PATTERN AVOIDING PERMUTATIONS AND INDEPENDENT SETS IN GRAPHS

CHRISTIAN BEAN*, MURRAY TANNOCK*, AND HENNING ULFARSSON*

Abstract. We encode certain pattern avoiding permutations as weighted independent sets in a family of graphs we call cores. For the classical case of 132-avoiding permutations we establish a bijection between the vertices of the cores and edges in a fully connected graph drawn on a convex polygon. We prove that independent sets in the core correspond to non-crossing subgraphs on the polygon, and then the well-known enumeration of these subgraphs transfers to an enumeration of 132-avoiding permutations according to left-to-right minima. We extend our results to the 123-, $(1324, 2143)$-, $(1234, 1324, 2143)$-, $(1234, 1324, 1432, 3214)$-avoiding permutations. We end by enumerating certain subsets of 1324-avoiding permutations that satisfy particular conditions on their left-to-right minima and right-to-left maxima.

Keywords: permutation patterns, non-crossing subgraphs, independent sets

1. Introduction

Recall that the *standardisation* of a string s of distinct integers is the unique permutation $\mathrm{st}(s)$ obtained by replacing the ith smallest entry of s with i. A permutation π of length n contains another permutation p of length k if there is a subsequence (not necessarily consisting of consecutive entries) $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ whose standardisation is p. In this context p is called a (*classical permutation*) *pattern*, and we say that π *contains* p. The subsequence in π is called an *occurrence* of p. If no occurrence exists then π *avoids* p. Take for example $\pi = 51324$, which contains $p = 123$ as the subsequences 134 and 124, but avoids $p = 231$.

Given a pattern p we define $\mathrm{Av}_n(p)$ as the set of permutations of length n that avoid p, and $\mathrm{Av}(p) = \cup_{n\geq 0} \mathrm{Av}_n(p)$. For a set P of patterns we also let $\mathrm{Av}_n(P) = \cap_{p\in P} \mathrm{Av}_n(p)$, and $\mathrm{Av}(P) = \cup_{n\geq 0} \mathrm{Av}_n(P)$. A

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permutation class is any set of permutations defined by the avoidance of a set of classical patterns.

We illustrate a connection between independent sets arising from the study of permutations avoiding the pattern 132 and the classical problem of enumerating non-crossing subgraphs in a complete graph drawn on a regular polygon. The methods developed can be extended to enumerate subclasses of permutations avoiding the pattern 1324, the only principal permutation class of length 4 that remains unenumerated.

2. Encoding permutations as weights on a grid

A letter $\pi(i)$ in a permutation π is called a *left-to-right minimum* if $\pi(j) > \pi(i)$ for all $j = 1, \ldots, i-1$. Note that $\pi(1)$ is always a left-to-right minimum in a non-empty permutation. *Left-to-right maxima*, *right-to-left-minima* and *right-to-left-maxima* are defined analogously.

The sequence of left-to-right minima of a permutation π will be called the *lrm-boundary* of the permutation and denoted $\text{lrm}(\pi)$, e.g. if $\pi = 845367912$ then $\text{lrm}(\pi) = 8431$. Given any permutation we can arrange the lrm-boundary on a NW-SE-diagonal and insert the remaining points of the permutation in a *staircase* (*grid*), B_a, where a is the number of lrms, above this diagonal. See Figure 1 for an example with the permutation 845367912.

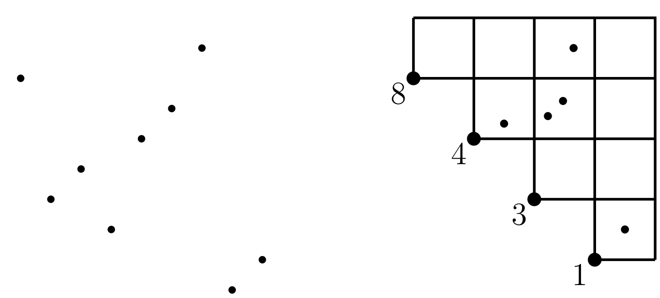


Figure 1. The permutation 845367912 drawn on the staircase grid, B_4. The left-to-right minima, 8, 4, 3 and 1, are drawn on a diagonal

We can give a coarser representation of this permutation by just writing how many points are in each box of the grid, see Figure 2. From now on we will call this the (*staircase*) *encoding* of the permutation. Note that it is not necessary to label the left-to-right minima, their values can be inferred from the number of points in each box.

It is important to notice that there are other permutations (of length 8) that have the encoding in Figure 2, e.g., 846379512. However, if

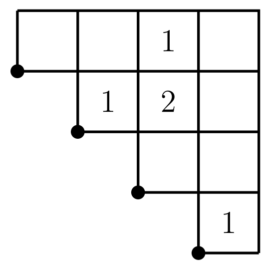


FIGURE 2. The permutation 845367912 encoded as the number of points in each box in the staircase grid

we restrict the permutations we encode to those avoiding the pattern 132 the encoding becomes unique. This is the subject of Section 3. The same is true for permutations avoiding 123 and we cover this in Section 4. In Section 5 we define a different encoding of permutations and use it to enumerate subclasses of Av(1324).

3. Permutations avoiding 132

If π is a permutation that avoids the pattern 132, such as the one in Figure 1, and we draw it on a staircase grid, notice that any row of the grid must contain an increasing sequence of numbers, e.g. the second row contains the increasing sequence 567. Moreover, notice that every rectangular region of boxes is also increasing. This implies the following remark.

Remark 3.1. The staircase encoding of a permutation avoiding 132 is unique.

We note that every staircase encoding uniquely determines a permutation in the set of permutations avoiding the bivincular patterns (see Bousquet-Mélou et al. [3]) $(132, \{2\}, \{\})$, $(132, \{\}, \{2\})$). The permutations avoiding 132 are a subset of this set.

Furthermore, for permutations avoiding 132, the presence of points in a box may constrain other boxes to be empty. Consider for example the encoding in Figure 2: The fact that the third box in row 1 is occupied constrains the rightmost boxes in rows 2 and 3 to be empty. These constraints are mutual. To capture these constraints we create a graph by placing a vertex for every box and an edge between boxes that exclude one another.

Definition 3.2. Let $n \geq 0$ be an integer. The 132-*core of size* n is the labelled, undirected graph D_n with vertex set $\{(ij) \colon i = 1, \ldots, n,\ j = i, \ldots, n\}$ and edges between (ij) and $(k\ell)$ if

- $k \in \{i+1, \ldots, j\}$ and $\ell \in \{j+1, \ldots, n\}$, or symmetrically
- $i \in \{k+1, \ldots, \ell\}$ and $j \in \{\ell+1, \ldots, n\}$.

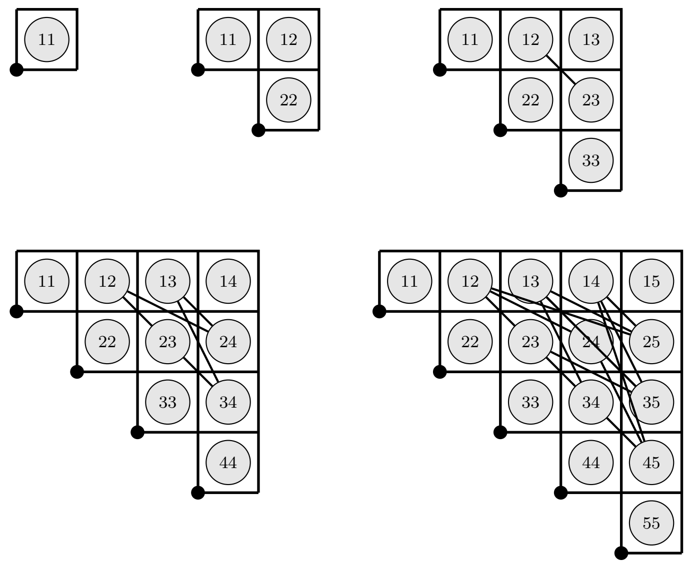


FIGURE 3. The 132-cores of sizes $1, \ldots, 5$

Remark 3.3. By construction, an independent set in D_n will correspond to a set of boxes in the staircase grid, B_n, that do no exclude one another.

Note that the 132-core is empty if $n = 0$. See Figure 3 for the 132-cores of sizes $n = 1, \ldots, 5$. We use the letter "D" in anticipation of Definition 5.1. Until we define another type of core in Section 4 we will refer to 132-cores as *cores*.

From the definition it is easy to see that D_n has $1 + 2 + \cdots + n = \binom{n+1}{2}$ vertices. To compute the number of edges consider a vertex $(1j)$ in the first row of D_n. It will have edges going to the southeast into the rectangular region $[2, \ldots, j] \times [j+1, \ldots, n]$, giving a total of $(j-1)(n-j) = (n+1)j - j^2 - n$ edges. The first row therefore has

$$(n+1)\binom{n+1}{2} - \frac{2n+1}{3}\binom{n+1}{2} - n^2 = \frac{n+2}{3}\binom{n+1}{2} - n^2 = \binom{n}{3}$$

edges. This can also be seen directly: The edge from $(1j)$ to $(k\ell)$ corresponds to the three element subset $\{k-1, j, \ell\}$ of $\{1, \ldots, n\}$. If we just consider edges going to the southeast then row i in the core of size n will look precisely like row 1 in a smaller instance of size $n - i + 1$. The total number of edges in D_n is therefore $\binom{n+1}{4}$. The enumerations of vertices and edges in the cores are special cases of the following result.

