A combinatorial Hopf algebra for the boson normal ordering problem

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Abstract

In the aim to understand the generalization of Stirling numbers occurring in the bosonic normal ordering problem, several combinatorial models have been proposed. In particular, Blasiak *et al.* defined combinatorial objects allowing to interpret the number of $S_{\mathbf{r},\mathbf{s}}(k)$ appearing in the identity $(a^\dagger)^{r_n}a^{s_n}\cdots(a^\dagger)^{r_1}a^{s_1} = (a^\dagger)^\alpha\sum S_{\mathbf{r},\mathbf{s}}(k)(a^\dagger)^k a^k$, where α is assumed to be non-negative. These objects are used to define a combinatorial Hopf algebra which specializes to the enveloping algebra of the Heisenberg Lie algebra. Here, we propose a new variant of this construction which admits a realization with variables. This means that we construct our algebra from a free algebra $\mathbb{C}\langle A\rangle$ using quotient and shifted product. The combinatorial objects (B-diagrams) are slightly different from those proposed by Blasiak *et al.*, but give also a combinatorial interpretation of the generalized Stirling numbers together with a combinatorial Hopf algebra related to Heisenberg Lie algebra. The main difference comes from the fact that the B-diagrams have the same number of inputs and outputs. After studying the combinatorics and the enumeration of B-diagrams, we propose two constructions of algebras called Fusion algebra $\mathcal{F}$, defined using formal variable and another algebra $\mathcal{B}$ constructed directly from the B-diagrams. We show the connection between these two algebras and that $\mathcal{B}$ can be endowed with a Hopf structure. We recognize two already known combinatorial Hopf subalgebras of $\mathcal{B}$: WSym the algebra of word symmetric functions indexed by set partitions and BWSym the algebra of biword symmetric functions indexed by set partitions into lists.

1 Introduction

In Quantum Field Theory, the concept of field allows the creation and the annihilation of particles in any point of the space. Like any quantum systems, a quantum field has an Hamiltonian H and the associated Hilbert space $\mathcal{H}$ is generated by the eigenvectors of H. In the bra-cket notation, the Hilbert space is generated by the vectors $|n\rangle$ assumed to be orthogonal ($\langle m|n\rangle = \delta_{n,m}$). This representation is usually called Fock space; the vector $|n\rangle$ means that there are n particles in the system. The creation and annihilation operators, denoted respectively by $a^\dagger$ and a are non-Hermitian operators acting on the Fock space by

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \text{ and } a|n\rangle = \sqrt{n}|n-1\rangle. \tag{1}$$

These operators generate the Heisenberg algebra abstractly defined as the free algebra generated by the elements a, $a^\dagger$ and quotiented by the relation

$$[a, a^\dagger] = 1. \tag{2}$$

The normal ordering problem consists in computing $\langle z|F(a^\dagger,a)|z\rangle$ where $F(a^\dagger,a)$ is an operator of the Heisenberg algebra, $|z\rangle$ is an eigenstate of the annihilation operator a ($a|z\rangle = z|z\rangle$), and $\langle z|a^\dagger = \langle z|z^*$. The strategy consists in sorting the letters on each terms of $F(a^\dagger,a)$ in such a way that all the letters $a^\dagger$ are in the left and all the letters a are in the right by using as many times as necessary the relation (2).

In a seminal paper, J. Katriel [13] pioneered the study of the combinatorial aspects of the normal ordering problem. He established the normal-ordered expression of $(a^\dagger a)^n$ in terms of Stirling numbers of second kind enumerating partitions of a set of n elements into k non empty subsets

$$(a^\dagger a)^n = \sum_{i=1}^{n} S(n,i)(a^\dagger)^i a^i. \tag{3}$$

The investigation of the normal ordered expression of $((a^\dagger)^r a^s)^n$ naturally gives rise to generalized Stirling numbers $S_{r,s}(n,k)$ and Bell polynomials [6, 7]. The interpretations of some special cases are well known and related to combinatorial numbers [11, 8]. For instance, for $r = 2$ and $s = 1$, the number $S_{2,1}(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}$ is the number partitions of $\{1,\dots,n\}$ into k lists (also called Lah numbers). More generally, Blasiak *et al.* [18] studied the bosonic normal ordering problem,

$$(a^\dagger)^{r_n} a^{s_n} \cdots (a^\dagger)^{r_1} a^{s_1} = (a^\dagger)^{\alpha_n} \sum_{k\geq s_1} S_{\mathbf{r},\mathbf{s}}(k)(a^\dagger)^k a^k, \tag{4}$$

with $\mathbf{r} = (r_1,\dots,r_n)$, $\mathbf{s} = (s_1,\dots,s_n)$ and $\alpha_n = \sum_{i=1}^n (r_i - s_i)$.

They gave also a combinatorial interpretation of the sequence $S_{\mathbf{r},\mathbf{s}}(k)$ in terms of graphs-like combinatorial objects called *bugs*. In a further work, Blasiak *et al.* [9] constructed a combinatorial Hopf algebras based on bugs to explain the computations. The aim of our paper is to investigate a variant of their construction. Our version allows a realization with variables which projects on the Heisenberg-Weyl algebra and the connections with some other combinatorial Hopf algebras appear clearly. Readers interested by the topic of combinatorial interpretations of the normal ordering should also refer to [5, 14, 15, 16, 17].

Our paper is organized as follow. In section 2, we introduce the combinatorial structure of B-diagrams and we give, in section 3, an algebraic structure based on these objects. Also, we describe a second algebra named *Fusion algebra* which specializes to the Heisenberg-Weyl algebra. We show that the B-diagram algebra is isomorphic to a subalgebra of $\mathcal{F}$ which allows us to describe the normal ordering problem in terms of B-diagram. We study the Hopf structure of B-diagrams in section 4 and we identify, in section 5, two already known subalgebras : WSym and BWSym.

2 B-diagrams

2.1 Definition and first examples

Let us first introduce notation. Let $\#E$ denote the cardinal of the set E, $[\![a,b]\!] := \{a, a+1, \dots, b-1, b\}$ for any pairs of integers $a \leq b$, $E = E' \uplus E''$ when $E = E' \cup E''[n]$ and $E' \cap E''[n] = \emptyset$, where $E''[n]$ means that we add $n = \#E'$ to each integer occurring in E''. For example, if we set $E = \{1,2,3\}$, thus $E[2] = \{3,4,5\}$.

Definition 1 *A* B-diagram *is a* 5*-tuple* $G = (n, \lambda, E^\uparrow, E^\downarrow, E)$ *such that*

1. $n \in \mathbb{N}$,
2. $\lambda = [\lambda_1,\dots,\lambda_n]$ *with* $\lambda_i \in \mathbb{N} \setminus \{0\}$ *for each* i,
3. $E^\uparrow, E^\downarrow \subset [\![1, \lambda_1 + \cdots + \lambda_n]\!]$,
4. $E \subset \{(a,b) : a \in E^\uparrow, b \in E^\downarrow, v(a) < v(b)\}$ *where* $v : [\![1, \lambda_1 + \cdots + \lambda_n]\!] \longrightarrow [\![1,n]\!]$ *is defined by* $v(k) = i$ *if* $k \in [\![\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_i]\!]$,
5. *for each* $a \in E^\uparrow$ *and* $b \in E^\downarrow$, *the sets* $\{(a,c) : (a,c) \in E\}$ *and* $\{(c,b) : (c,b) \in E\}$ *contain at most one element.*

Graphically, a B-diagram can be represented as a graph with n vertices. The vertex i has exactly λ_i inner (resp. outer) half-edges labelled by $[\![\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_i]\!]$. The inner (resp. outer) half edges which does not belong to $E^{\downarrow}$ (resp. $E^{\uparrow}$) are denoted by $\times$. An element of E is represented by an edge relying an outer half edge a of a vertex i to an inner half edge b of a vertex j with $i < j$.

Example 2 For instance, the B-diagram $G = (3, [3,1,2], [\![1,5]\!], [\![1,6]\!], \{(1,6),(2,4),(4,5)\})$ is represented in Figure 1

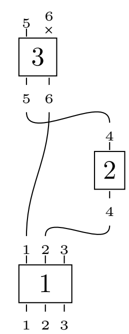


Figure 1: The B-diagram $(3, [3,1,2], [\![1,5]\!], [\![1,6]\!], \{(1,6),(2,4),(4,5)\})$

Also consider $G' = (4, [1,3,2,2], \{1,3,4,6\}, \{1,3,6,7\}, \{(1,6),(3,7)\})$ which is represented in Figure 2.

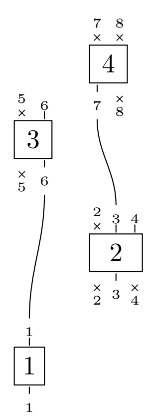


Figure 2: The B-diagram $(4, [1,3,2,2], \{1,3,4,6\}, \{1,3,6,7\}, \{(1,6),(3,7)\})$

Let $G = (n, \lambda, E^{\uparrow}, E^{\downarrow}, E)$, we define some tools in order to manipulate more easily the B-diagrams

1. the number of vertices is $|G| := n$,
2. we write $\omega(G) := \lambda_1 + \cdots + \lambda_n$, for the number of half-edge,
3. Let $\tau(G) := \#E$, be the number of edges,
4. the number of non used outer (resp. inner) half edges is $h^{\uparrow}(G) := \omega(G) - \#e^{\uparrow}(E)$ (resp. $h^{\downarrow}(G) := \omega(G) - \#e^{\downarrow}(E)$) where $e^{\uparrow}(a,b) = a$ (resp. $e^{\downarrow}(a,b) = b$),
5. the set of non used outer (resp. inner) non cut half edges is $H_f^{\uparrow}(G) := E^{\uparrow} \setminus e^{\uparrow}(E)$ (resp. $H_f^{\downarrow}(G) := E^{\downarrow} \setminus e^{\downarrow}(E)$),

6. the number of non used outer (resp. inner) non cut half edges is $h_f^\uparrow(G) := \#H_f^\uparrow(G)$ (resp. $h_f^\downarrow(G) := \#H_f^\downarrow(G)$),
7. the number of non used cut half edges is $h_c(G) := h^\uparrow(G) - h_f^\uparrow(G) + h^\downarrow(G) - h_f^\downarrow(G)$,
8. the set of half edges associated to a vertex i is $\hat{H}(i) := [\![\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_i]\!]$,
9. the set of outer (resp. inner) non cut half edges associated to a vertex i, $\hat{H}_f^\uparrow(i) := E^\uparrow \cap [\![\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_i]\!]$ (resp. $\hat{H}_f^\downarrow(i) := E^\downarrow \cap [\![\lambda_1 + \cdots + \lambda_{i-1} + 1, \lambda_1 + \cdots + \lambda_i]\!]$),
10. we will also use the map v of Definition 1; this map will be denoted v_G in case of ambiguity.

Example 3 Consider the B-diagram $G = (n, \lambda, E^\uparrow, E^\downarrow, E)$ represented in Figure 1. We have $|G| = 3$, $\omega(G) = 6$, and $\tau(G) = 3$. We also have $h^\uparrow(G) = 3$, $h_f^\uparrow(G) = 2$, $h^\downarrow(G) = h_f^\downarrow(G) = 3$, and $h_c(G) = 1$ since $H_f^\uparrow(G) = \{3, 5\}$ and $H_f^\downarrow(G) = \{1, 2, 3\}$. Furthermore $\hat{H}(1) = \hat{H}_f^\uparrow(1) = \hat{H}_f^\downarrow(1) = \{1, 2, 3\}$, $\hat{H}(2) = \hat{H}_f^\uparrow(2) = \hat{H}_f^\downarrow(2) = \{4\}$, $\hat{H}(3) = \hat{H}_f^\downarrow(3) = \{5, 6\}$, and $\hat{H}_f^\uparrow(3) = \{5\}$. Finally, $v(1) = v(2) = v(3) = 1$, $v(4) = 2$, and $v(5) = v(6) = 3$.

A special example B-diagram is given by the *empty diagram* $\varepsilon := (0, [], \varnothing, \varnothing, \varnothing)$. This is the only diagram of weight 0.

Definition 4 *Let $G = (n, \lambda, E^\uparrow, E^\downarrow, E)$ be a B-diagram. A* sub B-diagram *of G is completely characterized by a sequence $1 \le i_1 < \cdots < i_{n'} \le n$. More precisely, we define the B-diagram $G[i_1, \ldots, i_{n'}] = (n', \lambda', E'^\uparrow, E'^\downarrow, E')$ by*

1. $\lambda' = [\lambda_{i_1}, \ldots, \lambda_{i_{n'}}]$,
2. $E'^\uparrow = \phi\left(E^\uparrow \cap \bigcup_{\ell=1}^{n'} \hat{H}(i_\ell)\right)$ *and* $E'^\downarrow = \phi\left(E^\downarrow \cap \bigcup_{\ell=1}^{n'} \hat{H}(i_\ell)\right)$ *where ϕ is the only increasing bijection sending $\bigcup_{\ell=1}^{n'} \hat{H}(i_\ell)$ to $[\![1, \lambda_{i_1} + \cdots + \lambda_{i_{n'}}]\!]$.*
3. $E' = \phi\left(E \cap \bigcup_{\ell=1}^{n'} \hat{H}_f^\uparrow(i_\ell) \times \hat{H}_f^\downarrow(i_\ell)\right)$

Let $I = [i_1, \ldots, i_{n'}]$ be a sequence of vertices of G, we let $\complement_G I = [\![1, n]\!] \smallsetminus \{i_1, \ldots, i_{n'}\}$ denote the complement *of I in G.*

Example 5 Let G be the B-diagram of Figure 1. We have $\hat{H}(1) = \hat{H}_f^\uparrow(1) = \hat{H}_f^\downarrow(1) = \{1, 2, 3\}$, $\hat{H}(2) = \hat{H}_f^\uparrow(2) = \hat{H}_f^\downarrow(2) = \{4\}$, $\hat{H}(3) = \hat{H}_f^\downarrow(3) = \{5, 6\}$, and $\hat{H}_f^\uparrow(3) = \{5\}$. Set $i_1 = 1$ and $i_2 = 3$. Following Definition 4, we have

1. $\lambda' = [3, 2]$,
2. ϕ sends respectively $1, 2, 3, 5, 6$ to $1, 2, 3, 4, 5$. Hence, $E'^\uparrow = \phi(\{1, 2, 3, 5\}) = \{1, 2, 3, 4\}$ and $E'^\downarrow = \phi(\{1, 2, 3, 5, 6\}) = \{1, 2, 3, 4, 5\}$,
3. $E' = \phi(\{(1, 6)\}) = \{(1, 5)\}$.

We deduce that $G[1, 3] = (2, [3, 2], [\![1, 4]\!], [\![1, 5]\!], \{(1, 5)\})$ (see Figure 3).

