In this part of the appendix we give proofs for the mappings discussed in Section 11.

Lemma 12.1. *Let $\hat{x}$ be a modified ascent sequence with no two adjacent elements equal. Then*

(i) $\theta(\hat{x})$ avoids 321 if and only if $\hat{x}$ avoids 321,

(ii) $\theta(\hat{x})$ avoids q if and only if $\hat{x}$ avoids p and

(iii) $\theta(\hat{x})$ avoids p if and only if $\hat{x}$ avoids q and the letter playing the role of the 2 is the leftmost of that value in $\hat{x}$.

Proof. Let $\hat{x}$ be a modified ascent sequence with no two adjacent elements equal.

(i) Let abc be an occurrence of 321 in $\hat{x}$ so $c < b < a$. When we write $\hat{x}$ in two line notation we get

$$\cdots \genfrac{}{}{0pt}{}{a}{i} \cdots \genfrac{}{}{0pt}{}{b}{j} \cdots \genfrac{}{}{0pt}{}{c}{k} \cdots$$

with $i < j < k$. If we order the top then a will become the rightmost of a, b and c, followed by b, and finally c. Hence kji will form an occurrence of 321.

Conversely, assume that $\theta(\hat{x})$ contains 321 but $\hat{x}$ avoids 321. The 321 occurrence then corresponds to a set of letters a, b and c such that either $a = b$ or $b = c$. But notice since we break ties by ordering in ascending order this will not lead to a 321.

(ii) Let abc be an occurrence of p in $\hat{x}$, so $c < b < a$. When we write $\hat{x}$ in two line notation we have

$$\cdots \overset{a}{i} \overset{b}{(i{+}1)} \cdots \overset{c}{j} \cdots$$

with $j > i + 1$. Clearly from part (i) this is an occurrence of 321 in $\theta(\hat{x})$. It corresponds to $j(i + 1)i$, an occurrence of q. Moreover, it is clear that every occurrence of q in $\theta(\hat{x})$ corresponds to an occurrence of p in $\hat{x}$.

(iii) Let $a(b + 1)b$ be an occurrence of p in $\hat{x}$, so $a > b + 1$ and the $b + 1$ is the leftmost $b + 1$. Then in two line notation we have

$$\cdots \overset{a}{i} \cdots \overset{b+1}{j} \cdots \overset{b}{k} \cdots$$

and so when we apply the bijection to $\hat{x}$, we get a 321 occurrence by part (i). We first sort the top row, since $b + 1$ is the leftmost letter with value b it will appear immediately after the largest index of a b. Hence we have an occurrence of p.

Conversely assume that $\theta(\hat{x})$ contains an occurrence of p but $\hat{x}$ does not contain an occurrence of q with the 2 the as far left as possible. The first thing to consider, is if there was no $(b + 1)b$ but instead some cb where $c > b + 1$. If the $b + 1$ occurred either before the a or after the c then this would prevent the adjacency required. Therefore there must be no $b + 1$, a contradiction since every modified ascent sequence contains every number between 0 and the maximum in $\hat{x}$. □

Lemma 12.2. *Let $\hat{x}$ be a modified ascent sequence with no two adjacent elements equal. Then $\hat{x}$ avoids p if and only if $\phi(\hat{x})$ avoids s.*

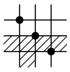
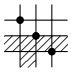
Proof. Let $\hat{x}$ be a modified ascent sequence with no two adjacent elements equal and let abc be an occurrence of p, so $c < b < a$. Then $\hat{x}$ has the form

$$\cdots \overset{a}{i} \overset{b}{(i{+}1)} \cdots \overset{c}{j} \cdots$$

where $j > i + 1$. On sorting the top row in ascending order (breaking ties with the bottom also in ascending order), we see that this will correspond to an occurrence of q. Consider an occurrence of r in $\phi(\hat{x})$. Notice that this will correspond to an occurrence of

$$\cdots \overset{a}{i} \overset{b}{(i{+}1)} \cdots \overset{b+1}{j} \cdots$$

where $b < a$, $j > i + 1$ and the b must be the leftmost b in $\hat{x}$. But then if we consider the bijection θ we will find that we have an occurrence of r in $\theta(\hat{x})$, a contradiction since θ is a bijection into $\mathrm{Av}_n(r)$.

By the shading lemma [10, Lemma 3.11], avoiding r is equivalent to avoiding . We have already shown that an occurrence of p corresponds to an occurrence of q, and since we avoid  we must have an occurrence of s.

Conversely, if $\phi(\bar{x})$ has an occurrence of s then it also has an occurrence of q which will correspond to an occurrence of p in $\bar{x}$. □

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