Proposition 3.4. *The number of cliques of size k in D_n is $\binom{n+1}{2k}$.*

Proof. The case of $k = 1$ (vertices) and $k = 2$ are given above. A clique of size $k > 2$ in D_n either has:

- no vertex in the first row of the core, or
- has exactly one vertex in the first row.

In the first case we have $\binom{n}{2k}$ cliques of size k, by induction. In the second case, assume the vertex in the first row is $(1j)$. The remaining vertices in the clique must then form a clique of size $k-1$ in the rectangular region $[2, \ldots, j] \times [j+1, \ldots, n]$. But such a clique is obtained by choosing $k-1$ rows and $k-1$ columns. This can be done in $\binom{j-1}{k-1}\binom{n-j}{k-1}$ ways. Thus the number of cliques is

$$\binom{n}{2k} + \sum_{j=k}^{n-k+1} \binom{j-1}{k-1}\binom{n-j}{k-1}.$$

The sum is easily seen to be $\binom{n}{2k-1}$ which completes the proof. □

As we will see below the number of independent sets in the core is more relevant for our purposes. In fact we will prove (in two different ways) the following theorem:

Theorem 3.5. *The number of independent sets of size k in the* 132-*core of size n is given by the coefficient of $x^n y^k$ in the generating function $F = F(x, y)$ satisfying the functional equation*

$$F = 1 + x \cdot F + \frac{xy \cdot F^2}{1 - y \cdot (F - 1)}.$$

This coefficient is

$$I(n, k) = \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{k-j}\binom{n}{j+1}\binom{n-1+j}{n-1}.$$

As noted in Remark 3.1 the encoding of a 132-avoiding permutation is sufficient to uniquely identify the permutation. Furthermore, if the permutation has n left-to-right minima, Remark 3.3 points out that the encoding will be a weighted independent set in D_n.

For a concrete example, consider the permutation 845367912 from Figure 1. We can encode it as the independent set $\{(13), (22), (23), (44)\}$ in D_4 along with the sequence of positive integer weights $(1, 1, 2, 1)$ recording the number of points at each vertex in the independent set.

Setting $y = y/(1-y)$, $x = xy$ in the generating function in Theorem 3.5, and collecting by powers of y gives a generating function

$$\begin{aligned}
&\frac{(2-xy)y + (1-y)(1-xy)}{2y} \\
(1)\qquad &-\frac{\sqrt{((xy)^2 - 4xy)y^2 + ((xy)^2 - 3xy)y(1-y) + (xy-1)^2(1-y)^2}}{2y}
\end{aligned}$$

for the well-known Narayana numbers [13, A001263] enumerating the permutations avoiding 132 by their number of left-to-right minima. These numbers are usually arranged in a triangle, whose n-th row (starting from 0) contains the number of 132-avoiding permutations of length n with k left-to-right minima. The triangle is often called a *Catalan-triangle* since the row sums are Catalan numbers, see Table 1.

1											
1											
1	1										
1	3	1									
1	6	6	1								
1	10	20	10	1							
1	15	50	50	15	1						
1	21	105	175	105	21	1					
1	28	196	490	490	196	28	1				
1	36	336	1176	1764	1176	336	36	1			
1	45	540	2520	5292	5292	2520	540	45	1		
1	55	825	4950	13860	19404	13860	4950	825	55	1	

TABLE 1. The Catalan triangle of Narayana numbers [13, A001263]

Setting $x = 1$ in (1), and thus forgetting the information about the left-to-right minima, gives the usual generating function of the Catalan numbers.

Instead of enumerating $\operatorname{Av}(132)$ by the number of left-to-right minima we can enumerate them by size of the independent set used to create the permutation.

Proposition 3.6. *The number of permutations avoiding* 132 *of length* ℓ *using an independent set of size* k *is*

$$\sum_{n=0}^{\ell} I(n,k)\binom{\ell-n-1}{k-1}$$

where $I(n,k)$ *is defined in Theorem 3.5.*

Proof. The sum is taken over the number of left-to-right minima and the binomial coefficient counts the number of partitions of the remaining points $n-\ell$ into k parts, corresponding to the vertices of the independent set. □

The proposition gives a new Catalan triangle, shown in Table 2, which has been added to the Online Encyclopedia of Integer Sequences [13, A262370].

1									
1									
1	1								
1	4								
1	10	3							
1	20	20	1						
1	35	77	19						
1	56	224	139	9					
1	84	546	656	141	2				
1	120	1176	2375	1104	86				
1	165	2310	7172	5937	1181	30			
1	220	4224	18953	24959	9594	830	5		
1	286	7293	45188	87893	56358	10613	380		
1	364	12012	99242	270452	264012	88472	8240	105	
1	455	19019	203775	747877	1044085	554395	100339	4480	14

TABLE 2. The Catalan triangle given by Proposition 3.6

In the triangle, the rightmost numbers in rows $2, 5, 8, 11, 14$ form the sequence

$$1, 1, 2, 5, 14.$$

Also, the rightmost numbers in rows $1, 4, 7, 10, 13$ form the sequence

$$1, 3, 9, 30, 105.$$

This leads to the following conjecture, which has been verified up to permutations of length 24.

Conjecture 3.7. *(1) The rightmost numbers in rows numbered* $2+3i$, $i = 0, 1\ldots$ *are the Catalan numbers.*

(2) The rightmost numbers in rows numbered $1+3i$, $i = 0, 1\ldots$ *are the central elements of the* $(1, 2)$*-Pascal triangle* [13, A029651].

We prove Theorem 3.5 in two ways: by connecting the independent sets in a core with non-crossing subgraphs of a complete graph drawn on a regular polygon; and by a direct analysis of the core. The second method will generalise to the sets $\mathrm{Av}(1324, 2143)$, and $\mathrm{Av}(1234, 1324, 2143)$.

3.1. **Proof of Theorem 3.5 using non-crossing subgraphs.** The idea is to establish a bijection ϕ_n from the vertex set D_n to the edge set of a complete graph drawn on a regular $(n+1)$-gon, satisfying the property:

$\star$ The vertices u, v in the core share an edge if and only if the edges $\phi_n(u)$, $\phi_n(v)$ cross in the complete graph.

Thus an independent set in the core will correspond to a non-crossing subgraph in the polygon.

Definition 3.8. Let $n \geq 1$ be an integer. The *polygon of size* n, K_n is the complete graph on n vertices drawn equally spaced on a circle. We label the vertices clockwise $1, \ldots, n$. The edge from i to j is denoted $e_{i,j}$.

Note that the polygon of size $n = 1$ is a single vertex. See Figure 4 for the polygons of sizes $n = 2, \ldots, 6$.

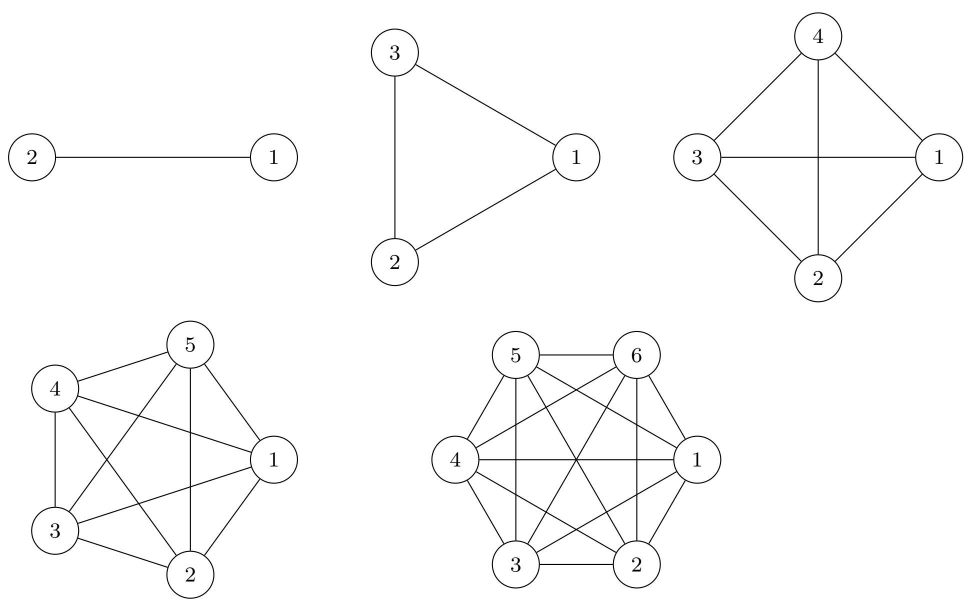


FIGURE 4. The polygons of sizes $2, \ldots, 6$

From the definition it is clear that K_n has $\binom{n}{2}$ edges. Note that edges $e_{i,j}$ and $e_{k,\ell}$ (assuming $i < k$) in the polygon cross if and only if $j \geq k$.