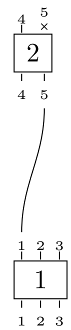


Figure 3: The sub B-diagram $G[1,3]$ of $G = (3, [3,1,2], [\![1,5]\!], [\![1,6]\!], \{(1,6),(2,4),(4,5)\})$

2.2 Connections and compositions

Definition 6 *A B-diagram $G = (n, \lambda, E^{\uparrow}, E^{\downarrow}, E)$ is* connected *if and only if for each $1 \le i < j \le n$, there exists a sequence of edges $(a_1, b_1), \dots, (a_k, b_k) \in E$ satisfying*

1. *$i \in \{v(a_1), v(b_1)\}$ and $j \in \{v(a_k), v(b_k)\}$,*
2. *for each $1 \le \ell < k$ one has $v(a_\ell) \in \{v(a_{\ell+1}), v(b_{\ell+1})\}$ or $v(b_\ell) \in \{v(a_{\ell+1}), v(b_{\ell+1})\}$.*

A *connected component* of a B-diagram G is a sequence $i_1 < \dots < i_{n'}$ such that $G[i_1, \dots, i_{n'}]$ is a connected sub B-diagram which is maximal in the sense that if we add any vertex i in the sequence $i_1 < \dots < i_{n'}$ then we obtain a sequence $i'_1 < \dots < i'_{n'+1}$ such that $G[i'_1, \dots, i'_{n'+1}]$ is not connected. Let $\mathrm{Connected}(G)$ denote the set of the connected components of G.

Example 7 The B-diagram in Figure 1 is connected whilst the B-diagram in Figure 2 has two connected components $[1,3]$ and $[2,4]$.

A sequence $1 \le i_1 < \dots, i_{n'} \le n$ is *isolated* in G if for each $(a,b) \in E$ implies that $v(a)$ and $v(b)$ are both in $\{i_1, \dots, i_{n'}\}$ or both in $\complement_G\{i_1, \dots, i_{n'}\}$.

Claim 8 *The following assertions are equivalent*

1. *I is isolated in G*
2. *$\complement_G I$ is isolated in G*
3. *There exist $j_1^1 < \dots < j_{k_1}^1, \dots, j_1^p < \dots < j_{k_p}^p$ such that $I = \{j_1^1, \cdots, j_{k_1}^1, \dots, j_1^p, \cdots, j_{k_p}^p\}$ and each sequence $[j_1^\ell, \dots, j_{k_\ell}^\ell]$ is a connected component of G.*

Remark 9
- *A B-diagram G and the empty diagram ε are both isolated in G.*
- *If I is a connected component of G then it is isolated in G.*

Let $\mathrm{Iso}(G)$ denote the set isolated sequences in G and set $\mathrm{Split}(G) = \{(I, \complement_G I) : I \in \mathrm{Iso}(G)\}$.

Definition 10 *Let $G = (n, \lambda, E^{\uparrow}, E^{\downarrow}, E)$ and $G' = (n', \lambda', E'^{\uparrow}, E'^{\downarrow}, E')$ be two B-diagrams. For any $k \ge 0$, any strictly increasing sequence $a_1 < \dots < a_k$ in $H_f^{\uparrow}(G)$ and any k-tuple of distinct integers*

$b_1, \ldots, b_k$ in $H_f^{\downarrow}(G')$, we define the composition $\overset{b_1,\ldots,b_k}{\underset{a_1,\ldots,a_k}{\star}}$ *by*

$$\begin{matrix} G' \\ \overset{b_1,\ldots,b_k}{\underset{a_1,\ldots,a_k}{\star}} \\ G \end{matrix} = G'',$$

where G'' is the 5 tuple $(n+n', [\lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_n], E''^{\uparrow}, E''^{\downarrow}, E'')$ with

1. $E''^{\uparrow} = E'^{\uparrow} \cup \{i + \omega(G) : i \in E'^{\uparrow}\}$ *and* $E''^{\downarrow} = E'^{\downarrow} \cup \{i + \omega(G) : i \in E'^{\downarrow}\}$,
2. $E'' = E \cup \{(a_\ell, b_\ell + \omega(G)) : 1 \leq \ell \leq k\} \cup \{(a + \omega(G), b + \omega(G)) : (a, b) \in E'\}$.

We easily check that the definition of composition is coherent with the structure of B-diagram

Claim 11 *Let $G = (n, \lambda, E^{\uparrow}, E^{\downarrow}, E)$ and $G' = (n', \lambda', E'^{\uparrow}, E'^{\downarrow}, E')$ be two B-diagrams. For any $k \geq 0$, any strictly increasing sequence $a_1 < \cdots < a_k$ in $H_f^{\uparrow}(G)$ and any k-tuple of distinct integers $b_1, \ldots, b_k$ in $H_f^{\downarrow}(G')$, the 5-tuple* $\begin{matrix} G' \\ \overset{b_1,\ldots,b_k}{\underset{a_1,\ldots,a_k}{\star}} \\ G \end{matrix}$ *is a B-diagram.*

Example 12 Figure 4 gives an example of composition.

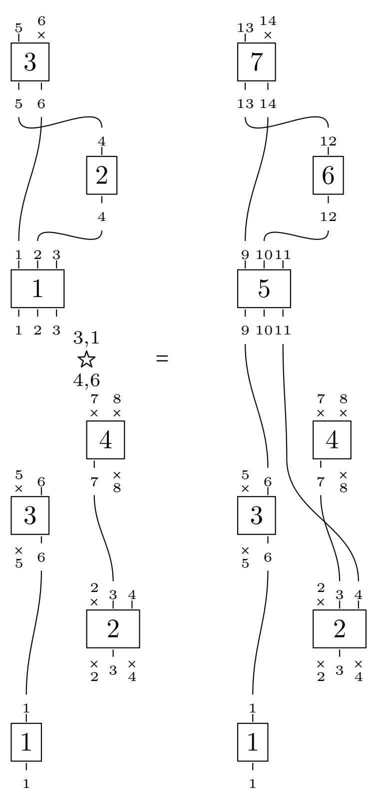


Figure 4: An example of composition

Note that the special case where $k = 0$ corresponds to a simple juxtaposition of the B-diagrams (see Figure 5 for an example). We set $G'|G'' := \underset{G'}{\overset{G''}{\star}}$. The operation $|$ endows the set of the B-diagrams $\mathbb{B}$ with a structure of monoid whose unity is ε.

Definition 13 *A B-diagram G is* indivisible *if $G = G'|G''$ implies $G' = G$ or $G'' = G$. Let $\mathbb{G}$ denote the set of indivisible B-diagrams.*

It is easy to check that the indivisible diagrams are algebraically independent. It follows

Proposition 14 *The monoid $(\mathbb{B}, |)$ is freely generated by $\mathbb{G}$: $\mathbb{B} \simeq \mathbb{G}^*$.*

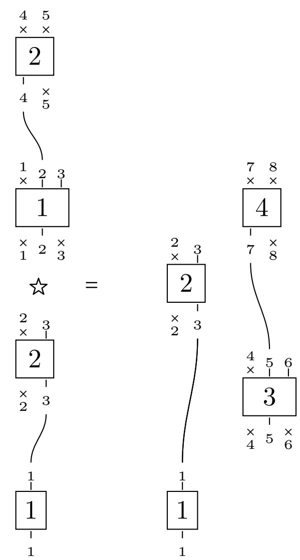


Figure 5: An example of composition when $k = 0$

We can give an alternative recursive definition for the B-diagrams using the compositions.

Lemma 15 *Let $G = (n, \lambda, E^{\uparrow}, E^{\downarrow}, E)$ be a B-diagram. Either $G = \varepsilon$ or there exists a B-diagram $V = (1, [p], E_v^{\uparrow}, E_v^{\downarrow}, E_v)$, a B-diagram $\tilde{G} = (n-1, \tilde{\lambda}, \tilde{E}^{\uparrow}, \tilde{E}^{\downarrow}, \tilde{E})$ and two sequences $1 \le a_1 < \cdots < a_k \le p$ and $1 \le b_1, \ldots, b_k \le \omega(\tilde{G})$ distinct satisfying*

$$G = \underset{\substack{a_1,\ldots,a_k \\ V}}{\overset{\substack{\tilde{G} \\ b_1,\ldots,b_k}}{\star}} .$$

Let us set

$$G \star G' := \left\{ \underset{\substack{a_1,\ldots,a_k \\ G}}{\overset{\substack{G' \\ b_1,\ldots,b_k}}{\star}} : a_1 < \cdots < a_k \in H_f^{\uparrow}(G) b_1, \ldots, b_k \in H_f^{\downarrow}(G') \text{ distinct and } k \ge 0 \right\}. \quad (5)$$

2.3 Enumeration

Let $d_{p,q}$ denote the number of B-diagram G such that $\omega(G) = p$ and $h_f^{\uparrow}(G) = q$ (see in Figure 6 the first values of $d_{p,q}$ or the sequence A265199 in [23]).

From lemma 15, we find the induction

$$d_{p,q} = \sum_{i=1}^{p} \sum_{j=0}^{i} \sum_{k=0}^{i} \sum_{\ell=0}^{j} \ell! \binom{j}{\ell} \binom{q-k+\ell}{\ell} \binom{i}{j} \binom{i}{k} d_{p-i,q-k+\ell}, \tag{6}$$

with the special cases $d_{0,0} = 1$ and $d_{p,q} = 0$ if $p, q \le 0$ and $(p,q) \ne (0,0)$. Indeed, we obtain a diagram with p half edges and q non used outer non cut half edges by branching ℓ inner half edges of an elementary B-diagram with i half edges, $j \le i$ non used inner non cut half edges, and $k \le i$ non used outer non cut half edges to a B-diagram with $p - i$ half edges and $q - k + \ell$ non used outer non cut half edges. The number of ways to do that is $\ell!\binom{j}{\ell}\binom{q-k+\ell}{\ell}\binom{i}{j}\binom{i}{k}$. Indeed, the factor $\ell!$ is the number of permutation of the inner half edges of the elementary B-diagram, the factor $\binom{j}{\ell}$ corresponds to the choice of ℓ half edges in the set of the j possible non cut half edges, the coefficients $\binom{q-k+\ell}{\ell}$ is the number of ways to choose ℓ outer half edges in the second B-diagram, and the factors $\binom{i}{j}\binom{i}{k}$ is the number of ways to select j inner half edges and k outer half edges in i half edges. The number α_p of

$d_{p,q}$	0	1	2	3	4	5	6
0	1						
1	2	2					
2	10	18	8				
3	62	154	124	32			
4	462	1426	1596	760	128		
5	3982	14506	20380	13680	4336	512	
6	38646	161042	269284	229448	104032	23520	2048

Figure 6: First values of $d_{p,q}$.

B-diagrams having exactly p half edges is given by

$$\alpha_p = \sum_{q=0}^{p} d_{p,q}. \tag{7}$$

The first values (see sequence A266093 in [23]) are

$$1, 4, 36, 372, 4372, 57396, 828020, 12962164, 218098356, \ldots$$

Example 16 Let us illustrate this enumeration by counting the B-diagrams of weight 3. A B-diagram of weight 3 is

1. either an elementary diagram of weight 3 (2^6 possibilities),
2. or it is composed by an elementary diagram of weight 2 and another elementary diagram of weight 1. The two diagrams can be juxtaposed (2^7 possibilities) or branched 2^6 possibilities.
3. or it is composed by three elementary diagrams of weight 1. The three diagrams can be juxtaposed (2^6 possibilities), or two of the three diagrams are connected (3×2^4 possibilities), or the three diagrams are connected (2^2 possibilities).

Hence, we obtain $2^6 + 2^7 + 2^6 + 2^6 + 3 \times 2^4 + 2^2 = 372$ as predicted by (7).

3 Algebraic aspects of B-diagrams

3.1 Fusion algebra

We consider the alphabets

- $\mathcal{A}_{\langle\times} = \left\{\mathtt{a}_{\langle\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}} : k\in\mathbb{N}, i_1,\dots,i_k,j_1,\dots,j_k\in\mathbb{N}\setminus\{0\}\right\}$,
- $\mathcal{A}_{\rangle\times} = \left\{\mathtt{a}_{\rangle\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}} : k\in\mathbb{N}, i_1,\dots,i_k,j_1,\dots,j_k\in\mathbb{N}\setminus\{0\}\right\}$,
- $\mathcal{A}_{\times\langle} = \left\{\mathtt{a}_{\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}\langle} : k\in\mathbb{N}, i_1,\dots,i_k,j_1,\dots,j_k\in\mathbb{N}\setminus\{0\}\right\}$,
- and $\mathcal{A}_{\times\rangle} = \left\{\mathtt{a}_{\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}\rangle} : k\in\mathbb{N}, i_1,\dots,i_k,j_1,\dots,j_k\in\mathbb{N}\setminus\{0\}\right\}$.

For simplicity, we also set $\mathcal{A}_{\langle\ \rangle} = \mathcal{A}_{\langle\times}\cup\mathcal{A}_{\times\rangle}$, $\mathcal{A}_{\rangle\ \langle} = \mathcal{A}_{\rangle\times}\cup\mathcal{A}_{\times\langle}$, $\mathcal{A}_{\bullet\times} = \mathcal{A}_{\langle\times}\cup\mathcal{A}_{\rangle\times}$, $\mathcal{A}_{\times\diamond} = \mathcal{A}_{\times\langle}\cup\mathcal{A}_{\times\rangle}$, and $\mathcal{A} = \mathcal{A}_{\langle\ \rangle}\cup\mathcal{A}_{\rangle\ \langle}$.

Definition 17 *Let $\mathcal{J}_F$ be the ideal of $\mathbb{C}\langle\mathcal{A}\rangle$, the free associative algebra generated by $\mathcal{A}$, generated by the polynomials*

1. $[\mathtt{e},\mathtt{f}] = \mathtt{ef}-\mathtt{fe}$ *for any* $\mathtt{e}\in\mathcal{A}_{\langle\ \rangle}$ *and* $\mathtt{f}\in\mathcal{A}$,
2. $\mathtt{ef}-\mathtt{fe}$ *for any* $\mathtt{e},\mathtt{f}\in\mathcal{A}_{\times\diamond}$ *and any* $\mathtt{e},\mathtt{f}\in\mathcal{A}_{\bullet\times}$,
3. $\mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\alpha\langle}\mathtt{a}_{\rangle\beta}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y} - \mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\rangle\beta}\mathtt{a}_{\alpha\langle}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y} - \mathtt{u}\mathtt{a}_{\bullet\alpha\beta}\mathtt{v}\mathtt{x}\mathtt{a}_{\alpha\beta\diamond}\mathtt{y}$ *for any* $\mathtt{u},\mathtt{v},\mathtt{x},\mathtt{y}\in\mathcal{A}^*$ *(the free monoid generated by $\mathcal{A}$),* $\alpha = \binom{i_1}{j_1}\cdots\binom{i_k}{j_k}$, $\beta = \binom{i'_1}{j'_1}\cdots\binom{i'_{k'}}{j'_{k'}}$ *(with* $k,k'\in\mathbb{N}$ *and* $i_1,\dots,i_k$, $j_1,\dots,j_k$, $i'_1,\dots,i'_{k'}$, $j'_1\dots,j'_{k'}\in\mathbb{N}\setminus\{0\}$*), and* $\bullet,\diamond\in\{\langle,\rangle\}$.

We call fusion algebra *the quotient* $\mathcal{F} := \mathbb{C}\langle\mathcal{A}\rangle/_{\mathcal{J}_F}$.