The following theorem shows that the bijection $\phi_n : (ij) \mapsto e_{i,j+1}$ has property $(\star)$ above.

Theorem 3.9. *There is an edge in the core D_n between vertices (ij) and $(k\ell)$ $(i < k)$ if and only if in the polygon K_{n+1} the edges $e_{i,j+1}$ and $e_{k,\ell+1}$ cross.*

Proof. The base cases are clear. Assume this is true for some $n \geq 1$ and consider D_{n+1}, the core of size $n+1$. This core can be obtained from D_n by appending a new column on the right, containing the vertices $1(n+1), 2(n+1), \ldots, (n+1)(n+1)$, and the appropriate edges. This corresponds to adding a new vertex labelled $n+2$ to the polygon of size $n+1$, to obtain the polygon of size $n+2$. By the inductive hypothesis we have that all edges in the core D_{n+1}, not incident to a vertex in the last column, correspond in the claimed manner with crossings in the polygon. To complete the proof we need to show that the same is true for the remaining edges.

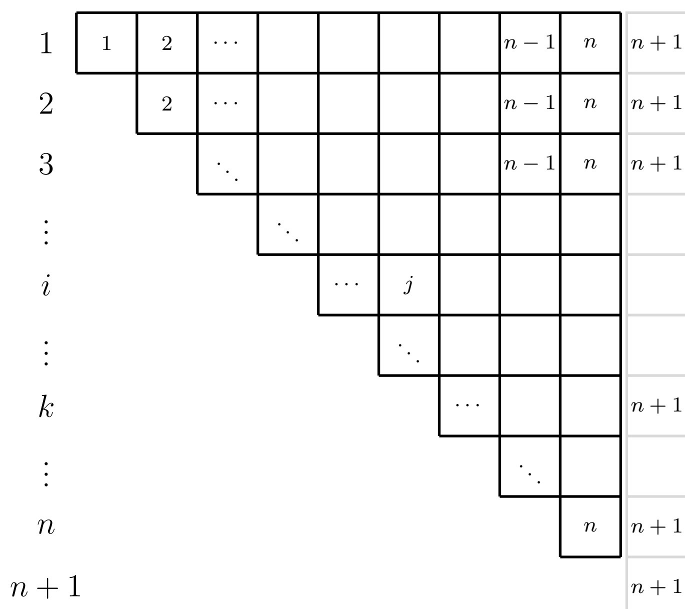


FIGURE 5. Visualising the proof of Theorem 3.9: Adding a column to the grid

Therefore take a vertex (ij), with $j < n+1$ in D_{n+1} and a vertex $(k(n+1))$ in the last column of D_{n+1}. From the definition of the core we see that there is an edge between these two vertices if and only if $i < k$ and $k \leq j (\leq n+1)$. This is equivalent to the edges $e_{i,(j+1)}$ and $e_{k,(n+2)}$ crossing in the polygon. $\square$

Enumerating non-crossing subgraphs is a classical problem, see e.g. Comtet [5]. For our purposes it is easy to apply the method in Section 2.1 of Flajolet and Noy [6] to arrive at the generating function equation for

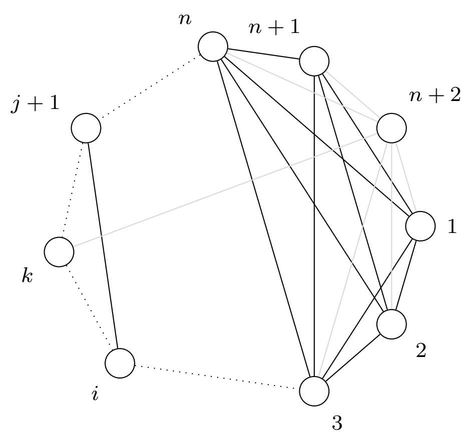


FIGURE 6. Visualising the proof of Theorem 3.9: Adding a vertex to the polygon

$F = F(x, y)$

$$F = 1 + xF + \frac{xyF^2}{1 - y(F - 1)}.$$

where the coefficient of $x^n y^k$ counts the number of non-crossing subgraphs with k edges in the polygon of size $n+1$. By Theorem 3.9 this is the same as the number of independent sets of size k in the core of size n. This proves the first part of Theorem 3.5. For the second part see Theorem 2 (ii) in Flajolet and Noy [6].

3.2. **Proof of Theorem 3.5: Finding $F(x, y)$ directly.** We now show how one can find the generating function $F(x, y)$ directly. This method will later be applied to other sets of permutations.

Consider an arbitrary core. Notice that any independent set can contain any number vertices of the form $1j$. This is equivalent to choosing boxes in the top row of the staircase grid and we call the number of vertices chosen the *degree*.

Let $F = F(x, y)$ be the generating function in which the coefficient of $x^n y^k$ counts the number of independent sets of size k in D_n. If the degree of a core is zero then the induced subgraph of the remaining vertices is isomorphic to some smaller core. If the degree, d, is greater than zero then the independent set contains the vertices $\{1x_1, 1x_2, \dots, 1x_d\}$ (with $x_i < x_j$ if $i < j$). The independent set must therefore not have any of vertices ℓm such that $\ell \in \{2, \dots, x_i\}$ and $m \in \{x_i + 1, \dots, n\}$ for any x_i. This can be visualised on the staircase grid, see Figure 7. We can then partition the remaining vertices into $d + 1$ sets,

$\{\ell m : 1 < \ell \leq x_1, \ell \leq m \leq x_1\}$, $\{\ell m : x_i < \ell \leq x_{i+1}, \ell \leq m \leq x_{i+1}\}$ for $i = 1, \ldots d-1$, and $\{\ell m : x_d < \ell \leq n, \ell \leq m \leq n\}$. Notice that the induced subgraphs on these sets of vertices are isomorphic to some core. Moreover, notice that the first two sets may be empty, but the rest contain at least one vertex. Therefore we obtain the following

$$F = 1 + xF + xyF^2 + \cdots + xy^n F^2 (F-1)^{n-1} + \ldots$$
$$= 1 + xF + \frac{xyF^2}{1 - y(F-1)},$$

where 1 is added for the core of size 0 as it has no top row, and so no degree.

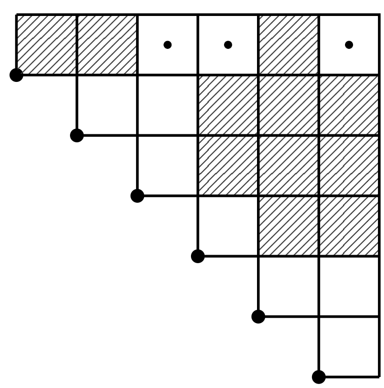


FIGURE 7. A staircase grid with induced shadings

4. PERMUTATIONS AVOIDING 123

Consider drawing a permutation that avoids 123 on a staircase grid. Take for example the permutation 639871542 shown in Figure 8. Notice

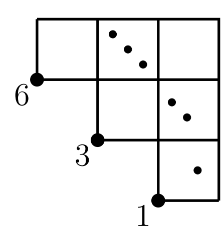


FIGURE 8. The permutation 639871542 drawn on a staircase grid. The left-to-right minima, 6, 3 and 1, are drawn on a diagonal

that now every rectangular region of boxes is decreasing, instead of increasing in the case of the 132-avoiding permutations, and the existence of points in a box produces constraints on boxes southwest and northeast of it. We are naturally lead to the following definition:

Definition 4.1. Let $n \geq 0$ be an integer. The 123-*core of size* n is the labelled, undirected graph U_n with vertex set $\{(ij)\colon i = 1, \ldots, n,\ j = i, \ldots, n\}$ and edges between (ij) and $(k\ell)$ if

- $i \in \{k+1, \ldots, n\}$ and $j \in \{\ell+1, \ldots, n\}$.

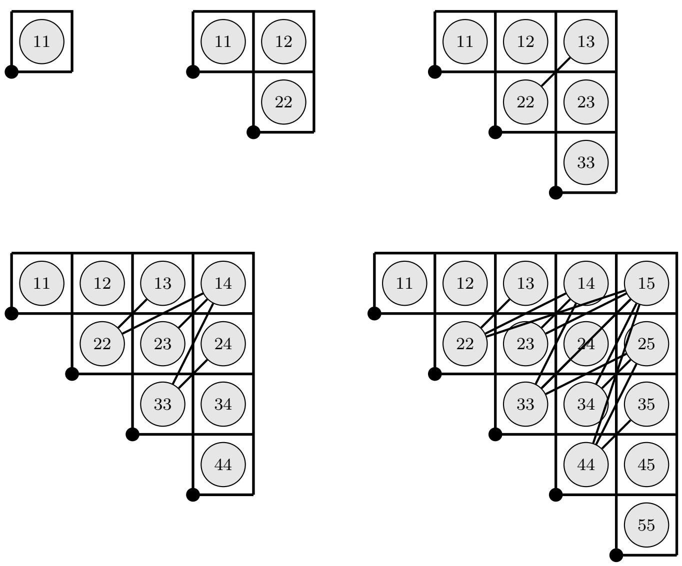


FIGURE 9. The 123-cores of sizes $1, \ldots, 5$

We use the letter "U" in anticipation of Definition 5.1. Note that U_n is isomorphic to D_n (the 132-core of size n) when $n = 0, 1, 2, 3$. However, D_4 is a 5-cycle, whereas U_4 is not. In general:

Proposition 4.2. *For $n \geq 4$, D_n is not isomorphic to U_n.*

Proof. In D_n the vertex $(1n)$ has degree $\frac{n(n-1)}{2}$ whereas in U_n the largest possible degree is $\lfloor \frac{n-1}{2} \rfloor \cdot \lceil \frac{n-1}{2} \rceil$. This can be seen by noting that the degree of the vertex (ij) in D_n is $(j-i)(i-1) + (n-j)(j-i) = (j-i)(n-1-(j-i))$. When n is odd this is maximised when $j-i = \frac{n-1}{2}$, while in the case where n is even when $j-i = \lfloor \frac{n-1}{2} \rfloor$ or $j-i \lceil \frac{n-1}{2} \rceil$. $\square$

This result implies in particular that we can not hope to establish a bijection from the vertices in the 123-core to the edges of the polygon like above.[1] See Figure 9 for the 123-cores of sizes $n = 1, \ldots, 5$.

Even though the 123-cores are not isomorphic to the 132-cores (for size greater than 3) they have the same number of cliques of each size:

Proposition 4.3. *The number of cliques of size k in U_n is $\binom{n+1}{2k}$.*

This can be proved with the same method as Proposition 3.4 or the following lemma, which shows that the two types of cores are glued together from isomorphic subgraphs:

Lemma 4.4. *(1) Let $n \geq 0$ be an integer. For any $1 \leq i \leq n$ let D_n^i be the subgraph of D_n consisting of the vertices, and edges, in the rectangular region $[i, \ldots, n] \times [1, \ldots, i]$. Define the subgraph U_n^i of U_n analogously. Then reflecting in a vertical axes, or more precisely*

$$\rho_i \colon D_n^i \to U_n^i, \quad (k\ell) \mapsto ((n+i-k)\ell)$$

is an isomorphism of graphs.

(2) Given a non-empty subset $S \subseteq \{1, \ldots, n\}$ the restriction of the isomorphism $\rho_{\max S}$, gives an isomorphism

$$\bigcap_{i \in S} D_n^i \to \bigcap_{i \in S} U_n^i.$$

This is clear from the definition of the cores D_n and U_n. See Figure 10 and Figure 11

Proof of Proposition 4.3. Let $f_k(G)$ count the number of k-cliques for a given graph G. Every k-clique in the core U_n appears in some U_n^i, but potentially multiple. Similarly k-cliques appear in some D_n^i. Let $I = \{1, \ldots, n\}$, then the number of k-cliques in U_n is given by

$$f_k(U_n) = f_k\left(\bigcup_{i \in I} U_n^i\right)$$

and similarly for D_n

$$f_k(D_n) = f_k\left(\bigcup_{i \in I} D_n^i\right).$$

For any $S \subseteq I$, define

$$U_n^S = \bigcap_{i \in S} U_n^i \text{ and } D_n^S = \bigcap_{i \in S} D_n^i$$

[1]See Section 6 for a common framework for 132- and 123-cores in terms of non-crossing and non-nesting partitions.

Then by the principle of inclusion and exclusion and Lemma 4.4(2) we get that

$$f_k(U_n) = \sum_{S \subseteq I} (-1)^{|I|} f_k(U_n^S) = \sum_{S \subseteq I} (-1)^{|I|} f_k(D_n^S) = f_k(D_n). \qquad \square$$

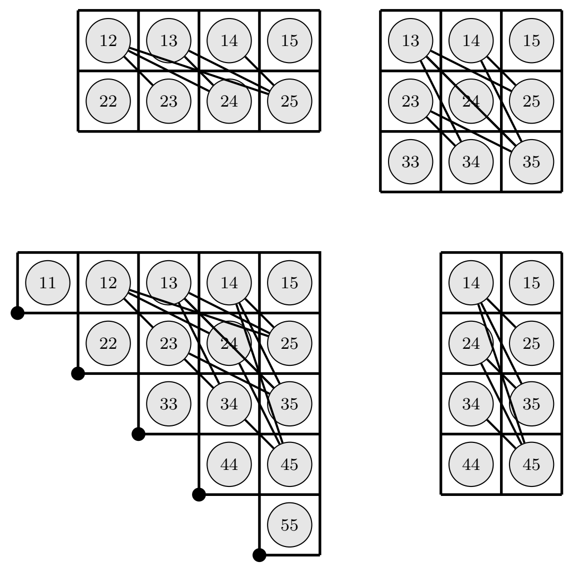


FIGURE 10. The 132-core of size 5 with the subgraphs D_5^2, D_5^3 and D_5^4

It turns out that the number of independent sets of each size is also the same:

Theorem 4.5. *The number of independent sets of size k in the* 123-*core of size n is given by the coefficient of $x^n y^k$ in the generating function $F = F(x, y)$ satisfying the functional equation*

$$F = 1 + xF + \frac{xyF^2}{1 - y(F-1)}.$$

This coefficient is

$$\frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{k-j} \binom{n}{j+1} \binom{n-1+j}{n-1}.$$

We leave it to the reader to prove this with the method used to prove Theorem 3.5 in Subsection 3.2, with the modification that we consider the number of non-empty boxes on the diagonal instead of the top row, see Figure 12.

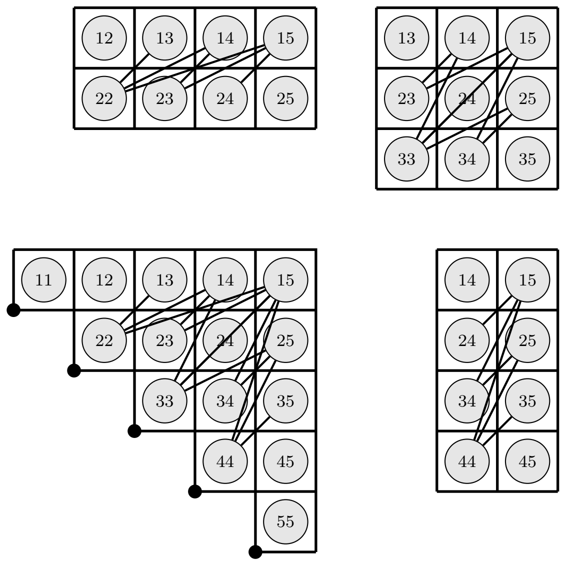


FIGURE 11. The 123-core of size 5 with the subgraphs U_5^2, U_5^3 and U_5^4

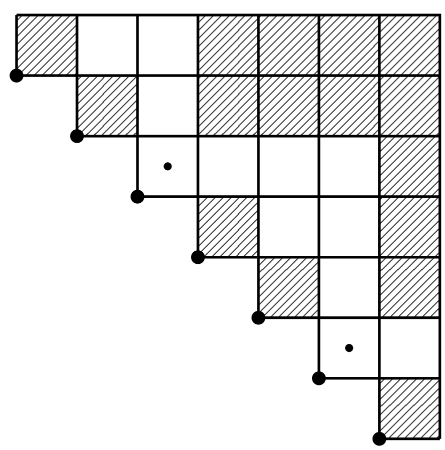


FIGURE 12. Illustration of how to prove Theorem 4.5 using a modification of the method in the proof of Theorem 3.5 in Subsection 3.2

Theorem 4.5 implies that the generating function in Equation (1), the enumeration in Proposition 3.6, and the Catalan triangles in Tables 1 and 2 also apply to permutations avoiding 123.

5. Permutations avoiding 1234 or 1324, or both

We next turn our attention to the permutation classes Av(1234), Av(1324) and their intersection Av(1234, 1324). For these classes of permutations the staircase encoding we defined above is no longer unique, the simplest example being the permutations 123 and 132 which belong to all three classes and have the same staircase encoding. To remedy this we also consider the right-to-left maxima (rlm) of a permutation, e.g. if $\pi = 213679845$ the lrm-sequence is 21 and the rlm-sequence is 985 and we can draw the *boundary grid* shown on the left in Figure 13. We define the (*boundary*) *encoding* of a permutation by writing the number of points in each box of its boundary grid; see the right part of Figure 13.