Remark that $\mathcal{A}_{\langle\ \rangle}$ is in the center of $\mathcal{F}$ and that the subalgebra of $\mathcal{F}$ generated by $\mathcal{A}_{\times\diamond}$ (resp. $\mathcal{A}_{\bullet\times}$) is commutative.
Using the rule 3 of Definition 17, we show by induction the following result

Lemma 18

$$\mathtt{a}_{\rangle\alpha_1}\cdots\mathtt{a}_{\rangle\alpha_n}\mathtt{a}_{\alpha_1\langle}\cdots\mathtt{a}_{\alpha_n\langle}\mathtt{a}_{\rangle\beta_1}\cdots\mathtt{a}_{\rangle\beta_m}\mathtt{a}_{\beta_1\langle}\cdots\mathtt{a}_{\beta_m\langle} = \sum_{k=0}^{\min\{n,m\}}\sum_{\substack{1\le i_1<\cdots<i_k\le n\\ 1\le j_1,\dots,j_k\le m\ \ distinct}} P\begin{pmatrix} i_1,\dots,i_k\\ j_1,\dots,j_k\end{pmatrix}\mathtt{a}_{\rangle\alpha_{i_1}\beta_{i_1}}\cdots\mathtt{a}_{\rangle\alpha_{i_k}\beta_{i_k}} Q\begin{pmatrix} i_1,\dots,i_k\\ j_1,\dots,j_k\end{pmatrix}\mathtt{a}_{\alpha_{i_1}\beta_{i_1}\langle}\cdots\mathtt{a}_{\alpha_{i_k}\beta_{i_k}\langle}, \quad (8)$$

where $P\binom{i_1,\dots,i_k}{j_1,\dots,j_k} = \prod_{j\notin\{j_1,\dots,j_k\}}\mathtt{a}_{\rangle\beta_j}\prod i\notin\{i_1,\dots,i_k\}\mathtt{a}_{\rangle\alpha_j}$ *and*
$Q\binom{i_1,\dots,i_k}{j_1,\dots,j_k} = \prod_{j\notin\{j_1,\dots,j_k\}}\mathtt{a}_{\beta_j\langle}\prod_{i\notin\{i_1,\dots,i_k\}}\mathtt{a}_{\alpha_j\langle}$.

Proof. We proceed by induction on m. If $m=0$ then the result is obvious. Otherwise applying successively many times the rule 3 of Definition 17 and hence the rule 2, one obtains

$$\begin{aligned}\mathtt{a}_{\rangle\alpha_1}\cdots\mathtt{a}_{\rangle\alpha_n}\mathtt{a}_{\alpha_1\langle}\cdots\mathtt{a}_{\alpha_n\langle}\mathtt{a}_{\rangle\beta_1}\cdots\mathtt{a}_{\rangle\beta_m}\mathtt{a}_{\beta_1\langle}\cdots\mathtt{a}_{\beta_m\langle} &= \mathtt{a}_{\rangle\beta_1}\prod_{i=1}^n\mathtt{a}_{\rangle\alpha_i}\prod_{i=1}^n\mathtt{a}_{\alpha_i\langle}\prod_{j=2}^n\mathtt{a}_{\rangle\beta_j}\prod_{j=2}^n\mathtt{a}_{\beta_j\langle}\mathtt{a}_{\beta_1\langle}\\ &+\sum_{i=1}^n\mathtt{a}_{\rangle\alpha_i\beta_1}\prod_{j\ne i}\mathtt{a}_{\rangle\alpha_j}\prod_{j\ne i}\mathtt{a}_{\alpha_j\langle}\prod_{j=2}^m\mathtt{a}_{\rangle\beta_j}\prod_{j=2}^m\mathtt{a}_{\beta_j\langle}\mathtt{a}_{\alpha_i\beta_1\langle}.\end{aligned} \quad (9)$$

Hence, applying the induction hypothesis to each term

$$\prod_{j=1}^{n}\mathtt{a}_{\rangle\alpha_i}\prod_{j=1}^{n}\mathtt{a}_{\alpha_i\langle}\prod_{j=2}^{n}\mathtt{a}_{\rangle\beta_i}\prod_{j=2}^{n}\mathtt{a}_{\beta_i\langle}\ \text{and}\ \prod_{j\neq i}\mathtt{a}_{\rangle\alpha_j}\prod_{j\neq i}\mathtt{a}_{\alpha_j\langle}\prod_{j=2}^{m}\mathtt{a}_{\rangle\beta_j}\prod_{j=2}^{m}\mathtt{a}_{\beta_j\langle}\mathtt{a}_{\alpha_i\beta_1\langle},\tag{10}$$

for $1\le i\le n$, and again the rule 2 of Definition 17 for writing each factor in the suitable order, we find the result. □

Whilst Formula (8) seems rather technical, it is easy to understand. Indeed, it means that each of the letters $\mathtt{a}_{\alpha_1\langle},\dots,\mathtt{a}_{\alpha_k\langle}$ can be paired or not with any of the letters $\mathtt{a}_{\rangle\beta_1},\dots,\mathtt{a}_{\rangle\beta_\ell}$.

Example 19 Let us illustrate Formula (8) on the following example

$$\begin{aligned}\mathtt{a}_{\rangle\alpha_1}\mathtt{a}_{\rangle\alpha_2}\mathtt{a}_{\alpha_1\langle}\mathtt{a}_{\alpha_2\langle}\mathtt{a}_{\rangle\beta_1}\mathtt{a}_{\rangle\beta_2}\mathtt{a}_{\beta_1\langle}\mathtt{a}_{\beta_2\langle}\quad=\quad&\mathtt{a}_{\rangle\beta_1}\mathtt{a}_{\rangle\beta_2}\mathtt{a}_{\rangle\alpha_1}\mathtt{a}_{\rangle\alpha_2}\mathtt{a}_{\beta_1\langle}\mathtt{a}_{\beta_2\langle}\mathtt{a}_{\alpha_1\langle}\mathtt{a}_{\alpha_2\langle}\\&+\mathtt{a}_{\rangle\beta_2}\mathtt{a}_{\rangle\alpha_2}\mathtt{a}_{\rangle\alpha_1\beta_1}\mathtt{a}_{\beta_2\langle}\mathtt{a}_{\alpha_2\langle}\mathtt{a}_{\alpha_1\beta_1\langle}+\mathtt{a}_{\rangle\beta_2}\mathtt{a}_{\rangle\alpha_1}\mathtt{a}_{\rangle\alpha_2\beta_1}\mathtt{a}_{\beta_2\langle}\mathtt{a}_{\alpha_1\langle}\mathtt{a}_{\alpha_2\beta_1\langle}\\&+\mathtt{a}_{\rangle\beta_1}\mathtt{a}_{\rangle\alpha_2}\mathtt{a}_{\rangle\alpha_1\beta_2}\mathtt{a}_{\beta_1\langle}\mathtt{a}_{\alpha_2\langle}\mathtt{a}_{\alpha_1\beta_2\langle}+\mathtt{a}_{\rangle\beta_1}\mathtt{a}_{\rangle\alpha_1}\mathtt{a}_{\rangle\alpha_2\beta_2}\mathtt{a}_{\beta_1\langle}\mathtt{a}_{\alpha_1\langle}\mathtt{a}_{\alpha_2\beta_2\langle}\\&+\mathtt{a}_{\rangle\alpha_1\beta_1}\mathtt{a}_{\rangle\alpha_2\beta_2}\mathtt{a}_{\alpha_1\beta_1\langle}\mathtt{a}_{\alpha_2\beta_2\langle}+\mathtt{a}_{\rangle\alpha_2\beta_1}\mathtt{a}_{\rangle\alpha_1\beta_2}\mathtt{a}_{\alpha_2\beta_1\langle}\mathtt{a}_{\alpha_1\beta_2\langle}\end{aligned}.$$

Now, we define $\widetilde{\mathcal{F}}$ as the subspace of $\mathcal{F}$ generated by the elements under the form

$$\mathtt{a}_{\bullet_1\alpha_1}\cdots\mathtt{a}_{\bullet_k\alpha_k}\mathtt{a}_{\alpha_1\diamond_1}\cdots\mathtt{a}_{\alpha_k\diamond_k},$$

with $\bullet_i,\diamond_i\in\{\langle,\rangle\}$, for any $1\le i\le k$ and $k\in\mathbb{N}$.

Proposition 20 *$\widetilde{\mathcal{F}}$ is a subalgebra of $\mathcal{F}$.*

Proof. It suffices to prove that is stable for the product. Let $\mathtt{u}=\mathtt{a}_{\bullet_1\alpha_1}\cdots\mathtt{a}_{\bullet_k\alpha_k}\mathtt{a}_{\alpha_1\diamond_1}\cdots\mathtt{a}_{\alpha_k\diamond_k}$ and $\mathtt{v}=\mathtt{a}_{\bullet'_1\alpha'_1}\cdots\mathtt{a}_{\bullet'_{k'}\alpha'_{k'}}\mathtt{a}_{\alpha'_1\diamond'_1}\cdots\mathtt{a}_{\alpha'_{k'}\diamond'_{k'}}$. Let $i_1,\dots,i_\ell$ be the indices such that $\diamond_{i_1}=\cdots=\diamond_{i_\ell}=\rangle$ and $\{j_1,\dots,j_{k-\ell}\}=\{1,\dots,k\}\setminus\{i_1,\dots,i_\ell\}$. In the same way, let $i'_1,\dots,i'_{\ell'}$ be the indices such that $\bullet_{i'_1}=\cdots=\bullet_{i'_{\ell'}}=\langle$ and $\{j'_1,\dots,j'_{k'-\ell'}\}=\{1,\dots,k'\}\setminus\{i'_1,\dots,i'_{\ell'}\}$. From the relations 1 and 2 of Definition 17, one has

$$\mathtt{uv}=\mathtt{a}_{\bullet_{i_1}\alpha_{i_1}}\cdots\mathtt{a}_{\bullet_{i_\ell}\alpha_{i_\ell}}\mathtt{a}_{\langle\alpha_{i'_1}}\cdots\mathtt{a}_{\langle\alpha_{i'_{\ell'}}}\mathtt{u}'\mathtt{v}'\mathtt{a}_{\alpha_{i'_1}\diamond_{i'_1}}\cdots\mathtt{a}_{\alpha_{i'_{\ell'}}\diamond_{i'_{\ell'}}}\mathtt{a}_{\alpha_{i_1}\rangle}\cdots\mathtt{a}_{\alpha_{i_\ell}\rangle},\tag{11}$$

with $\mathtt{u}'=\mathtt{a}_{\bullet_{j_1}\alpha_{j_1}}\cdots\mathtt{a}_{\bullet_{j_{k-\ell}}\alpha_{i_{k-\ell}}}\mathtt{a}_{\alpha_{j_1}\langle}\cdots\mathtt{a}_{\alpha_{j_{k-\ell}}\langle}$ and $\mathtt{v}'=\mathtt{a}_{\rangle\alpha'_{j'_1}}\cdots\mathtt{a}_{\rangle\alpha_{j'_{k'-\ell'}}}\mathtt{a}_{\alpha'_{j'_1}\diamond_{j'_1}}\cdots\mathtt{a}_{\alpha_{j'_{k'-\ell'}}\diamond_{j'_{k'-\ell'}}}$.
Hence, one has only to prove that $\mathtt{u}'\mathtt{v}'\in\widetilde{\mathcal{F}}$ which is a direct consequence of Equality (8). □

Example 21 We consider the element $\mathtt{w}_1=\mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}$, $\mathtt{w}_2=\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{2}{2}\rangle}\in\widetilde{\mathcal{F}}$ and $\mathtt{w}=\mathtt{w}_1\mathtt{w}_2$. First from the point (1) of Definition 17, $\mathtt{a}_{\binom{1}{2}\rangle}$ is in the center of the algebra $\mathcal{F}$. Hence,

$$\mathtt{w}=\mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}.$$

Now we use the point (3) and compute

$$\begin{aligned}\mathtt{w}&=\overbrace{}^{\mathtt{u}}\mathtt{a}_{\rangle\binom{1}{1}}\overbrace{\mathtt{a}_{\rangle\binom{1}{2}}}^{\mathtt{v}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\rangle\binom{2}{1}}\overbrace{\mathtt{a}_{\rangle\binom{2}{2}}}^{\mathtt{x}}\mathtt{a}_{\binom{2}{1}\langle}\overbrace{\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}}^{\mathtt{y}}\\&=\mathtt{ua}_{\rangle\binom{1}{1}}\mathtt{va}_{\rangle\binom{2}{1}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{xa}_{\binom{2}{1}\langle}\mathtt{y}+\mathtt{ua}_{\rangle\binom{1}{1}\binom{2}{1}}\mathtt{vxa}_{\binom{1}{1}\binom{2}{1}\rangle}\mathtt{y}\\&=\mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{1}{1}\binom{2}{1}\rangle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}.\end{aligned}$$

We use again the point (3) on the first terms of the last sum and find

$$\begin{aligned}\mathtt{w}\ =\ &\mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{2}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{2}\langle}\mathtt{a}_{\binom{1}{1}\binom{2}{2}\rangle}\\&+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{1}{1}\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}.\end{aligned}$$

Finally, using the point (2) of Definition 17, we show $w\in\widetilde{\mathcal{F}}$. Indeed,

$$\begin{aligned}\mathtt{w}\ =\ &\mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{2}{2}\rangle}+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{2}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{1}}\mathtt{a}_{\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{1}\binom{2}{2}\rangle}\mathtt{a}_{\binom{1}{2}\rangle}\\&+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{2}}\mathtt{a}_{\binom{1}{1}\binom{2}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{2}\rangle}.\end{aligned}$$

To each element $\mathtt{b} \in \mathcal{A}$ we define $|\mathtt{b}| := \max\{i_\ell : 1 \le \ell \le k\}$ and $\omega(n) := \max\{j_\ell : 1 \le \ell \le k\}$ for any $\mathtt{b} \in \mathcal{A}$ with $\mathtt{b} = \mathtt{a}_{\bullet\binom{i_1}{j_1}\cdots\binom{i_1}{j_1}}$ or $\mathtt{b} = \mathtt{a}_{\binom{i_1}{j_1}\cdots\binom{i_1}{j_1}\diamond}$ $(\bullet, \diamond \in \{\langle, \rangle\})$. Also we set, for any word $\mathtt{w}$ in $\mathcal{A}^*$, $|\mathtt{w}| = \omega(\mathtt{w}) = 0$ if $\mathtt{w}$ is the empty word, $|\mathtt{w}| = \max\{|\mathtt{u}|, |\mathtt{b}|\}$ and $\omega(\mathtt{w}) = \max\{\omega(\mathtt{u}), \omega(\mathtt{b})\}$ for $\mathtt{w} = \mathtt{ub}$ with $\mathtt{b} \in \mathcal{A}$.
We define on $\mathbb{C}\langle\mathcal{A}\rangle$ the shifted product as the only associative product satisfying, for each $\mathtt{u}, \mathtt{v} \in \mathcal{A}^*$, $\mathtt{u} \star \mathtt{v} = \mathtt{uv}[|\mathtt{u}|, \omega(\mathtt{u})]$, where $\mathtt{a}_{\bullet\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}}[m,n] = \mathtt{a}_{\bullet\binom{i_1+n}{j_1+m}\cdots\binom{i_k+n}{j_k+m}}$, $\mathtt{a}_{\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}\diamond}[m,n] = \mathtt{a}_{\binom{i_1+n}{j_1+m}\cdots\binom{i_k+n}{j_k+m}\diamond}$ $(\bullet, \diamond \in \{\langle, \rangle\})$, and $\mathtt{w}[n,m] = \mathtt{u}[n,m]\mathtt{b}[n,m]$ if $\mathtt{u} \in \mathcal{A}^*$ and $\mathtt{b} \in \mathcal{A}$.

Claim 22 *1. Let $P \in \mathcal{J}_F$. For any $Q \in \mathbb{C}\langle\mathcal{A}\rangle$ we have $P \star Q, Q \star P \in \mathcal{J}_F$. In consequence, the operation $\star$ is well defined on $\mathcal{F}$.*

2. If $P, Q \in \widetilde{\mathcal{F}}$ then $P \star Q \in \widetilde{\mathcal{F}}$.

These properties implies that we can endow the space $\mathcal{F}$ with the associative product $\star$. Let $\hat{\mathcal{F}}$ denote this algebra.