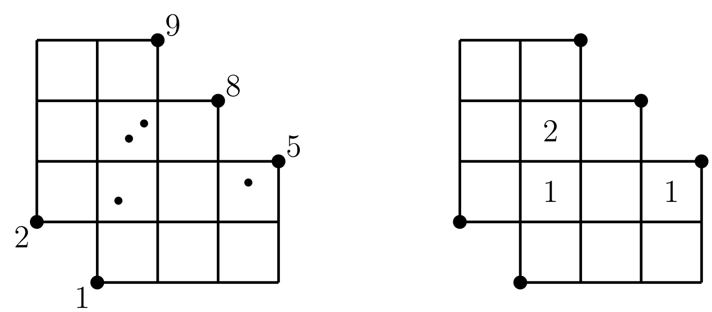


Figure 13. On the left the permutation 213679845 is drawn on a boundary grid highlighting its lrm's and rlm's. On the right we have the boundary encoding of the same permutation

More generally, given a permutation π we define its *boundary*, $\partial(\pi)$, as the standardisation of the subsequence of π containing the left-to-right minima and the right-to-left maxima. For example if $\pi = 213679845$, as in Figure 13, then $\partial(\pi) = 21543$. By construction $\partial(\pi)$ avoids 123, and every permutation that avoids 123 is its own boundary.

Given a permutation π that avoids 123 we define its *boundary grid*, $\mathrm{bg}(\pi)$, as the set of 1-by-1 boxes whose corners have integer coordinates, with the requirement that the lower left corner of each box is north-west of a left-to-right minimum, and the upper right corner of each box is south-east of a right-to-left maximum. For example, the boundary grid of 21543 is the boundary grid in Figure 13.

We now generalise the construction of the 132- and 123-cores above:

Definition 5.1. A *boundary grid* is a collection of 1-by-1 boxes whose corners have integer coordinates. Given such a boundary grid B we define the

(1) *up-core* as the graph $U(B)$ whose vertices are the boxes in the grid and with edges between boxes (ij), $(k\ell)$ if $i < k$, $j < \ell$ and the rectangle $\{i, i+1, \dots, k\} \times \{j, j+1, \dots, \ell\}$ is a subset of B.
(2) *down-core* $D(B)$ whose vertices are the boxes in the grid B and with edges between boxes (ij), $(k\ell)$, if $i < k$, $\ell < j$ and the rectangle $\{i, i+1, \dots, k\} \times \{\ell, \ell+1, \dots, j\}$ is a subset of B.
(3) *updown-core* $UD(B)$ whose vertices are the boxes in the grid B and with edges between boxes (ij), $(k\ell)$, if $i \leq k$, $\ell \leq j$ (with at least one of the inequalities strict) and the rectangle $\{i, i+1, \dots, k\} \times \{\ell, \ell+1, \dots, j\}$ is a subset of B.

We note that the 132-core is the down-core for the staircase grid while the 123-core is the up-core for the same boundary grid. A permutation is *skew-decomposable* if it can be written as a non-trivial skew-sum. If a permutation π avoids 123 and has length greater than 1 then it is skew-indecomposable if and only if $UD(\mathrm{bg}(\pi))$ is connected and every left-to-right minimum is attached to some 1-by-1 box in $\mathrm{bg}(\pi)$. By convention the empty permutation and 1 are skew-decomposable.

Remark 5.2. Every permutation avoiding 123 can be represented by a unique skew-sum of skew-indecomposable permutations.

Given a boundary grid B we let $\mathrm{uperms}(B)$ be the set of permutations that can be obtained by choosing a weighted independent set in the up-core $U(B)$ and inflating the vertices of the independent set into a decreasing sequence of points, whose length is determined by the weight of the vertex. Likewise, we define $\mathrm{dperms}(B)$ using the down-core, by inflating weights into increasing sequences; $\mathrm{udperms}(B)$ using the updown-core, by inflating an (unweighted) independent set into a single point for each vertex. Given these definitions it is clear that

$$\mathrm{Av}(1324) = \bigsqcup_{\pi \in \mathrm{Av}(123)} \mathrm{dperms}(\mathrm{bg}(\pi)),$$

$$\mathrm{Av}(1234) = \bigsqcup_{\pi \in \mathrm{Av}(123)} \mathrm{uperms}(\mathrm{bg}(\pi)),$$

and

$$\mathrm{Av}(1234, 1324) = \bigsqcup_{\pi \in \mathrm{Av}(123)} \mathrm{udperms}(\mathrm{bg}(\pi)).$$

Our main focus will be on subclasses of the first and third permutation classes in the equations above, as their enumerations are unknown. We start by considering the subclass Av(1324, 2143) of Av(1324).

5.1. **Down-cores and smooth permutations.** The smooth permutations are those that correspond to smooth Schubert varieties. Sandhya and Lakshmibai [9] showed that these permutations are the class Av(1324, 2143). The enumeration was first done in the unpublished preprint Haiman [8] and is given by the generating function

$$\frac{1-5x+3x^2+x^2\sqrt{1-4x}}{1-6x+8x^2-4x^3}. \tag{2}$$

Bousqet-Mélou and Butler [2] provided an independent proof of this generating function, and Slofstra and Richmond [10] have found the enumeration of smooth Schubert varieties of all classical finite types, the case of permutations being type A. Below we will rederive the generating function (2) while also keeping track of the number of boundary points and size of the independent set.

Our approach uses the equation

$$\mathrm{Av}(1324, 2143) = \bigsqcup_{\pi \in \mathrm{Av}(123,2143)} \mathrm{dperms}(\mathrm{bg}(\pi)). \tag{3}$$

It is well-known that the enumeration of the boundaries Av(123, 2143) is given by the alternate Fibonacci numbers [13, A001519]. The following lemma gives a structural description of the boundaries.

Lemma 5.3. *A boundary grid of a skew-indecomposable permutation in* Av(123, 2143) *(of length at least two) is a sequence of staircase grids, whose every other member has been reflected in a diagonal, sharing their north-westernmost box with the south-easternmost box of the next grid; see Figures 14 and 15.*

Proof. Consider the staircase grid of a permutation $\pi \in \mathrm{Av}(123, 2143)$. Any rectangular region in the grid must be decreasing as π avoids 123. Moreover any rectangular region not including a box from the leading diagonal can contain at most one point in order for π to avoid 2143. If the grid is skew-indecomposable then we must have a point not in the leading diagonal appearing north-east of each left-to-right minimum. This forces the structure described. □

Let EB_a be the boundary grid obtained by doubling the right-most column in the staircase grid B_a, and let ED_a be the down-core of this grid.[2]

[2]"E" is for extended.

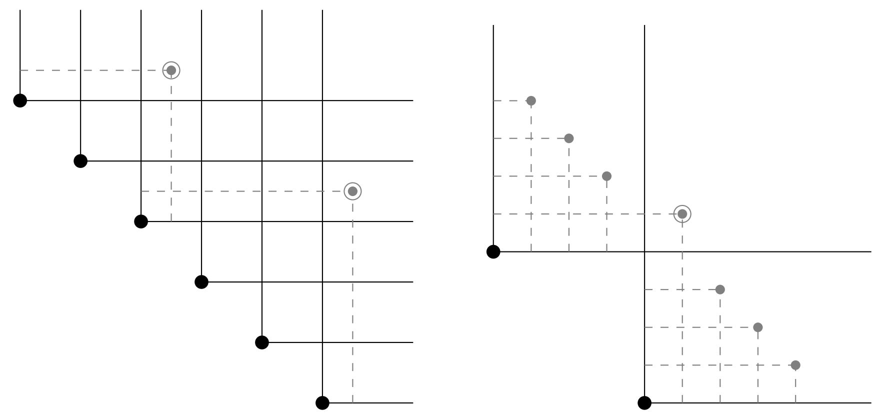


FIGURE 14. Two skew-indecomposable boundaries in Av(123, 2143) drawn on the staircase diagram

To prove Equation (2) for the enumeration of smooth permutations we need the following two lemmas:

Lemma 5.4. *Let $G = G(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in ED_a. Then*

$$G = \frac{F - 1}{(1 + y)x},$$

where $F = F(x, y)$ is the generating function in Theorem 3.5.

Proof. Notice that if we add a box to the bottom of the last column of EB_a we obtain B_{a+1}. Therefore $F = (1 + y)xG + 1$. Solving for G gives the claimed equation. □

Remark 5.5. Let B be a boundary grid, then if B' is the grid obtained by reflecting along the line $y = x$ then $D(B)$, $D(B')$ are isomorphic graphs and so are $U(B)$, $U(B')$.

Lemma 5.6. *Let $P_{ind} = P_{ind}(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in down-cores on boundaries of skew-indecomposable permutations from* $\mathrm{Av}_a(123, 2143)$. *Then*

$$P_{ind} = \frac{x(F - 1)}{2 - G} + x,$$

where $F = F(x, y)$ is the generating function in Theorem 3.5, and $G = G(x, y)$ is the generating function in Lemma 5.4. Setting $y = x/(1-x)$ gives the generating function for skew indecomposable permutations in Av(1324, 2143).