Example 23 Consider the element $\mathtt{w} = \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle} \star \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}$. One has

$$\begin{aligned}
\mathtt{w} &= \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\rangle\binom{2}{3}}\mathtt{a}_{\rangle\binom{2}{4}}\mathtt{a}_{\binom{2}{3}\langle}\mathtt{a}_{\binom{2}{4}\rangle}\\
&= \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{3}}\mathtt{a}_{\rangle\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{3}\langle}\mathtt{a}_{\binom{2}{4}\rangle} + \mathtt{a}_{\rangle\binom{1}{1}\binom{2}{4}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{3}}\mathtt{a}_{\binom{1}{1}\binom{2}{4}\rangle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{3}\langle}\\
&\quad + \mathtt{a}_{\rangle\binom{1}{1}\binom{2}{3}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\binom{2}{3}\langle}\mathtt{a}_{\binom{1}{2}\rangle}\mathtt{a}_{\binom{2}{4}\rangle}.
\end{aligned}$$

3.2 The algebra of B-diagrams

We consider the algebra $\mathcal{B}$ consisting in the space formally generated by the B-diagrams and endowed with the product $\star$ defined by

$$G \star G' = \sum_{G'' \in G \text{☆} G'} G'' = \sum_{\substack{a_1 < \cdots < a_k \in H^\uparrow(G)\\ b_1,\dots,b_k \in H^\downarrow(G'),\ \text{distinct}}} \begin{array}{c} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \text{☆} \\ {\scriptstyle a_1,\dots,a_k} \\ G \end{array}. \tag{12}$$

For simplicity, for any $1 \le a_1 < \cdots < a_k \le \omega(G)$ and $1 \le b_1, \dots, b_k \le \omega(G')$ distinct, we set

$$\begin{array}{c} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \star \\ {\scriptstyle a_1,\dots,a_k} \\ G \end{array} = \begin{cases} \begin{array}{c} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \text{☆} \\ {\scriptstyle a_1,\dots,a_k} \\ G \end{array} & \text{if } a_1 < \cdots < a_k \in H^\uparrow(G) \text{ and } b_1, \dots, b_k \in H^\downarrow(G') \\ 0 & \text{otherwise} \end{cases}$$

With this notation we have

$$G \star G' = \sum_{\substack{1 \le a_1 < \cdots < a_k \le \omega(G)\\ 1 \le b_1,\dots,b_k \le \omega(G'),\ \text{distinct}}} \begin{array}{c} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \star \\ {\scriptstyle a_1,\dots,a_k} \\ G \end{array}. \tag{13}$$

We set $\mathcal{B}_k = \operatorname{span}\{G : \omega(G) = k\}$. Note that $\mathcal{B}$ splits into the direct sum $\mathcal{B} = \bigoplus_k \mathcal{B}_k$ and the dimension of each space $\mathcal{B}_k$ is finite.
Straightforwardly from the definition, we check

Claim 24 $(\mathcal{B}, \star)$ *is a graded algebra with finite dimensional graded component. The unit of this algebra is the empty B-diagram ε.*

Example 25 See figures 7 and 8 for two examples of product.

We remark that the product is triangular in the sense that all the B-diagrams different from $G'' = G|G'$ which appear in the product $G \star G'$ have strictly less connected components than G''. Hence, since $\mathcal{B} = \mathbb{C}[\mathbb{B}]$ and $\mathbb{B}$ is isomorphic to the free monoid generated by $\mathbb{G}^*$ (see Proposition(14)), we obtain the following result.

Proposition 26 *The algebra $\mathcal{B}$ is free on the indivisible B-diagrams. In other words, it is isomprphic to $\mathcal{B} \coloneqq \mathbb{C}\langle\mathbb{G}\rangle$.*

3.3 From B-diagrams algebra to Fusion algebra

The aim of this section is to prove that the algebra $\mathcal{B}$ is isomorphic to a subalgebra of $\hat{\mathcal{F}}$.

Definition 27 *A* path *in a B-Diagram $G = (n, \lambda, E^\uparrow, E^\downarrow, E)$ is an increasing sequence of integers $1 \le i_1 < \cdots < i_k \le \omega(G)$ such that $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k) \in E$, $i_1 \in H_f^\downarrow(G)$ and $i_k \in H_f^\uparrow(G)$. Let* $\mathrm{Paths}(G)$ *denote the set of the paths in G.*

Let $p = (i_1, \dots, i_k) \in \mathrm{Path}(G)$. For simplicity, we define

- $p^\downarrow = i_1$ *and* $p^\uparrow = i_k$,
- $\bullet_p \coloneqq \begin{cases} \rangle & \text{if } i_1 \in \mathcal{H}_f^\downarrow \\ \langle & \text{otherwise} \end{cases}$ *and* $\diamond_p = \begin{cases} \langle & \text{if } i_k \in \mathcal{H}_f^\uparrow \\ \rangle & \text{otherwise} \end{cases}$
- $\mathrm{seq}_G(p) = \binom{v_G(i_1)}{i_1} \cdots \binom{v_G(i_k)}{i_k}$

Remark 28 *It is easy to check that*

1. $h_f^\uparrow = \#\{p \in \mathrm{Paths}(G) : \diamond_p = \langle\}$,
2. $h_f^\downarrow = \#\{p \in \mathrm{Paths}(G) : \bullet_p = \rangle\}$,
3. $h_c = \#\{p \in \mathrm{Paths}(G) : \diamond_p = \rangle\} + \#\{p \in \mathrm{Paths}(G) : \bullet_p = \langle\}$.

Example 29 Consider the B-diagram G of Figure 1. One has

$$\mathrm{Paths}(G) = \{(1,6), (2,4,5), (3)\}.$$

Suppose that $p = (1,6)$, one has

$$\bullet_p = \rangle,\ \diamond_p = \rangle,\ \text{and } \mathrm{seq}_G(p) = \binom{1}{1}\binom{3}{6}.$$

We have $h_f^\uparrow = 2 = \#\{(2,4,5), (3)\}$, $h_f^\downarrow = 3 = \#\{(1,6), (2,4,5), (3)\}$, and $h_c = 1 = \#\{(1,6)\}$.

For any B-diagram G we define

$$\mathtt{w}(G) \coloneqq \prod_{p \in \mathrm{Paths}(G)} \mathsf{a}_{\bullet_p \mathrm{seq}_G(p)} \prod_{p \in \mathrm{Paths}(G)} \mathsf{a}_{\mathrm{seq}(p)_G \diamond_p} \in \hat{\mathcal{F}}. \tag{14}$$

Clearly, a B-diagram G is completely characterized by the values of $\mathtt{seq}(p)$, $\bullet_p$ and $\diamond_p$ associated to each of its paths p. So $\mathtt{w}$ is into. Furthermore, $\mathtt{w}$ sends the empty B-diagram ε to the unit 1 of $\hat{\mathcal{F}}$.

Example 30 If we consider the B-diagram $G = (1,6)$ in Figure 1, one has

$$\mathtt{w}(G) = \mathsf{a}_{\rangle\binom{1}{1}\binom{3}{6}} \mathsf{a}_{\rangle\binom{1}{2}\binom{2}{4}\binom{3}{5}} \mathsf{a}_{\rangle\binom{1}{3}} \mathsf{a}_{\binom{1}{1}\binom{3}{6}\rangle} \mathsf{a}_{\binom{1}{2}\binom{2}{4}\binom{3}{5}\rangle} \mathsf{a}_{\binom{1}{3}\rangle}.$$

Definition 31 *Let $\mathcal{F}_{\mathcal{H}}$ denote the subalgebra of $\hat{\mathcal{F}}$ generated by the elements $\mathtt{w}(G)$ where G is a B-diagram.*

Lemma 32 *The set $\{\mathtt{w}(G) : G \text{ is a B-diagram}\}$ is a basis of the space $\mathcal{F}_{\mathcal{H}}$.*

Proof. One has to prove that for any $G_1, \dots, G_n$ distinct, $\sum \alpha_i \mathtt{w}(G_i) = 0$ implies $\alpha_1 = \dots = \alpha_n = 0$. We proceed as follows: first we consider the partially commutative free algebra $\mathbb{C}\langle A, \theta\rangle = \mathbb{C}\langle A\rangle/_{\mathcal{J}_1}$ where $\mathcal{J}_1$ is the ideal generated by the polynomials of the points 1 and 2 of Definition 17. Notice that $\mathbb{C}\langle A, \theta\rangle = \mathbb{C}[\mathbb{M}(\mathcal{A}, \theta)]$ is the algebra of the free partially commutative monoid $\mathbb{M}(\mathcal{A}, \theta) = \mathcal{A}^*/_{\equiv_\theta}$ where $\equiv_\theta$ is the congruence generated by $\mathtt{uefv} = \mathtt{ufev}$ for each $\mathtt{u}, \mathtt{v} \in \mathcal{A}^*$, ($\mathtt{e} \in \mathcal{A}_{\langle\ \rangle}$ and $\mathtt{f} \in \mathcal{A}$) or $\mathtt{e}, \mathtt{f} \in \mathcal{A}_{|\diamond}$ or $\mathtt{e}, \mathtt{f} \in \mathcal{A}_{\bullet|}$. Hence, we define $\mathcal{J}_2$ the ideal of $\mathbb{C}\langle A, \theta\rangle$ generated by the polynomials

$$\mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\alpha\langle}\mathtt{a}_{\rangle\beta}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y} - \mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\rangle\beta}\mathtt{a}_{\alpha\langle}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y} - \mathtt{u}\mathtt{a}_{\bullet\alpha\beta}\mathtt{v}\mathtt{x}\mathtt{a}_{\alpha\beta\diamond}\mathtt{y} \tag{15}$$

for any $\mathtt{u}, \mathtt{v}, \mathtt{x}, \mathtt{y} \in \mathbb{M}(\mathcal{A}, \theta)$, $\alpha = \binom{i_1}{j_1} \cdots \binom{i_k}{j_k}$, $\beta = \binom{i'_1}{j'_1} \cdots \binom{i'_{k'}}{j'_{k'}}$ (with $k, k' \in \mathbb{N}$ and $i_1, \dots, i_k$, $j_1, \dots, j_k$, $i'_1, \dots, i'_{k'}$, $j'_1 \dots, j'_{k'} \in \mathbb{N} \setminus \{0\}$), and $\bullet, \diamond \in \{\langle, \rangle\}$. Observe that $\mathcal{F} = \mathbb{C}[\mathbb{M}(\mathcal{A}, \theta)]/_{\mathcal{J}_2}$. We define $\widetilde{\mathtt{w}}(G) = \prod_{p \in \mathrm{Paths}(G)} \mathtt{a}_{\bullet_p \mathrm{seq}_G(p)} \prod_{p \in \mathrm{Paths}(G)} \mathtt{a}_{\mathrm{seq}(p)_G \diamond_p} \in \mathbb{C}\langle A, \theta\rangle$. Let $\mathcal{W}$ be the subspace generated by the monomials $\widetilde{\mathtt{w}}(G)$. Remarking that the map $\widetilde{\mathtt{w}}$ is into, our statement is equivalent to $\mathcal{W} \cap \mathcal{J}_2 = 0$. Let $P \in \mathcal{W} \cap \mathcal{J}_2$. We have

$$\begin{aligned} P &= Q\left(\mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\alpha\langle}\mathtt{a}_{\rangle\beta}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y} - \mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\rangle\beta}\mathtt{a}_{\alpha\langle}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y} - \mathtt{u}\mathtt{a}_{\bullet\alpha\beta}\mathtt{v}\mathtt{x}\mathtt{a}_{\alpha\beta\diamond}\mathtt{y}\right) R \\ &= \sum_i \alpha_i \prod_{p \in \mathrm{Paths}(G_i)} \mathtt{a}_{\bullet_p \mathrm{seq}_{G_i}(p)} \prod_{p \in \mathrm{Paths}(G_i)} \mathtt{a}_{\mathrm{seq}(p)_{G_i} \diamond_p}. \end{aligned}$$

Since $\mathtt{u}\mathtt{a}_{\bullet\alpha}\mathtt{v}\mathtt{a}_{\alpha\langle}\mathtt{a}_{\rangle\beta}\mathtt{x}\mathtt{a}_{\beta\diamond}\mathtt{y}$ can not be written under the form $\prod_j \mathtt{a}_{\bullet_j \alpha_j} \prod_j \mathtt{a}_{\alpha_j \diamond_j}$, this is not possible unless $P = 0$ (because of the factor $\mathtt{a}_{\alpha\langle}\mathtt{a}_{\rangle\beta}$). □

The behaviour of the paths with respect to the composition is summarized as follows:

$$\begin{aligned} \mathrm{Paths}\left(\begin{matrix} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \star \\ {\scriptstyle a_1,\dots,a_k} \\ G \end{matrix}\right) &= \left\{p \in \mathrm{Paths}(G) : p^\uparrow \notin \{a_1, \dots, a_k\}\right\} \cup \left\{p'[\omega(G)] : p' \in \mathrm{Paths}(G') : p^\downarrow \notin \{b_1, \dots, b_k\}\right\} \\ &\cup \left\{pp'[\omega(G)] : p \in \mathrm{Paths}(G), p' \in \mathrm{Paths}(G'), p^\uparrow \in \{a_1, \dots, a_k\}, p'^\downarrow \in \{b_1, \dots, b_k\}\right\} \end{aligned} \tag{16}$$

where pp' denotes the sequence obtained by catening p and p' and $p[n]$ is the sequence obtained from p by adding n to each element.

Example 33 Examine Figure 4. Let G denote the lower B-diagram in the left hand sides of the equality and by G' . We have

$$\mathrm{Paths}(G) = \{(1,6),(2),(3,7),(4),(5),(8)\} \text{ and } \mathrm{Paths}(G') = \{(1,6),(2,4,5),(3)\}$$

Hence

$$\mathrm{Paths}\left(\begin{matrix} G' \\ {\scriptstyle 3,1} \\ \star \\ {\scriptstyle 4,6} \\ G \end{matrix}\right) = \{(2),(3,7),(5),(8)\} \cup \{(10,12,13)\} \cup \{(1,6,9,14),(4,11)\}\}$$

Theorem 34 *The algebras $\mathcal{B}$ and $\mathcal{F}_{\mathcal{H}}$ are isomorphic and an explicit isomorphism sends each B-diagram G to $\mathtt{w}(G)$.*

Proof. First let us prove that $\mathtt{w}$ can be extended as a morphism of algebra. In other words, we first extend $\mathtt{w}$ as linear map and we prove that $\mathtt{w}(G \star G') = \mathtt{w}(G) \star \mathtt{w}(G')$. Observe that from (12), we obtain

$$\mathtt{w}(G\star G') = \sum_{\substack{i_1<\cdots<i_k\in H_f^{\uparrow}(G)\\ j_1,\dots,j_k\in H_f^{\downarrow}(G'),\ \text{distinct}}} \mathtt{w}\left(\begin{array}{c} G' \\ {\scriptstyle j_1,\dots,j_k} \\ \star \\ {\scriptstyle i_1,\dots,i_k} \\ G \end{array}\right). \tag{17}$$

But if $G'' = \begin{array}{c} G' \\ {\scriptstyle j_1,\dots,j_k} \\ \star \\ {\scriptstyle i_1,\dots,i_k} \\ G \end{array}$, Equality (16) implies

$$\begin{aligned}\mathtt{w}(G'') \quad = \quad & \prod^{(*)} {\color{red}\mathtt{a}}_{\bullet_p \mathtt{seq}_{G''}(p)} \prod^{(**)} {\color{red}\mathtt{a}}_{\bullet_{p'}\mathtt{seq}_{G''}(p')} \prod^{(***)} {\color{red}\mathtt{a}}_{\bullet_p \mathtt{seq}_{G''}(pp')} \\ & \times \prod^{(*)} {\color{blue}\mathtt{a}}_{\mathtt{seq}_{G''}(p)\diamond_p} \prod^{(**)} {\color{blue}\mathtt{a}}_{\mathtt{seq}_{G''}(p')\diamond_{p'}} \prod^{(***)} {\color{blue}\mathtt{a}}_{\mathtt{seq}_{G''}(pp')\diamond_{p'}}\end{aligned} \tag{18}$$

where the products $\prod^{(*)}$ are over the paths $p\in \mathrm{Paths}(G)$ such that $p^{\uparrow}\notin\{i_1,\dots,i_k\}$, the products $\prod^{(**)}$ are over the paths $p'\in\mathrm{Paths}(G')$ such that $p'^{\downarrow}\notin\{j_1,\dots,j_k\}$, and the products $\prod^{(***)}$ are over the pairs of paths $p\in\mathrm{Paths}(G)$ and $p'\in\mathrm{Paths}(G')$ with $p^{\uparrow}=i_h$ and $p'^{\downarrow}=j_h$ for some $1\le h\le k$.

Now, we examine $\mathtt{w}(G)\star\mathtt{w}(G')$. One has

$$\begin{aligned}\mathtt{w}(G)\star\mathtt{w}(G') \quad = \quad & \left(\prod_{p\in\mathrm{Paths}(G)} {\color{red}\mathtt{a}}_{\bullet_p \mathrm{seq}_G(p)} \prod_{p\in\mathrm{Paths}(G)} {\color{blue}\mathtt{a}}_{\mathrm{seq}_G(p)\diamond_p}\right)\star\left(\prod_{p'\in\mathrm{Paths}(G')} {\color{red}\mathtt{a}}_{\bullet_{p'}\mathrm{seq}_{G'}(p')} \prod_{p'\in\mathrm{Paths}(G')} {\color{blue}\mathtt{a}}_{\mathrm{seq}_{G'}(p')\diamond_{p'}}\right) \\ = \quad & \prod_{\substack{p\in\mathrm{Paths}(G)\\ \diamond_p=\rangle}} {\color{red}\mathtt{a}}_{\bullet_p \mathrm{seq}_G(p)} \prod_{\substack{p'\in\mathrm{Paths}(G')\\ \bullet_{p'}=\langle}} \left({\color{red}\mathtt{a}}_{\langle \mathrm{seq}_{G'}(p)}[|G|,\omega(G)]\right)(\mathtt{u}\star\mathtt{u}') \\ & \times \prod_{\substack{p\in\mathrm{Paths}(G)\\ \diamond_p=\rangle}} {\color{blue}\mathtt{a}}_{\mathrm{seq}_G(p)\rangle} \prod_{\substack{p'\in\mathrm{Paths}(G')\\ \bullet_{p'}=\langle}} \left({\color{blue}\mathtt{a}}_{\mathrm{seq}_{G'}(p')\diamond_{p'}}[|G|,\omega(G)]\right)\end{aligned}$$

where

$$\mathtt{u} = \prod_{\substack{p\in\mathrm{Paths}(G)\\ \diamond_p=\langle}} {\color{red}\mathtt{a}}_{\bullet_p \mathrm{seq}_G(p)} \prod_{\substack{p\in\mathrm{Paths}(G)\\ \diamond_p=\langle}} {\color{blue}\mathtt{a}}_{\mathrm{seq}_G(p)\langle} \text{ and } \mathtt{u}' = \prod_{\substack{p'\in\mathrm{Paths}(G')\\ \bullet_{p'}=\rangle}} {\color{red}\mathtt{a}}_{\rangle \mathrm{seq}_{G'}(p')} \prod_{\substack{p'\in\mathrm{Paths}(G')\\ \bullet_{p'}=\rangle}} {\color{blue}\mathtt{a}}_{\mathrm{seq}_{G'}(p')\diamond_{p'}} \tag{19}$$

Hence we apply equality (8) to $\mathtt{u}\star\mathtt{u}'$. Observing that the pairs of sequences $i_1<\cdots<i_k$ and $j_1,\dots,j_k$ in (8) are in a one to one correspondence with the B-diagrams $\begin{array}{c} G' \\ {\scriptstyle j_1,\dots,j_k} \\ \star \\ {\scriptstyle i_1,\dots,i_k} \\ G \end{array}$, we obtain

$$\begin{aligned}\mathtt{u}\star\mathtt{u}' = \sum_{G''} \Bigg(& \prod^{(*)'} {\color{red}\mathtt{a}}_{\bullet_p \mathtt{seq}_{G''}(p)} \prod^{(**)'} {\color{red}\mathtt{a}}_{\langle \mathtt{seq}_{G''}(p'[\omega(G)])} \prod^{(***)'} {\color{red}\mathtt{a}}_{\bullet_p \mathtt{seq}_{G''}(pp'[\omega(G)])} \\ & \times \prod^{(*)'} {\color{blue}\mathtt{a}}_{\mathtt{seq}_{G''}(p)\langle} \prod^{(**)'} {\color{blue}\mathtt{a}}_{\mathtt{seq}_{G''}(p'[\omega(G)])\diamond_{p'}} \prod^{(***)'} {\color{blue}\mathtt{a}}_{\mathtt{seq}_{G''}(pp'[\omega(G)])\diamond_{p'}}\Bigg),\end{aligned} \tag{20}$$

where the sum is over the B-diagram $G'' = \begin{array}{c} G' \\ {\scriptstyle j_1,\dots,j_k} \\ \star \\ {\scriptstyle i_1,\dots,i_k} \\ G \end{array}$ with $i_1<\cdots<i_k\in H_f^{\uparrow}(G)$ and $j_1,\dots,j_k\in H_f^{\downarrow}(G)$ distinct, the products $\prod^{(*)'}$ are over the paths $p\in\mathrm{Paths}(G)$ such that $p^{\uparrow}\notin\{i_1,\dots,i_k\}$ and $\diamond_p=\langle$, the products $\prod^{(**)'}$ are over the paths $p'\in\mathrm{Paths}(G')$ such that $p'^{\downarrow}\notin\{j_1,\dots,j_k\}$ and $\bullet_{p'}=\rangle$, and the products

$\prod\limits^{(***)'}$ are over the pairs of paths $p \in \mathrm{Paths}(G)$ and $p' \in \mathrm{Paths}(G')$ with $p^{\uparrow} = i_h$ and $p'^{\downarrow} = j_h$ for some $1 \leq h \leq k$.
Notice that for a given G'' the paths which do not appear in the product

$$\begin{aligned}\mathcal{P}(G'') \quad = \quad & \prod^{(*)'} \mathtt{a}_{\bullet_p \mathtt{seq}_{G''}(p)} \prod^{(**)'} \mathtt{a}_{\langle \mathtt{seq}_{G''}(p'[\omega(G)])} \prod^{(***)'} \mathtt{a}_{\bullet_p \mathtt{seq}_{G''}(pp'[\omega(G)])} \\ & \times \prod^{(*)'} \mathtt{a}_{\mathtt{seq}_{G''}(p)\langle} \prod^{(**)'} \mathtt{a}_{\mathtt{seq}_{G''}(p'[\omega(G)])\diamond_{p'}} \prod^{(***)'} \mathtt{a}_{\mathtt{seq}_{G''}(pp'[\omega(G)])\diamond_{p'}}\end{aligned} \tag{21}$$

are exactly the paths p of G such that $\diamond_p = \rangle$ and the paths $p'[\omega(G)]$ where $p' \in \mathtt{Paths}(G')$ and $\bullet_{p'} = \langle$.
Hence, comparing (21) to (18), one obtains

$$\begin{aligned}\mathtt{w}(G'') = & \prod_{\substack{p \in \mathrm{Paths}(G) \\ p^{\uparrow} = \rangle}} \mathtt{a}_{\bullet_p \mathrm{seq}_G(p)} \prod_{\substack{p' \in \mathrm{Paths}(G') \\ \bullet_{p'} = \langle}} \left(\mathtt{a}_{\langle \mathrm{seq}_{G'}(p)}[|G|, \omega(G)]\right) \mathcal{P}(G'') \\ & \times \prod_{\substack{p \in \mathrm{Paths}(G) \\ \diamond p = \rangle}} \mathtt{a}_{\mathrm{seq}_G(p)\rangle} \prod_{\substack{p' \in \mathrm{Paths}(G') \\ \bullet_{p'} = \langle}} \left(\mathtt{a}_{\mathrm{seq}_{G'}(p')\bullet_{p'}}[|G|, \omega(G)]\right).\end{aligned} \tag{22}$$

So (19) becomes

$$\mathtt{w}(G) \star \mathtt{w}(G') = \sum_{G''} \mathtt{w}(G'')$$

where the sum is over the B-diagram $G'' = \overset{G'}{\underset{G}{\overset{j_1,\ldots,j_k}{\underset{i_1,\ldots,i_k}{\star}}}}$ with $i_1 < \cdots < i_k \in H^{\uparrow}_f(G)$ and $j_1, \ldots, j_k \in H^{\downarrow}_f(G)$ distinct. In other words,

$$\mathtt{w}(G) \star \mathtt{w}(G') = \mathtt{w}(G \star G').$$

Lemma 32 allows us to conclude. □

Example 35 Compare the computation in Figure 7 with Example 23.

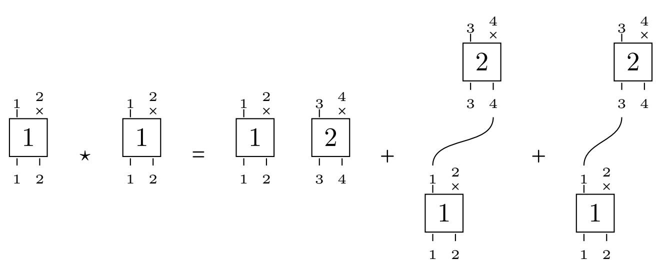


Figure 7: An example of computation in $\mathcal{B}$

Also compare Figure 8 to

$$\begin{aligned}&\left(\mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\langle}\right)^{\star 2} = \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{3}}\mathtt{a}_{\rangle\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\langle}\mathtt{a}_{\binom{2}{3}\langle}\mathtt{a}_{\binom{2}{4}\langle} \\ &+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{3}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\binom{2}{3}\langle}\mathtt{a}_{\binom{1}{2}\langle}\mathtt{a}_{\binom{2}{4}\langle} + \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}\binom{2}{3}}\mathtt{a}_{\rangle\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\binom{2}{3}\langle}\mathtt{a}_{\binom{2}{4}\langle} \\ &+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{2}}\mathtt{a}_{\rangle\binom{1}{2}}\mathtt{a}_{\rangle\binom{2}{3}}\mathtt{a}_{\binom{1}{1}\binom{2}{2}\langle}\mathtt{a}_{\binom{1}{2}\langle}\mathtt{a}_{\binom{2}{3}\langle} + \mathtt{a}_{\rangle\binom{1}{1}}\mathtt{a}_{\rangle\binom{1}{2}\binom{2}{4}}\mathtt{a}_{\rangle\binom{2}{3}}\mathtt{a}_{\binom{1}{1}\langle}\mathtt{a}_{\binom{1}{2}\binom{2}{4}\langle}\mathtt{a}_{\binom{2}{3}\langle} \\ &+\mathtt{a}_{\rangle\binom{1}{1}\binom{2}{3}}\mathtt{a}_{\rangle\binom{1}{2}\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\binom{2}{3}\langle}\mathtt{a}_{\binom{1}{2}\binom{2}{4}\langle} + \mathtt{a}_{\rangle\binom{1}{1}\binom{2}{4}}\mathtt{a}_{\rangle\binom{1}{3}\binom{2}{4}}\mathtt{a}_{\binom{1}{1}\binom{2}{4}\langle}\mathtt{a}_{\binom{1}{2}\binom{2}{3}\langle}\end{aligned}$$

As a consequence, one has

Corollary 36 *The algebra $\mathcal{B}$ is associative.*

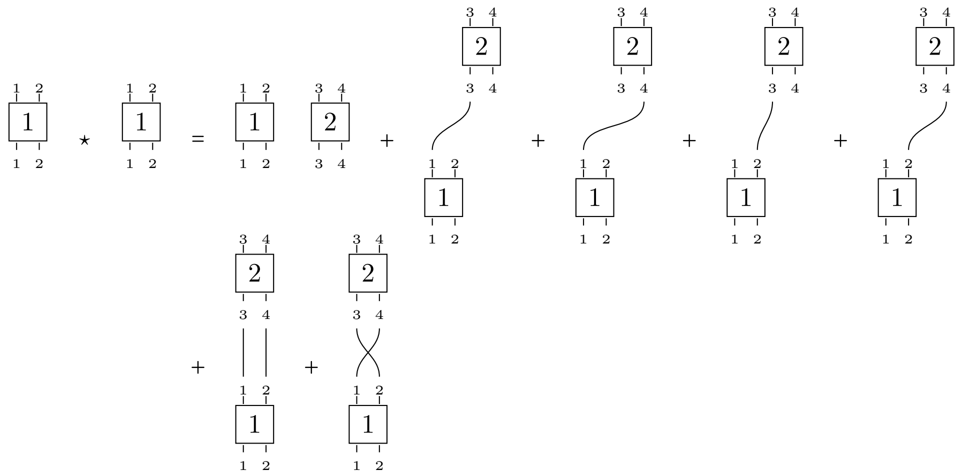


Figure 8: A second example of computation in $\mathcal{B}$

Alternatively, this result can be shown in a combinatorial way. First, we extend bi-linearly the composition $\underset{a_1,\dots,a_k}{\overset{b_1,\dots,b_k}{\star}}$. Hence, we observe

$$\begin{matrix} G' \\ \underset{\alpha_1,\dots,\alpha_p}{\overset{\beta_1,\dots,\beta_p}{\star}} \\ G \end{matrix} \star G'' = \sum \begin{matrix} G' \star G'' \\ \underset{a_1,\dots,a_k}{\overset{b_1,\dots,b_k}{\star}} \\ G \end{matrix}, \tag{23}$$

where the sum is over the sequences $1 \le a_1 < \cdots < a_k \le \omega(G)$ and the sequences of distinct elements $1 \le b_1, \dots, b_k \le \omega(G') + \omega(G'')$ such that there exists $1 \le i_1 < \cdots < i_p$ such that $a_{i_1} = \alpha_1$, $b_{i_1} = \beta_1$, $\dots$, $a_{i_p} = \alpha_p$, $b_{i_p} = \beta_p$.