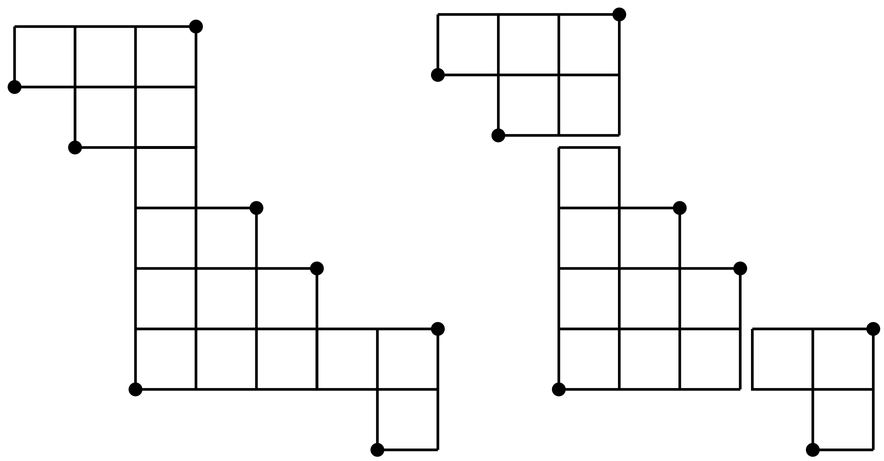


FIGURE 15. A skew-indecomposable boundary in $\mathrm{Av}(123, 2143)$

Proof. Consider the structure given in Lemma 5.3. We first show how to build the grids with at least one box. The rightmost grid can be seen as some non-empty B_a or as some non-empty B_a reflected along the line $y = x$. We then build the rest by alternating between a non-empty EB_a and a non-empty EB_a reflected along the line $y = x$, see Figure 15. This can be seen as a skew-sum of non-empty EB_a's and so by Remark 5.5 we can use the generating function G as in Lemma 5.4. We always first place a B_a since if we first place a box, this together with the first reflected ED_a will form a reflected B_a. This leads to the generating function

$$P_{ind} = \frac{x(F-1)}{1-(G-1)} + x$$

where we add x for the empty grid representing the permutation 1. □

Proposition 5.7. *Let $P = P(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in down-cores on boundaries given by a permutation in $\mathrm{Av}_a(123, 2143)$. Then*

$$P = \frac{1}{1-P_{ind}}$$

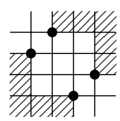
By setting $y = x/(1-x)$ we recover the generating function (2) for the number of $(1324, 2143)$-avoiding permutations.

Proof. Taking the skew sum of skew-indecomposable permutations in the set $\mathrm{Av}(1324, 2143)$ gives us the entire set $\mathrm{Av}(1324, 2143)$. □

5.2. **Replacing 2143 with an arbitrary 123-avoiding pattern.** We can replace the pattern 2143 in Equation (3) with any pattern (or set of patterns), p, avoiding 123 and ask whether our methods extend to enumerate the set

$$\bigsqcup_{\pi \in \mathrm{Av}(123,p)} \mathrm{dperms}(\mathrm{bg}(\pi)).$$

First of all one would hope that this set equals $\mathrm{Av}(1324, p)$ as in the case of $p = 2143$ but this is not true in general, e.g., if $p = 3412$, then

$$\mathrm{Av}\left(1324, \right.$$

$$\left.\right) = \bigsqcup_{\pi \in \mathrm{Av}(123,3412)} \mathrm{dperms}(\mathrm{bg}(\pi)).$$

In general, given a pattern p that avoids 123 we define $\mathrm{bdms}(p)$ as the set of mesh patterns, defined by Brändén and Claesson [4], all of whose underlying classical patterns are p and shaded in such a way that it forces every point to be either a left-to-right minimum or a right-to-left maximum.

There is one other case, with p a pattern of length 4, where the resulting set of permutations is a permutation class, besides $p = 2143$, and that is $p = 1432$, giving $\mathrm{Av}(1324, 1432)$, conjectured to be enumerated by a non-D-finite generating function, see [13, A165542] and Albert et. al [1, Section 6.2].

5.3. **Updown-cores and the class** $\mathrm{Av}(1234, 1324, 2143)$**.** The permutation class $\mathrm{Av}(1234, 1324)$ is conjectured to have a non-D-finite generating function by Albert et. al [1, Section 6.4]. In this section we show that the subclass that also avoids 2143 has the rational generating function

$$\frac{1 - 3x - 2x^3}{1 - 4x + 2x^2 - 2x^3 + x^4}. \tag{4}$$

As before we have the equation

$$\mathrm{Av}(1234, 1324, 2143) = \bigsqcup_{\pi \in \mathrm{Av}(123,2143)} \mathrm{udperms}(\mathrm{bg}(\pi)).$$

so Lemma 5.3 still describes the boundaries of these permutations.

Lemma 5.8. *Let $R = R(x, y)$ be be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in updown-cores of B_a. Then*

$$R = 1 + xR + \frac{xyR}{1 - x}.$$

Proof. Consider an independent set in $UD(B_a)$. It can contain at most one vertex from the top row. If it contains no vertex then the remaining vertices form a graph isomorphic to $UD(B_{a-1})$. Otherwise, we have one vertex in the top row. If the vertex is in the i^{th} box from the left then there will be i left-to-right minima to the left of it. The remaining vertices will form a graph isomorphic to some $UD(B_b)$ such that $b < a$. □

Lemma 5.9. *Let $Q_{ind} = Q_{ind}(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in updown-cores on boundaries of a skew-indecomposable permutations in* $\mathrm{Av}_a(123, 2143)$. *Then*

$$Q_{ind} = Q_{up} + Q_{down} + x + x^2 + x^2 y,$$

where $Q_{up} = Q_{up}(x, y)$, $Q_{down} = Q_{down}(x, y)$ satisfy the equations

$$Q_{up} = xR - x - x^2 - x^2 y + \frac{(xR - x)Q_{down}}{x} + \frac{x^2 y R(Q_{up} + Q_{down} + x^2 + x^2 y)}{1 - x},$$

$$Q_{down} = xR - x - x^2 - x^2 y + \frac{(xR - x)Q_{up}}{x} + \frac{x^2 y R(Q_{up} + Q_{down} + x^2 + x^2 y)}{1 - x},$$

and $R = R(x, y)$ is given in Lemma 5.8. Setting $y = x$ gives the generating function for skew indecomposable permutations in $\mathrm{Av}(1234, 1324, 2143)$.

Proof. Recall Lemma 5.3. We define Q_{up} to be the generating function with coefficient $x^a y^k$ whose rightmost part in its skew-sum is some B_b with $b > 1$ and Q_{down} to be the generating with coefficient $x^a y^k$ whose rightmost part in its skew-sum is some B_b reflected along the line $y = x$ with $b > 1$. Therefore if a grid has only one part in their skew-sum in either of these (recall Remark 5.2) it will be counted by $xR - x - x^2 - x^2 y$ where we subtract B_0 and B_1.

We will first count Q_{up} by choosing vertices for the independent set in the top row determined by the rightmost part of its skew-sum. This row has at most one vertex in the independent set. If it contains none then the rightmost boxes will be of the shape of some B_c such that $c > 0$ and the the lefthand boxes must be some grid defined by a skew-indecomposable permutation in $\mathrm{Av}(123, 2143)$ with a skew-sum starting with a B_b reflected along the line $y = x$ with $b > 1$. If it contains a vertex then the leftmost boxes will be equivalent to a B_c such that $c \geq 0$. The rightmost summand can be of the form of any grid defined by some skew-indecomposable permutation in $\mathrm{Av}(123, 2143)$. This leads to the generating function as above. Q_{down} is derived analogously. Therefore Q_{ind} is as claimed. □

Proposition 5.10. *Let $Q = Q(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in updown-cores on boundaries from* $\mathrm{Av}_a(123, 2143)$*. Then*

$$Q = \frac{1}{1 - Q_{ind}}$$

By setting $y = x$ we get the generating function (4) *for the number of* $(1234, 1324, 2143)$*-avoiding permutations.*

Proof. Taking the skew sum of skew-indecomposable permutations in the set $\mathrm{Av}(1234, 1324, 2143)$ gives us the entire set $\mathrm{Av}(1234, 1324, 2143)$. □

The generating function gives the enumeration

$$1, 1, 2, 6, 21, 75, 268, 958, 3425, 12245, 43778, 156514, 559565, \ldots$$

and has been added to the Online Encyclopedia of Integer Sequences [13, A263790].