From (13) we obtain,

$$(G \star G') \star G'' = \sum_{\substack{1\le\alpha_1<\cdots<\alpha_p\le\omega(G) \\ 1\le\beta_1,\dots,\beta_p\le\omega(G')\text{distinct}}} \begin{matrix} G' \\ \underset{\alpha_1,\dots,\alpha_p}{\overset{\beta_1,\dots,\beta_p}{\star}} \\ G \end{matrix} \star G'' = \sum_{\substack{1\le a_1<\cdots<a_k\le\omega(G) \\ 1\le b_1,\dots,b_k\le\omega(G')+\omega(G'')\text{distinct}}} \begin{matrix} G' \star G'' \\ \underset{a_1,\dots,a_k}{\overset{b_1,\dots,b_k}{\star}} \\ G \end{matrix} = G \star (G' \star G'').$$

3.4 Application to the boson normal ordering

The Heisenberg-Weyl algebra is defined as the quotient $\mathcal{HW} = \mathbb{C}\langle\{\mathsf{a}, \mathsf{a}^\dagger\}\rangle/_{\mathcal{J}_{\mathcal{HW}}}$, where $\mathcal{J}_{\mathcal{HW}}$ is the ideal generated by $\mathsf{a}\mathsf{a}^\dagger - \mathsf{a}^\dagger\mathsf{a} - 1$. The algebra $\mathcal{HW}$ is classically related to the boson normal ordering problem. We consider a slightly different algebra $\mathcal{H}$ which is obtained by adding a central element e to $\mathcal{HW}$. We will see that e has a natural combinatorial interpretation. Indeed, let us define the map $\mathfrak{p} : \mathcal{A} \longrightarrow \{\mathsf{a}, \mathsf{a}^\dagger, \mathsf{e}\}$ sending each element of $\mathcal{A}_{\rangle\times}$ to $\mathsf{a}^\dagger$, each element of $\mathcal{A}_{\times\langle}$ to a, and each element of $\mathcal{A}_{\langle\ \rangle}$ to e. The map $\mathfrak{p}$ can be extended as linear maps $\widetilde{\mathfrak{p}} : \widetilde{\mathcal{F}} \longrightarrow \mathcal{H}$, $\hat{\mathfrak{p}} : \hat{\mathcal{F}} \longrightarrow \mathcal{H}$, and $\mathfrak{p}_{\mathcal{H}} : \mathcal{F}_{\mathcal{H}} \longrightarrow \mathcal{H}$. We also define the linear map $\mathfrak{p}_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{H}$ by

$$\mathfrak{p}_{\mathcal{B}}(G) = \left(\mathsf{a}^\dagger\right)^{\#\{p\in\mathtt{Paths}(G):\bullet_p=\rangle\}} \mathsf{a}^{\#\{p\in\mathtt{Paths}(G):\diamond_p=\langle\}} \mathsf{e}^{\#\{p\in\mathtt{Paths}(G):\bullet_p=\langle\}+\#\{p\in\mathtt{Paths}(G):\diamond_p=\rangle\}}. \tag{24}$$

Equivalently, from Remark 28, one has

$$\mathfrak{p}_{\mathcal{B}}(G) = \left(\mathsf{a}^\dagger\right)^{h_f^{\downarrow}(G)} \mathsf{a}^{h_f^{\uparrow}(G)} \mathsf{e}^{h_c(G)}. \tag{25}$$

Example 37 If G is the B-diagram of Figure 1, one has

$$\mathfrak{p}_{\mathcal{B}}(G) = \left(\mathtt{a}^{\dagger}\right)^{3}\mathtt{a}^{2}\mathtt{e}.$$

From (24) it is easy to check that $\mathfrak{p}_{\mathcal{B}}$ factorizes through $\mathcal{F}_{\mathcal{H}}$. More precisely

$$\mathfrak{p}_{\mathcal{B}} = \mathfrak{p}_{\mathcal{H}} \circ \mathtt{w}. \tag{26}$$

Furthermore, one has

Proposition 38 *The maps* $\widetilde{\mathfrak{p}}$, $\hat{\mathfrak{p}}$, $\mathfrak{p}_{\mathcal{H}}$, *and* $\mathfrak{p}_{\mathcal{B}}(G)$ *are morphisms of algebras.*

Proof. The fact that $\widetilde{\mathfrak{p}}$, $\hat{\mathfrak{p}}$, and $\mathfrak{p}_{\mathcal{H}}$ are morphisms is straightforward from their definitions. The map $\mathfrak{p}_{\mathcal{B}}(G)$ is a morphism of algebras because it is the composition of two morphisms of algebras (see Formula (26)). □

We easily check the product formula

$$\left(\mathtt{a}^{\dagger}\right)^{m}\mathtt{a}^{n}\mathtt{e}^{q}.\left(\mathtt{a}^{\dagger}\right)^{r}\mathtt{a}^{s}\mathtt{e}^{t} = \sum_{i=0}^{\min\{n,r\}} i!\binom{q}{i}\binom{r}{i}\left(\mathtt{a}^{\dagger}\right)^{m+r-i}\mathtt{a}^{n+s-i}\mathtt{e}^{q+t}. \tag{27}$$

This formula has the following combinatorial interpretation. Consider two B-diagrams G and G', there are $i!\binom{h_f^{\uparrow}(G)}{i}\binom{h_f^{\downarrow}(G')}{i}$ ways to compose G with G' and obtain a B-diagram G'' such that $h_f^{\downarrow}(G'') = h_f^{\downarrow}(G) + h_f^{\downarrow}(G') - i$, $h_f^{\uparrow}(G'') = h_f^{\uparrow}(G) + h_f^{\uparrow}(G') - i$, and $h_c(G'') = h_c(G) + h_c(G)$.

Example 39 Compare Figure 8 to

$$\left(\mathtt{a}^{\dagger}\right)^{2}\mathtt{a}^{2}.\left(\mathtt{a}^{\dagger}\right)^{2}\mathtt{a}^{2} = \left(\mathtt{a}^{\dagger}\right)^{4}\mathtt{a}^{4} + 4\left(\mathtt{a}^{\dagger}\right)^{3}\mathtt{a}^{3} + 2\left(\mathtt{a}^{\dagger}\right)^{2}\mathtt{a}^{2}.$$

The relationships between the algebras defined in this section are summarized in Figure 9 where the dashed arrow indicates that we replace the product in $\widetilde{\mathcal{F}}$ by a shifted product.

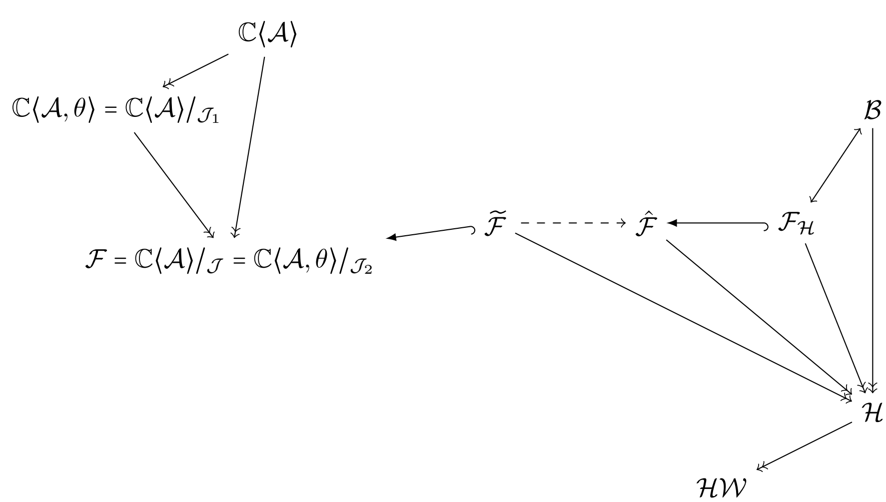


Figure 9: The different algebras related to the B-diagrams

4 Cogebraic aspects of B-diagrams

4.1 The Hopf algebras of B-diagrams

We define the linear map $\Delta : \mathcal{B} \longrightarrow \mathcal{B}\otimes\mathcal{B}$ by setting

$$\Delta(G) = \sum_{I\in \mathrm{Iso}(G)} G[I]\otimes G[\complement I]. \tag{28}$$

Equivalently, from claim 8, one has

$$\Delta(G) = \sum_{\mathcal{I}\subset \mathrm{Connected}(G)} G\left[\bigcup_{I\in\mathcal{I}} I\right]\otimes G\left[\complement_G \bigcup_{I\in\mathcal{I}} I\right]. \tag{29}$$

For simplicity we set

$$G\langle I\rangle = \begin{cases} G[I] & \text{if } I\in \mathrm{Iso}(G)\\ 0 & \text{otherwise}\end{cases}.$$

With this notation, one has

$$\Delta(G) = \sum_{I\uplus J=[\![1,|G|]\!]} G\langle I\rangle\otimes G\langle J\rangle. \tag{30}$$

We also define $\epsilon : \mathcal{B}\longrightarrow\mathbb{C}$ as the projection to $\mathcal{B}_0$. Obviously, Δ is a coassociative, cocomutative product and ϵ is its counity. So,

Proposition 40 $(\mathcal{B},\Delta,\epsilon)$ *is a connected graded co-commutative cogebra.*

Proof. Since $\dim\mathcal{B}_0 = 1$, we have only to check that Δ is graded. That is $\Delta(\mathcal{B}_k)\subset\bigoplus_{i+j=k}\mathcal{B}_i\otimes\mathcal{B}_j$. This is straightforward from the definition of Δ. □

Since $\mathcal{B}$ is a connected cogebra and an algebra with finite graded dimension, if it is a bigebra then it is a Hopf algebra. Hence, one has only to prove that Δ is a morphism of algebra.

Let us prove that for each pair (G,G') of B-diagrams one has $\Delta(G\star G') = \Delta(G)\star\Delta(G')$. We start from the equality

$$\Delta(G\star G') = \sum_{I\uplus J=[\![1,|G|+|G'|]\!]} G\star G'\langle I\rangle\otimes G\star G'\langle J\rangle.$$

Each I appearing in the sum splits into two sets $(I\cap[\![1,|G|]\!])\uplus(I\cap[\![|G|+1,|G|+|G'|]\!])$. Also, it is the case for the set J. Hence,

$$\Delta(G\star G') = \sum_{\substack{I\uplus J=[\![1,|G|]\!]\\ I'\uplus J'=[\![|G|+1,|G|+|G'|]\!]}} (G\star G')\langle I\cup I'\rangle\otimes(G\star G')\langle J\cup J'\rangle.$$

Applying (13), we obtain

$$\Delta(G\star G') = \sum_{\substack{1\leqslant a_1<\cdots<a_k\leqslant\omega(G)\\ 1\leqslant b_1,\dots,b_k\leqslant\omega(G)\text{distinct}}}\ \sum_{\substack{I\uplus J=[\![1,|G|]\!]\\ I'\uplus J'=[\![|G|+1,|G|+|G'|]\!]}} \begin{pmatrix} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \star \\ {\scriptstyle a_1,\dots,a_k} \\ G\end{pmatrix}\langle I\cup I'\rangle\otimes\begin{pmatrix} G' \\ {\scriptstyle b_1,\dots,b_k} \\ \star \\ {\scriptstyle a_1,\dots,a_k} \\ G\end{pmatrix}\langle J\cup J'\rangle.$$

Observe that the elements of I and J define two complementary subdiagrams of G and the elements of I' and J' define two complementary subdiagrams of G'. Hence,

$$\Delta(G\star G') = \sum_{\substack{I\uplus J=[\![1,|G|]\!]\\ I'\uplus J'=[\![1,|G'|]\!]}}\ \sum_{\substack{1\leqslant a_1<\cdots<a_k\leqslant\omega(G)\\ 1\leqslant b_1,\dots,b_k\leqslant\omega(G)\text{distinct}}} \begin{pmatrix} G'\langle I'\rangle \\ {\scriptstyle [b_1,\dots,b_k]^{I'}} \\ \star \\ {\scriptstyle [a_1,\dots,a_k]^{I}} \\ G\langle I\rangle\end{pmatrix}\otimes\begin{pmatrix} G'\langle J'\rangle \\ {\scriptstyle [b_1^{J'},\dots,b_k]^{J'}} \\ \star \\ {\scriptstyle [a_1,\dots,a_k]^{J}} \\ G\langle J\rangle\end{pmatrix}$$

where the sequence $[a_1,\dots,a_k]^I$ (resp. $[a_1,\dots,a_k]^J$, $[b_1,\dots,b_k]^{I'}$ and $[b_1,\dots,b_k]^{J'}$) denotes the image of subsequence of $[a_1,\dots,a_k]$ (resp. $[a_1,\dots,a_k]$, $[b_1,\dots,b_k]$ and $[b_1,\dots,b_k]$) in $G[I]$ (resp. $G[J]$, $G[I']$ and $G[J']$). In other words, each sequence $1\leqslant a_1<\dots<a_k\leqslant\omega(G)$ (resp. $b_1,\dots,b_k$) splits into two subsequences one corresponding to half edges in $G[I]$ (resp. $G[I']$) and the second to half edges in $G[J]$ (resp. $G[J']$). Hence, we deduce

$$\Delta(G\star G')=\sum_{\substack{I\uplus J=[\![1,|G|]\!]\\ I'\uplus J'=[\![1,|G'|]\!]}}\ \sum_{\substack{1\leqslant a_1<\dots<a_l\leqslant\omega(G\langle I\rangle)\\ 1\leqslant b_1,\dots,b_l\leqslant\omega(G'\langle I'\rangle)\text{ distinct}}}\ \sum_{\substack{1\leqslant c_1<\dots<c_t\leqslant\omega(G\langle J\rangle)\\ 1\leqslant d_1,\dots,d_t\leqslant\omega(G'\langle J'\rangle)\text{ distinct}}}\begin{pmatrix}G'\langle I'\rangle\\ {\scriptstyle b_1,\dots,b_l}\\ \star\\ {\scriptstyle a_1,\dots,a_l}\\ G\langle I\rangle\end{pmatrix}\otimes\begin{pmatrix}G'\langle J'\rangle\\ {\scriptstyle d_1,\dots,d_t}\\ \star\\ {\scriptstyle c_1,\dots,c_t}\\ G\langle J\rangle\end{pmatrix}.$$

Finally, one computes

$$\begin{aligned}\Delta(G\star G')&=\sum_{\substack{I\uplus J=[\![1,|G|]\!]\\ I'\uplus J'=[\![1,|G'|]\!]}}(G\langle I\rangle\star G'\langle I'\rangle)\otimes(G\langle J\rangle\star G'\langle J'\rangle)\\ &=\left(\sum_{I\uplus J=[\![1,|G|]\!]}G\langle I\rangle\otimes G\langle J\rangle\right)\star\left(\sum_{I'\uplus J'=[\![1,|G'|]\!]}G'\langle I'\rangle\otimes G'\langle J'\rangle\right)\\ &=\Delta(G)\star\Delta(G').\end{aligned}$$

This shows that $\mathcal{B}$ is a graded bialgebra with finite dimensional graded component. Hence,

Theorem 41 $(\mathcal{B},\star,\Delta)$ *is a graded Hopf algebra.*

4.2 Primitive elements

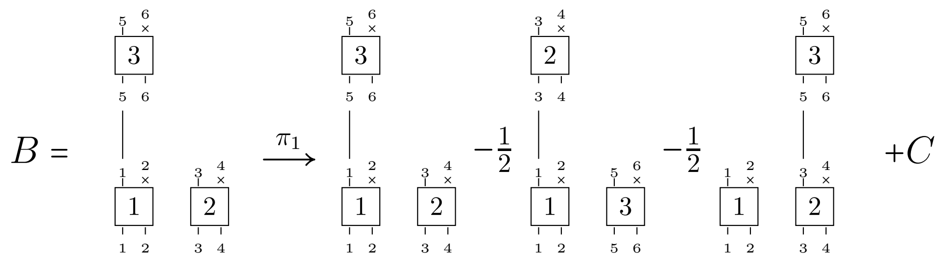
It is classical and easy to show that the primitive elements endowed with the bracket product $[P,Q]=P\star Q-Q\star P$ form a Lie algebra. The Cartier-Quillen-Milnor-Moore Theorem (see e.g., [21]) assures that each graded, connected, and cocommutative Hopf algebra is isomorphic to the universal enveloping of the Lie algebra of its primitive elements. Obviously, it is the case for $\mathcal{B}$. More precisely, $\mathcal{B}$ is a Hopf algebra which is graded by the number of half edges of the B-diagrams and the dimensions of its graded components are finite. Since $\mathcal{B}$ is free on the indivisible B-diagrams, the dimensions of the graded components of the space $\mathtt{Prim}(\mathcal{B})$ of the primitive elements are necessarily the same than those of the free Lie algebra $\mathcal{L}(\mathbb{G})$ generated by the indivisible elements.

Now, let us recall a few facts about the Eulerian idempotent. The convolution product is classically defined on $Hom(\mathcal{B},\mathcal{B})$ by $f*g=\mu\circ(f\otimes g)\circ\Delta$ where μ denotes the linear map from $\mathcal{B}\otimes\mathcal{B}$ to $\mathcal{B}$ sending $P\otimes Q$ to $P\star Q$. The Eulerian idempotent is defined by (see e.g., [22])

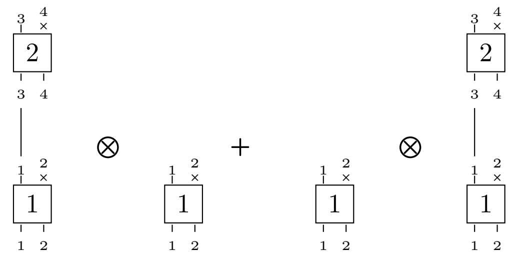
$$\pi_1:=\log_*(Id)=\sum_{k>0}\frac{(-1)^{k+1}}{k}(Id-\xi)^{*k}\tag{31}$$

where ξ denotes the unity of the convolution algebra, that is the projection on the space generated by the empty B-diagram ε. Remarking that $(Id-\xi)^{*k}$ sends to 0 any B-diagram that can be written as $G_1|\cdots|G_k$ with each $G_i\neq\varepsilon$, π_1 maps each B-diagram to a finite linear combination of B-diagrams which is known to belong in $\mathtt{Prim}(\mathcal{B})$.

Example 42 Examine the following example



where C is a linear combination of connected B-diagrams. Setting $\hat{\Delta} = \Delta - Id \otimes \varepsilon - \varepsilon \otimes Id$, we observe the three B-diagrams occurring in the above formula have the same image by $\hat{\Delta}$



Hence, $\pi_1(B)$ is primitive.

A fast examination of $\pi_1(G)$, where G is a indivisible B-diagram, shows that $\pi_1(G) = G + \cdots$, where $\cdots$ means a linear combination of B-diagrams which are non indivisible or have less connected components than G. Since, the indivisible B-diagrams are algebraically independent, this shows that $\texttt{Prim}(\mathcal{B})$ contains a subalgebra which is isomorphic to $\mathcal{L}(\mathbb{G})$. Hence, the dimensions of the graded components of $\texttt{Prim}(\mathcal{B})$ being the same than those of $\mathcal{L}(\mathbb{G})$, we deduce the following result.

Theorem 43 *The Lie algebra of the primitive elements of $\mathcal{B}$ is isomorphic to the free Lie algebra generated by the indivisible B-diagrams. Indeed,* $\mathrm{Prim}(\mathcal{B})$ *is freely generated as a Lie algebra by* $\{\pi_1(G) : G \in \mathbb{G}\}$.

5 Two subalgebras

5.1 Word symmetric functions

In this section, we investigate a combinatorial Hopf algebra whose bases are indexed by set partitions. A *set partitions* is a partition of $[\![1, n]\!]$ for a given $n \in \mathbb{N}$. Let $\pi \vDash n$ denote the fact that π is a set partition of $[\![1, n]\!]$ and define $\pi \uplus \pi' := \pi \cup \{\{i_1 + n, \ldots, i_k + n\} : \{i_1, \ldots, i_k\} \in \pi'$ for $\pi \vDash n$. The set of set partitions is endowed with the partial order $\leq$ of refinement defined by $\pi \leq \pi'$ if the sets of π' are the union of sets of π. Finally, we say that a set partition π is *indivisible* if $\pi = \pi' \uplus \pi''$ where π' and π'' are set partitions implies either $\pi' = \pi$ or $\pi'' = \pi$.

The algebra of word symmetric functions was introduced by Wolf [24] as a noncommutative analogue of the algebra of symmetric functions. The Hopf structure of this algebra was described in [4, 2] (see also [3] for the polynomial realization with finite alphabets). In order to avoid confusion with some other analogues of symmetric functions (see e.g. [12]), we denote this algebra by WSym which is described (as an abstract bigebra) as the space spanned by the elements M_π, where π is a set partitions, and endowed with the product

$$M_\pi M_{\pi'} = \sum_{\pi \uplus \pi' \leq \pi'' \leq \{\{1,\ldots,n\},\{n+1,\ldots,n'+1\}\}} M_{\pi''}, \tag{32}$$

if $\pi \vDash n$ and $\pi' \vDash n'$, and the coproduct

$$\Delta(M_\pi) = \sum_{\pi = e \uplus f} M_{\mathrm{std}(e)} \otimes M_{\mathrm{std}(f)}, \tag{33}$$

with $\mathrm{std}(e) = \{\{\phi(i_1), \ldots, \phi(i_k)\} : \{i_1, \ldots, i_k\} \in e\}$ where ϕ is the unique increasing bijection from $\bigcup_{\alpha \in e} \alpha$ to $[\![1, \sum_{\alpha \in e} \#\alpha]\!]$. The following result is well known and is an easy consequence of (32).

Proposition 44 *The algebra* WSym *is freely generated as an algebra by*

$$\{M_\pi : \pi \textit{ is an indivisible set partition}\}.$$

To each set partition $\pi \vDash n$, we associate a B-diagram $b_\pi = (n, [1,\dots,1], [\![1,n]\!], [\![1,n]\!], E_\pi)$ where E_π is the set of the pairs (i,j) such that $i,j \in e \in \pi$, and $j = \{\min(\ell) \in e : i < \ell\}$. Graphically, the components of π corresponds to the connected sub B-diagrams of b_π.

Example 45 Consider in Figure 10, the graphical representation of

$$b_{\{\{1,3\},\{2\},\{4,7,8\},\{5,6\}\}} = (8, [1,1,1,1,1,1,1], [\![1,8]\!], [\![1,8]\!], \{(1,3),(4,7),(7,8),(5,6)\}).$$

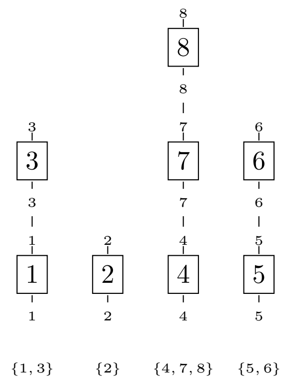


Figure 10: Graphical representation of $b_{\{\{1,3\},\{2\},\{4,7,8\},\{5,6\}\}}$

Obviously, the space $\mathcal{B}_1^1$ generated by the elements b_π is stable by product and Δ sends $\mathcal{B}_1^1$ to $\mathcal{B}_1^1 \otimes \mathcal{B}_1^1$. So, $\mathcal{B}_1^1$ is a sub Hopf algebra of $\mathcal{B}$. More precisely, one checks

$$b_\pi \star b_{\pi'} = \sum_{\pi \uplus \pi' \leq \pi'' \leq \{\{1,\dots,n\},\{n+1,\dots,n'+1\}\}} b_{\pi''}, \tag{34}$$

if $\pi \vDash n$ and $\pi' \vDash n'$, and

$$\Delta(b_\pi) = \sum_{\pi = e \uplus f} b_{\mathrm{std}(e)} \otimes b_{\mathrm{std}(f)}. \tag{35}$$

Remarking that π is indivisible if and only if the B-diagram b_π is indivisible, we deduce that $\mathcal{B}_1^1$ is the free subalgebra of $\mathcal{B}$ generated by the set $\{b_\pi : \pi \text{ is an indivisible set partition}\}$. Hence, comparing (34) to (32) and (35) to (33), Proposition 44 implies the following result.

Theorem 46 *The subalgebra $\mathcal{B}_1^1$ generated by $\{b_\pi : \pi$ is an indivisible set partition$\}$ is a Hopf algebra isomorphic to* WSym*.*

Example 47 For instance, compare

$$\begin{aligned} M_{\{\{1,3\},\{2\}\}} M_{\{\{1\},\{2\}\}} &= M_{\{\{1,3\},\{2\},\{4\},\{5\}\}} + M_{\{\{1,3\},\{2,4\},\{5\}\}} + M_{\{\{1,3,4\},\{2\},\{5\}\}} + M_{\{\{1,3\},\{2,5\},\{4\}\}} \\ &+ M_{\{\{1,3,5\},\{2\},\{4\}\}} + M_{\{\{1,3,4\},\{2,5\}\}} + M_{\{\{1,3,5\},\{2,4\}\}}, \end{aligned}$$

to the product pictured in Figure 11.

5.2 Bi-words symmetric functions

In this section, we consider the Hopf algebra BWSym of set partitions into lists defined in [10, 1]. A set partition into lists is a set of lists $\Pi = \{[i_1^1, \dots, i_{\ell_1}^1], \dots, [i_1^k, \dots, i_{\ell_k}^k]\}$ satisfying $k \geq 0$, $\ell_j \leq 1$ for

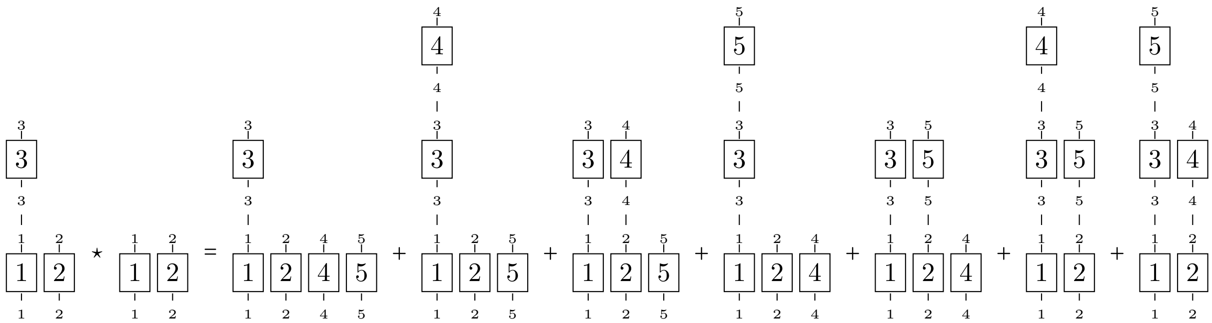


Figure 11: An example of product in $\mathcal{B}_1^1$

each $1 \le j \le k$, the integers $i_1^1, \dots, i_{\ell_1}^1, \dots, i_1^k, \dots, i_{\ell_k}^k$ are distinct, and $[\![1, n]\!] = \{i_t^j : 1 \le j \le k, 1 \le t \le \ell_j\}$ for a certain $n \in \mathbb{N}$; we let $\Pi \vDash n$ denote this property. As for the set partitions we define $\Pi \uplus \Pi' = \Pi \cup \{[i_1 + n, \dots, i_k + n] : [i_1, \dots, i_k] \in \Pi'\}$ for $\Pi \vDash n$ and we will say that Π is *indivisible* if $\Pi = \Pi' \uplus \Pi''$ for some set partitions in lists Π', Π'' implies either $\Pi' = \Pi$ or $\Pi'' = \Pi$.

The underlying space of BWSym is freely generated by the set $\{\Phi^\Pi : \Pi \vDash n, n \ge 0\}$. Whence endowed with the product

$$\Phi^\Pi \Phi^{\Pi'} = \Phi^{\Pi \uplus \Pi'} \tag{36}$$

and the coproduct

$$\Delta(\Phi^\Pi) = \sum_{\Pi = E \uplus F} \Phi^{\mathrm{std}(E)} \otimes \Phi^{\mathrm{std}(F)}, \tag{37}$$

with $\mathrm{std}(E) = \{[\phi(i_1), \dots, \phi(i_k)] : [i_1, \dots, i_k] \in e\}$ where ϕ is the unique increasing bijection from $\bigcup_{[\alpha_1,\dots,\alpha_p] \in E} \{\alpha_1, \dots, \alpha_p\}$ to $[\![1, \sum_{[\alpha_1,\dots,\alpha_p] \in E} p]\!]$, BWSym is a Hopf algebra.

We need the following property which is straightforward from (36).

Proposition 48 BWSym *is freely generated as an algebra by the set* $\{\Phi^\Pi : \Pi$ *is indivisible*$\}$.

Let $\mathcal{B}_1^2$ be the subspace of $\mathcal{B}$ generated by the set $\mathcal{G}_1^2$ of B-diagrams under the form

$$(n, [2, \dots, 2], [\![1, 2n]\!], \{1, 3, 5, \dots, 2n - 1\}, E).$$

Remarking that $\mathcal{G}_1^2$ is exactly the set of the B-diagrams whose each vertex has two outer non cut half edges, one inner non cut half edge and one inner cut half edge, the space $\mathcal{B}_1^2$ is stable for the product $\star$ and Δ sends $\mathcal{B}_1^2$ to $\mathcal{B}_1^2 \otimes \mathcal{B}_1^2$. In other words, $\mathcal{B}_1^2$ is a sub Hopf algebra of $\mathcal{B}$.

Let us describe a graded one-to-one correspondence between the set partitions into lists and the B-diagrams of $\mathcal{G}_1^2$. First we associate to each permutation $\sigma \in \mathfrak{S}_n$ a connected B-diagram $m_\sigma = (n, [2, \dots, 2], [\![1, 2n]\!], \{1, 3, 5, \dots, 2n - 1\}, E_\sigma)$ where $(2i - 1, 2j - 1) \in E_\sigma$ if and only if $i < j$ and $j = \min\{k : \alpha_i < \sigma^{-1}(k) < \sigma^{-1}(i)\}$ where $\alpha_i = \sup\{\sigma^{-1}(k) : k < i, \sigma^{-1}(k) < \sigma^{-1}(i)\}$ and $(2i, 2j - 1) \in E_\sigma$ if and only if $i < j$ and $j = \min\{k : \sigma^{-1}(i) < \sigma^{-1}(k) < \beta_i\}$ where $\beta_i = \inf\{\sigma^{-1}(k) : k < i, \sigma^{-1}(k) > \sigma^{-1}(i)\}$.

Example 49 Consider the permutation $\sigma = [5, 2, 4, 1, 3, 7, 6]$. We have $\alpha_1 = \alpha_2 = \alpha_5 = -\infty$, $\alpha_3 = 4$, $\alpha_4 = 2$, $\alpha_6 = \alpha_7 = 5$, $\beta_1 = \beta_3 = \beta_6 = +\infty$, $\beta_2 = \beta_4 = 4$, $\beta_5 = 2$, and $\beta_7 = 7$. Hence,

$$m_\sigma = (7, [2, 2, 2, 2, 2, 2, 2], [\![1, 14]\!], \{1, 3, 5, 7, 9, 11, 13\}, \{(1, 3), (2, 5), (3, 9), (4, 7), (6, 11), (11, 13)\}).$$

See Figure 12 for a graphical representation.