5.4. **The class** $\mathrm{Av}(1234, 1324, 1432, 3214)$**.** Notice that when we require permutations in $\mathrm{Av}(1234, 1324)$ to avoid 2143 we are forcing the skew-indecompable permutations to have a boundary grid made out of B_a's joined by single boxes, see Figure 15. Here we replace avoidance of 2143 with avoidance of 1432 and 3214 which allows slightly larger joints. We show that this subclass has the rational generating function

$$\frac{1 - x - x^2 - x^3}{1 - 2x - x^2 - 2x^3 - 4x^4 - 8x^5 + 15x^7 + 14x^8 + 7x^9}. \tag{5}$$

We have that

$$\mathrm{Av}(1234, 1324, 1432, 3214) = \bigsqcup_{\pi \in \mathrm{Av}(123,1432,3214)} \mathrm{udperms}(\mathrm{bg}(\pi))$$

since in the updown cores there is at most one vertex in each row and column and if we have an occurrence of 1432 using this inside vertex we would also have an occurrence on the boundary by choosing the corresponding maxima. Similarly for 3214 we could choose a minima.

Lemma 5.11. *A boundary grid of a skew-indecomposable permutation in* $\mathrm{Av}(123, 1432, 3214) \setminus \{1, 12, 2143\}$ *is an alternating sequence of grids of the form* $\mathrm{bg}(132)$ *and* $\mathrm{bg}(213)$ *sharing their north-westernmost box with the south-easternmost box of the next grid.*

Proof. To the right and above any left-to-right minima there can be at most two right-to-left maxima as we are avoiding 1432. Similarly below and to left of any right-to-left maxima there can be at most two left-to-right minima. This forces the structure described. □

Lemma 5.12. *Let $S_{ind} = S_{ind}(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in updown-cores on boundaries of a skew-indecomposable permutations in* $\mathrm{Av}_a(123, 1432, 3214)$*. Then*

$$S_{ind} = S_{up} + S_{down} + x + x^2(1 + y) + x^4(1 + 7y + 7y^2),$$

where $S_{up} = S_{up}(x, y)$, $S_{down} = S_{down}(x, y)$ satisfy the equations

$$S_{up} = x^3(1 + 3y + y^2) + xS_{down} + xyS_{down} + x^2y(S_{up} + (1 + y)x^2),$$

$$S_{down} = x^3(1 + 3y + y^2) + xS_{up} + xyS_{up} + x^2y(S_{down} + (1 + y)x^2).$$

Setting $y = x$ gives the generating function for skew indecomposable permutations in $\mathrm{Av}(1234, 1324, 1432, 3214)$*.*

Proof. Recall Lemma 5.11. We define S_{up} to be the generating function for the skew-indecomposable grids whose rightmost grid is of the form bg(213) and S_{down} for the skew-indecomposable grids whose rightmost grid is of the form bg(132). Therefore if either have only one part it will be counted by $x^3(1 + 3y + y^2)$.

We first count S_{up} by choosing a vertex in the bottom row of the rightmost part of its skew sum. We can contain at most one of these two vertices in an independent set. If we choose neither or the right vertex, the rightmost boxes are defined by some skew-indecomposable permutation in $\mathrm{Av}(123, 1432, 3214)$ with a skew sum starting with a bg(132). If we choose the left vertex then the rightmost boxes are defined by some skew-indecomposable permutation in $\mathrm{Av}(123, 1432, 3214)$ with a skew sum starting with a bg(213) or is a single box. This leads the generating function above. S_{down} is derived analogously. Therefore S_{ind} is as claimed where we add in x for the single point, $x^2(1 + y)$ for bg(12) and $x^4(1 + 7y + 7y^2)$ for bg(2143). □

Proposition 5.13. *Let $S = S(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in updown-cores on boundaries from* $\mathrm{Av}_a(123, 1432, 3214)$*. Then*

$$S = \frac{1}{1 - S_{ind}}$$

By setting $y = x$ we get the generating function (5) *for the number of* $(1234, 1324, 1432, 3214)$*-avoiding permutations.*

Proof. Taking the skew sum of skew-indecomposable permutations in the set $\mathrm{Av}(1234, 1324, 1432, 3214)$ gives us the entire set $\mathrm{Av}(1234, 1324, 1432, 3214)$. □

The generating function gives the enumeration

$1, 1, 2, 6, 20, 62, 172, 471, 1337, 3846, 11030, 31442, 89470, 254934 \ldots$

and has been added to the Online Encyclopedia of Integer Sequences [13, A260696].

5.5. **Non-intersecting boundaries.** Having looked at the 2143-avoiding permutations inside the class Av(1324) we next turn our attention to permutations that are forced to contain 2143.

Definition 5.14. We say that a permutation has a *non-intersecting boundary of type* (a, b) if it has a left-to-right minima, b right-to-left maxima and these two sequences do not intersect, in the sense that the smallest lrm is to the left of the largest rlm, and the first lrm is smaller than the last rlm.

Note that our example in Figure 13 has a non-intersecting boundary of type $(2, 3)$.

These boundary points determine a *non-intersecting boundary grid of type* (a, b), denoted $B_{a,b}$. The down-core of this grid is

$$D_{a,b} = D(B_{a,b})$$

and will be called the *non-intersecting core of type* (a, b).

We use techniques analogous to those in earlier sections to find the generating function for 1324-avoiders with a non-intersecting boundary of type (a, b) where *either* a or b is at most 3. Note that the non-intersecting core of type $(a, 1)$ is precisely the same as the 132-core of size a. First we consider those permutations with non-intersecting boundary of type $(a, 2)$.

Definition 5.15. Let $EB_{a,b}$ be $B_{a,b}$ with the final column doubled. Let $ED_{a,b} = D(EB_{a,b})$.

Note that $EB_{a,1} = EB_a$, defined above Lemma 5.4.

Lemma 5.16. *Let $H = H(x, y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in $D_{a,2}$. Then*

$$H = \frac{G}{1 - y(G - 1)},$$

where $G = G(x, y)$ is the generating function in Lemma 5.4, for the down-core EB_a.

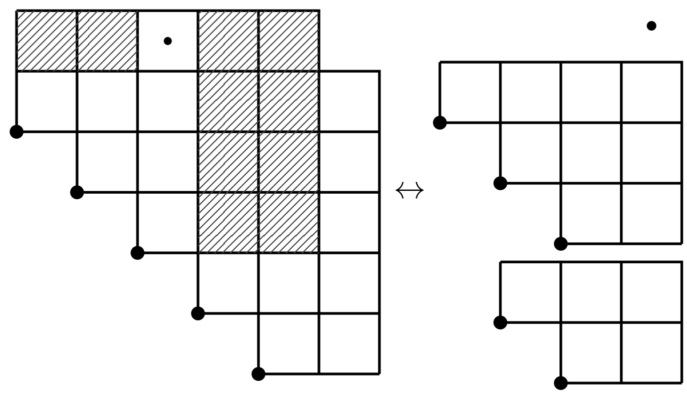


FIGURE 16. An illustration of the proof of Lemma 5.16 by selecting vertices in the top row of $D_{5,2}$

Proof. Consider the graph $D_{a,2}$. Every vertex in the top row can be chosen to be a member of an independent set with other vertices in the top row. Similar to before, selection of these vertices imposes restrictions. If we choose n vertices in the top row we see $n+1$ graphs equivalent to $ED_{c,1}$ (for some non-negative integer c). One of these graphs is empty if the rightmost vertex in the top row is chosen. Hence the generating function is given by

$$\begin{aligned} H &= G + yG(G-1) + y^2G(G-1)^2 + \cdots + y^nG(G-1)^n + \cdots \\ &= \frac{G}{1-y(G-1)}. \end{aligned}$$ □

Proposition 5.17. *The number of* 1324*-avoiding permutations of length* n *with a non-intersecting boundary of type* $(a,2)$*, for some integer* $a \geq 1$*, is given by the coefficient of* x^n *of the generating function*

$$x^2 H\left(x, \frac{x}{1-x}\right).$$

Proof. In order to generate the permutations with this boundary it is necessary to multiply by x^2 since we need to add in the two right-to-left maxima. □

This gives the enumeration

$$0, 0, 1, 1, 4, 14, 49, 174, 626, 2276, 8346, 30821, 114495, 427481, \ldots$$

of these permutations (starting from the empty permutation of length 0).

We next consider the case of three right-to-left maxima.

Lemma 5.18. *Let $I = I(x,y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in $ED_{a,2}$. Then*

$$I = H + \frac{yG(H-1)}{1-y(G-1)}$$

where $G = G(x,y)$ and $H = H(x,y)$ are the generating functions in Lemmas 5.4 and 5.16, respectively.