Noticing that any connected B-diagram G in $\mathcal{G}_1^2$ satisfies that $h_f^\uparrow(G) = |G| + 1$, we construct $n + 1$ different B-diagrams by adding one vertex to G. A quick reasoning by induction on the number of

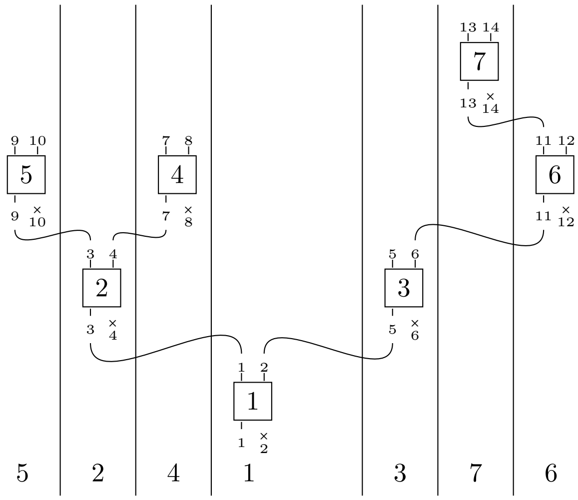


Figure 12: The B-diagram $m_{[5,2,4,1,3,7,6]}$

vertices shows that the set of the connected B-diagram in $\mathcal{G}_1^2$ with n vertices is exactly the set of the B-diagrams m_σ for $\sigma \in \mathfrak{S}_n$ and the correspondence $\sigma \longrightarrow m_\sigma$ is one to one.
We extend the construction to the set partitions into lists as follows. To each integer sequence $I = [i_1, \dots, i_n]$ of n distinct integers, we associate the permutation $\mathrm{std}(I)$ obtained by replacing each j_k by k in I for each k where $j_1 < \dots < j_n$ and $\{j_1, \dots, j_n\} = \{i_1, \dots, i_n\}$. To each set partition into lists $\Pi = \{L_1, \dots, L_k\}$, we associate the unique B-diagram m_Π in $\mathcal{G}_1^2$ with k connected components obtained by replacing the integers ℓ in each $m_{\mathrm{std}(L_t)}$ by j_ℓ^t in L_t where $L_t = [i_1^t, \dots, i_{n_t}^t]$, $j_1^t < \dots < j_{n_t}^t$ and $\{i_1^t, \dots, i_{n_t}^t\} = \{j_1^t < \dots < j_{n_t}^t\}$. Hence, $\mathcal{G}_1^2 = \{m_\Pi : \Pi \text{ is a set partition into lists}\}$ and the correspondence $\Pi \longrightarrow m_\Pi$ is one to one. Furthermore, by construction, Π is indivisible if and only if m_Π is indivisible.

We deduce from the above discussion that the algebra $\mathcal{B}_1^2$ has the same graded dimension than BWSym and that it is freely generated as a an algebra by the set

$$\{m_\Pi : \Pi \text{ is an indivisible set partition into lists}\}.$$

Also, notice that (12) allows us to interpret the product $m_\Pi \star m_{\Pi'}$ in terms of set partitions in lists. Let $\Pi = [L_1, \dots, L_k] \vDash n$ and $\Pi' = [L'_1, \dots, L'_{k'}] \vDash n'$ be two set partitions into lists. One has

$$m_\Pi \star m_{\Pi'} = m_{\Pi \uplus \Pi'} + \sum m_{\Pi''} \tag{38}$$

where the sum is over the set partitions in list $\Pi'' = [L''_1, \dots, L''_{k''}] \vDash n + n'$ such that each L''_i is

1. either a list L_j,
2. or a list $[j_1 + n, \dots, j_\ell + n]$ with $[j_1, \dots, j_\ell] \in \Pi'$,
3. or a list $[i_1, \dots, i_{p_1}, j_1^1 + n, \dots, j_{\ell_1}^1 + n, i_{p_1} + 1, \dots, i_{p_2}, j_1^2 + n, \dots, j_{\ell_2}^2 + n, \dots, i_{p_{t-1}} + 1, \dots, i_{p_t}, j_1^t + n, \dots, j_{\ell_t}^1 + n, i_{p_t} + 1, \dots, i_{p_{t+1}}]$ where $t > 1$, $p_1 < p_2 < \dots < p_{t+1}$, $[i_1, \dots, i_{p_{t+1}}] \in \Pi$, and for each $1 \le s \le t$, $[j_1^s, \dots, j_{\ell_s}^s] \in \Pi''$.

Example 50 For instance, compare

$$\begin{aligned} m_{\{[3,1],[2]\}} m_{\{[1,2]\}} &= m_{\{[3,1],[2],[4,5]\}} + m_{\{[3,1],[2,4,5]\}} + m_{\{[3,1],[4,5,2]\}} \\ &\quad + m_{\{[3,1,4,5],[2]\}} + m_{\{[3,4,5,1],[2]\}} + m_{\{[4,5,3,1],[2]\}}. \end{aligned}$$

to the product pictured in Figure 13.

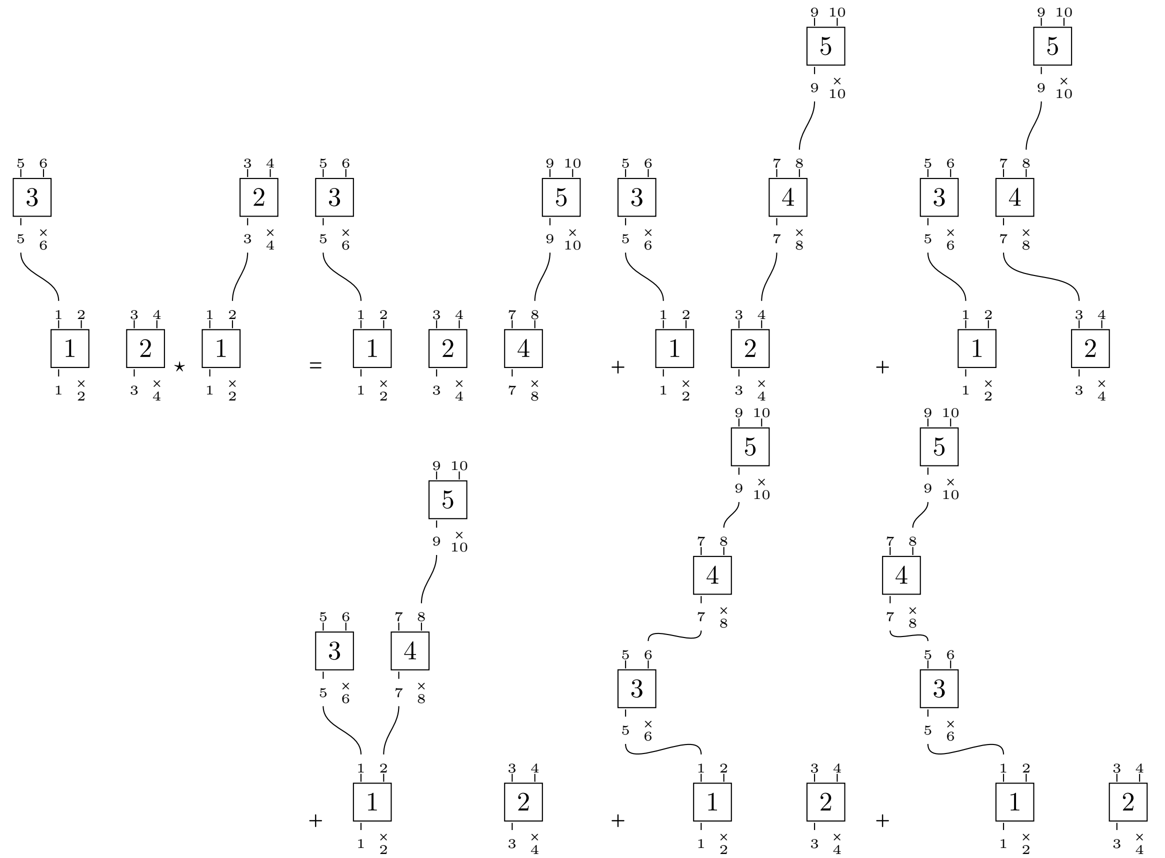


Figure 13: An example of product in $\mathcal{B}_1^2$

We deduce the following result.

Theorem 51 *The Hopf algebras* BWSym *and* $\mathcal{B}_1^2$ *are isomorphic.*

Proof. From the above discussion and Proposition 48, the algebras BWSym and $\mathcal{B}_1^2$ are isomorphic. An explicit isomorphism η sends Φ^Π to m_Π for each nonsplitable set partition into lists Π. It remains to prove that it is a morphism of cogebras. Remarking that the lists of a set partition into lists Π correspond to the connected components of m_Π, we show that equality (29) implies

$$(\eta\otimes\eta)(\Delta(\Phi^\Pi)) = (\eta\otimes\eta)(\Phi^\Pi\otimes\epsilon+\epsilon\otimes\Phi^\Pi) = m_\Pi\otimes\epsilon+\epsilon\otimes m_\Pi = \Delta(m_\Pi) \qquad (39)$$

for each nonsplitable partition Π. So since $\eta\otimes\eta : \mathrm{BWSym}\otimes\mathrm{BWSym}\longrightarrow\mathcal{B}_1^2\otimes\mathcal{B}_1^2$ is a morphism of algebras, the equality $(\eta\otimes\eta)(\Delta(\Phi^{\Pi_1}\cdots\Phi^{\Pi_k})) = \Delta(m_{\Pi_1}\star\cdots\star m_{\Pi_k})$ holds for any k-tuples $(\Pi_1,\dots,\Pi_k)$ of nonsplitable set partitions into lists. This proves that η is a morphism of bigebra and implies our statement. □

6 Conclusion

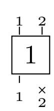
In this paper, we have described a combinatorial Hopf algebra which gives a diagrammatic representation of the calculations involved in the normal boson ordering. This algebra is rather closed to the one proposed by Blasiak *et al.*, but there are many differences which can be exploited to understand soundly the combinatorial aspects of these computations. First, the underlying objects are slightly different. Our objects are graphs with labeled vertices whilst those of Blasiak *et al.* are unlabeled; furthermore our graph have vertices which have the same number of inner edges and outer edges and each edge is either valid or stump. So our algebra is bigger. The algebra of Blasiak *et al.* specializes to the enveloping algebra of the Heisenberg Lie algebra which is described in terms of quotient as follows: $\mathcal{U}(\mathcal{L}_{\mathcal{H}}) \sim \mathbb{C}\langle \mathtt{a}^\dagger, \mathtt{a}, \mathtt{e}' \rangle / _{\mathcal{J}'}$ where $\mathcal{J}'$ is the ideal generated by the three polynomials $[\mathtt{a}^\dagger, \mathtt{a}] - \mathtt{e}'$, $[\mathtt{a}^\dagger, \mathtt{e}']$, and $[\mathtt{a}, \mathtt{e}']$. The role of the letter $\mathtt{e}'$ consists in collecting the statistic of the number of edges denoted by $|\Gamma_0|$ in [9] and by $\tau(G)$ in our paper. Consider the algebra $\mathcal{H}'$ obtained by adding a central element $\mathtt{e}$ to $\mathcal{U}(\mathcal{L}_{\mathcal{H}})$. This algebra allows us to take into account both the statistics $h_c(G)$ and $\tau(G)$. Indeed, it suffices to consider the morphism of algebra $\mathfrak{p}'_{\mathcal{H}}$ sending each $\mathtt{a}_{\rangle\binom{i_1}{j_1}\cdots\binom{i_k}{j_k}}$ to $\mathtt{a}^\dagger \mathtt{e}'^{k-1}$, each element of $\mathcal{A}_{\times\langle}$ to $\mathtt{a}$, and each element of $\mathcal{A}_{\langle\ \rangle}$ to $\mathtt{e}$. Remarking that $\sum_{p\in\mathrm{Paths}(G)} (\ell(\mathrm{seq}(p)) - 1) = \tau(G)$ where $\ell(s)$ denotes the length of the sequence s, we obtain

$$\mathfrak{p}'_{\mathcal{H}}(\mathtt{w}(G)) = \left(\mathtt{a}^\dagger\right)^{h_f^{\downarrow}(G)} \mathtt{a}^{h_f^{\uparrow}(G)} \mathtt{e}^{h_c(G)} \mathtt{e}'^{\tau(G)}. \tag{40}$$

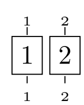
Now, the multiplication formula reads

$$\left(\mathtt{a}^\dagger\right)^m \mathtt{a}^n \mathtt{e}^q \mathtt{e}'^v . \left(\mathtt{a}^\dagger\right)^r \mathtt{a}^s \mathtt{e}^t \mathtt{e}'^w = \sum_{i=0}^{\min\{n,r\}} i! \binom{q}{i} \binom{r}{i} \left(\mathtt{a}^\dagger\right)^{m+r-i} \mathtt{a}^{n+s-i} \mathtt{e}^{q+t} \mathtt{e}'^{v+w+i}. \tag{41}$$

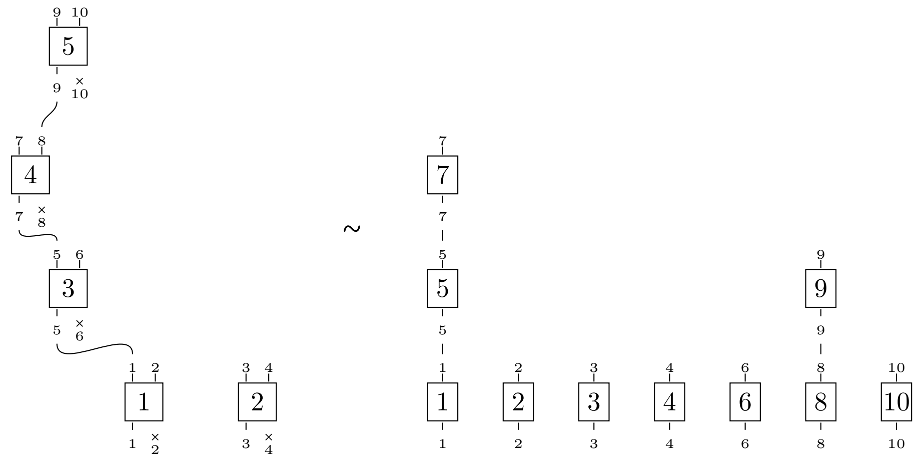
Section 5 gave two examples of subalgebras which are related to combinatorial objects. These construction can be generalized to a family of Hopf combinatorial algebras generated by colored partitions [1]. Some combinatorial properties of these objects can be deduced from simple manipulations. For instance, the number of set partitions into lists of n is equal to the number of set partitions of $2n$ such that each part contains at most one even number and, in that case, this number is the minimum of the part. This comes from the morphism $\mathcal{B}_1^2 \longrightarrow \mathcal{B}_1^1$ sending the element



to



together with the interpretations in terms of set partitions and set partitions into lists described in section 5. For instance, the explicit isomorphism sends $\{[4,5,3,1],[2]\}$ to $\{\{1,5,7\},\{2\},\{3\},\{4\},\{6\},\{8,9\},\{10\}\}$. This comes from the correspondence



Applying this strategy to various subalgebras, we find interpretations of some generalizations of Bell polynomials, like r-Bell [19] or $(r_1,\dots,r_p)$-Bell [20], in terms of B-diagrams. All these investigations are relegated to a forthcoming paper.

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