Proof. Consider the graph $ED_{a,2}$. Every vertex in the final column can be chosen to be a member of an independent set with any other vertices in the final column. Selecting the vertices in the final column places restrictions on the remainder of the grid. If the independent set contains no vertices from the final column, then we see a graph of the form $D_{a,2}$. If we chose in the final column then the graph breaks down into smaller graphs. The rightmost graph is equivalent to some $D_{c,2}$, all other graphs seen are equivalent to some $ED_{d,1}$ by Remark 5.5. If we chose n vertices in the final column, we see n such smaller graphs, with the exception that if we chose the uppermost vertex in the column, $n-1$ graphs are seen, this is equivalent to seeing an empty graph above this point. Therefore the generating function is given by

$$\begin{aligned} I &= H + y(H-1)G + y^2(H-1)G(G-1) + \cdots \\ &\qquad \cdots + y^n(H-1)G(G-1)^{n-1} + \cdots \\ &= H + \frac{yG(H-1)}{1-y(G-1)}. \end{aligned}$$

□

Lemma 5.19. *Let $J = J(x,y)$ be the generating function where the coefficient of $x^a y^k$ is the number of independent sets of size k in $D_{a,3}$. Then*

$$J = (1+y)I - y\frac{y(1+y)^2(G-1)I}{1-y(G-1)},$$

where $G = G(x,y)$ and $I = I(x,y)$ are the generating functions in

Proof. Consider the graph $D_{a,3}$. Every vertex in the second from top row can be chosen to be a member of an independent set with other vertices in the same row. We note that the rightmost box, b_r, (marked with $\circ$ in Figure 18) is independent of any other boxes in the grid and thus can be chosen, or not, independently of everything else. Similar to before selection of every other vertex imposes restrictions on the remainder of the grid. If we select no vertex to the left of b_r we get a graph of the form $ED_{a,2}$. Now we consider choosing at least one vertex from the second from top row excluding the rightmost box. In doing

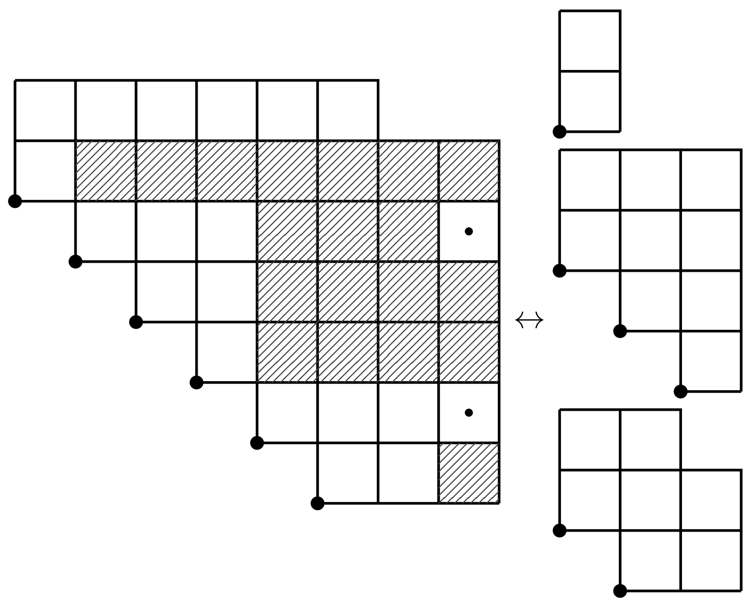


FIGURE 17. An illustration of the proof of Lemma 5.18 by selecting vertices in the final column of $ED_{6,2}$

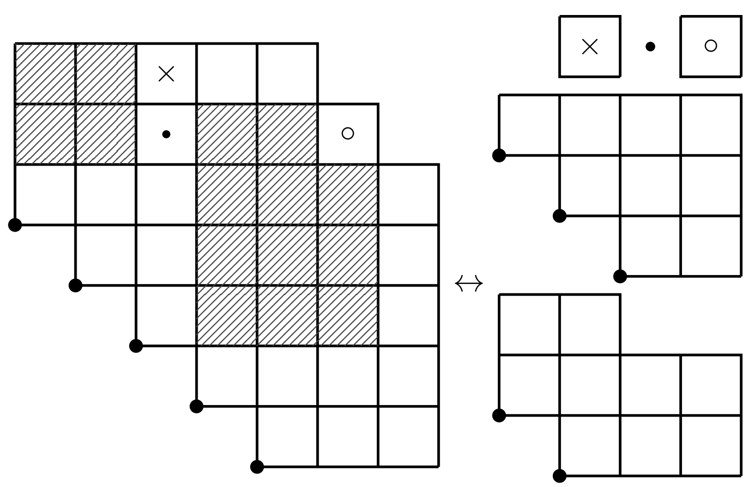


FIGURE 18. An illustration of the proof of Lemma 5.19 by selecting vertices in the second row of $D_{5,3}$

this we see a smaller $ED_{e,2}$ to the right of the rightmost vertex selected, a single independent box directly above the rightmost vertex (marked with $\times$), and smaller graphs of form $ED_{f,1}$ directly below each point. The $ED_{e,2}$ by the rightmost point can be empty when we choose the vertex second from right, but none of the $ED_{f,1}$ graphs can be empty.

Therefore the generating function is given by

$$J = \underbrace{(1+y)I - y}_{\text{Empty row}} + y(1+y)^2 I(G-1) + y^2(1+y)^2 I(G-1)^2 + \cdots$$
$$\cdots + y^n(1+y)^2 I(G-1)^n + \cdots$$
$$= (1+y)I - y + \frac{y(1+y)^2 I(G-1)}{1-y(G-1)}. \qquad \square$$

Proposition 5.20. *The number of* 1324*-avoiding permutations of length* n *with a non-intersecting boundary of type* $(a, 3)$*, for some integer* $a \geq 1$*, is given by the coefficient of* x^n *of the generating function*

$$x^3 J\left(x, \frac{x}{1-x}\right).$$

Proof. In order to generate the permutations with this boundary it is necessary to multiply by x^3 since we need to add in the three right-to-left maxima. $\square$

This gives the enumeration

$$0, 0, 0, 1, 1, 7, 33, 139, 566, 2279, 9132, 36488, 145500, 579318, \ldots$$

of these permutations (starting from the empty permutation of length 0).

One would hope that the method applied above would extend to arbitrary many right-to-left maxima, but we have not succeeded in generalising our methods.

6. Future work

6.1. **Non-crossing and non-nesting partitions.** As pointed out to us by Galashin [7], bump-diagrams, see e.g.Rubey and Stump [11], for non-crossing and non-nesting partitions provide an alternative framework for the independent sets in 132- and 123-cores: An independent set in a 132-core (123-core) corresponds to a non-crossing (non-nesting) partition; see Figure 19.

We note that the independent sets in the 132-core are the vertices of the non-crossing complex and the independent sets in the 123-core are the vertices of the non-nesting complex, see e.g. Santos et al. [12]. Both of these complexes are pure. From empirical testing it seems that the down-cores on boundaries avoiding 123 are pure if and only if the boundary avoids 2143. We record this as a conjecture:

Conjecture 6.1. *Let* π *be a* 123*-avoiding permutation. The independent set complex of the down-core* $D(\mathrm{bg}(\pi))$ *is pure if and only if* π *avoids* 2143.

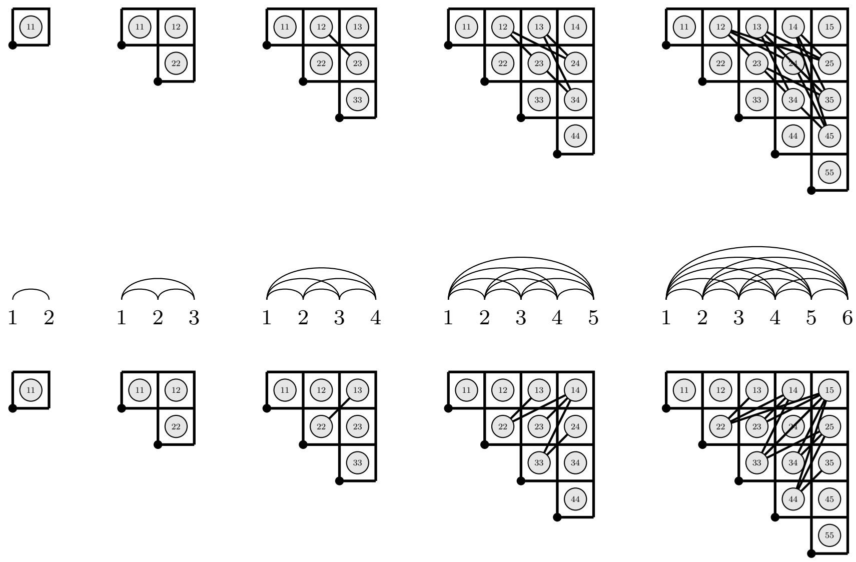


FIGURE 19. The first (last) line shows the 132-cores (123-cores) of sizes $1, \ldots, 5$. In between is the amalgamation of the bump-diagrams of all set partitions of $\{1, \ldots, n+1\}$, where n is the size of the corresponding core

6.2. **Vincular patterns.** If we consider the sets $\mathrm{Av}(1\underline{23}4)$ and $\mathrm{Av}(1\underline{32}4)$ with our methods, we see first of all that they have exactly the same boundary permutations and there is an obvious bijection, preserving the boundary (as well as the encoding), that reverses each column.

$$|\,\mathrm{Av}(1\underline{23}4)\,| = |\,\mathrm{Av}(1\underline{32}4)\,|.$$

A similar bijection can be obtained for the bivincular patterns where the 2 and the 3 in the patterns are also required to be consecutive values.

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