ON DEGREE SEQUENCES OF UNDIRECTED, DIRECTED, AND BIDIRECTED GRAPHS

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Abstract. Bidirected graphs generalize directed and undirected graphs in that edges are oriented locally at every node. The natural notion of the degree of a node that takes into account (local) orientations is that of net-degree. In this paper, we extend the following four topics from (un)directed graphs to bidirected graphs:

- Erdős–Gallai-type results: characterization of net-degree sequences,
- Havel–Hakimi-type results: complete sets of degree-preserving operations,
- Extremal degree sequences: characterization of uniquely realizable sequences, and
- Enumerative aspects: counting formulas for net-degree sequences.

To underline the similarities and differences to their (un)directed counterparts, we briefly survey the undirected setting and we give a thorough account for digraphs with an emphasis on the discrete geometry of degree sequences.

1. Introduction

Let $G = (V, E)$ be an undirected simple graph. The degree $\mathbf{d}(G)_v$ of a node $v \in V$ is the number of edges $e \in E$ incident to v. The **degree sequence** of G is the vector $\mathbf{d}(G) \in \mathbb{Z}^V$. Degree sequences of undirected graphs are basic combinatorial statistics that have received considerable attention (starting as early as 1874 with Cayley [4]) that find applications in many areas such as communication networks, structural reliability, and stereochemistry; c.f. [18]. A groundbreaking result, due to Erdős and Gallai [8], is a characterization of degree sequences among all sequences of nonnegative integers. In a complementary direction, Havel [10] and Hakimi [9] described a complete set of degree preserving operations on graphs. These two fundamental results have led to a thorough understanding of degree sequences. A geometric perspective on degree sequences, pioneered by Peled and Srinivasan [13], turns out to be particularly fruitful. Here, degree sequences of graphs on n labelled nodes are identified with points in $\mathbb{Z}^n$ and the collection of all such degree sequences can be studied by way of *polytopes of degree sequences*. Among the many results obtained from this approach, we mention the formulas for counting degree sequences due to Stanley [16] and the classification of uniquely realizable degree sequences by Koren [11]; see Section 2.

In this paper, we consider notions of degree sequences for other classes of graphs. A **directed graph** $D = (V, A)$ is a simple undirected graph together with an orientation of each edge, that is, every edge (or arc) is an ordered tuple (u, v) and is hence is oriented from u to v. The natural notion of degree that takes into account orientations is the **net-degree** of a node v

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defined by

$$\mathbf{d}(D)_v \ := \ |\{u \in V : (u,v) \in A\}| - |\{u \in V : (v,u) \in A\}|. \tag{1}$$

The net-degree is sometimes called the *imbalance*; see [12, 14].

Of particular interest in this paper will be the class of bidirected graphs. Bidirected graphs were introduced by Edmonds and Johnson [7] in 1970 and studied under the name of oriented signed graphs by Zaslavsky [20].

Definition 1.1. A **bidirected graph** or **bigraph**, for short, is a pair $B = (G, \tau)$ where $G = (V, E)$ is a simple undirected graph and τ assigns to every pair $(v, e) \in V \times E$ a local orientation $\tau(v, e) \in \{-1, 1\}$ if v is incident to e and $\tau(v, e) = 0$ otherwise.

We call $|B| := G$ the **underlying graph** of B and τ its **bidirection function**. The local orientation of an edge $e = uv$ can be interpreted such that e points locally towards v if $\tau(v, e) = 1$ and away from v otherwise. Graphically, we draw the edges with two arrows, one at each endpoint; see Figure 1. The notion of net-degree has a natural extension to bigraphs where the net-degree of a node v is given by $\mathbf{d}(B)_v = \sum_{e \in E} \tau(v, e)$.

Figure 1. A bigraph B on three nodes.

The objective of this paper is to extend the following four topics from (un)directed graphs to bigraphs:

- Erdős–Gallai-type results: Characterization of net-degree sequences.
- Havel–Hakimi-type results: Complete sets of degree-preserving operations.
- extremal degree sequences: Characterization of uniquely realizable sequences.
- enumerative aspects: Counting formulas for net-degree sequences.

Bigraphs simultaneously generalize undirected and directed graphs which makes the study of these four topics particularly interesting. As we will show in Section 4, the characterization of net-degree sequences is simpler for bigraphs as compared to graphs and digraphs but the degree-preserving operations are more involved and combine those for the undirected and directed case.

The paper is organized as follows. In *Section 2* we survey the results for undirected graphs. Our point of view will be more geometric by highlighting the polytope of degree sequences introduced by Stanley [16]. This point of view allows for a simple treatment of extremal degree sequences such as uniquely realizable and 'boundary' sequences, i.e., degree sequences that attain equality in the Erdős–Gallai conditions. In *Section 3* we give a coherent treatment of net-degree sequences for directed graphs. Again, our approach draws from the geometry of net-degree sequences. We introduce the classes of weakly-split digraphs and width-2 posets and show that they correspond to boundary and uniquely realizable digraphs, respectively. In particular, we give explicit formulas for the number of width-2 posets. *Section 4* is dedicated to bigraphs and we completely address the four topics above.

For sake of simplicity, we restrict to graphs without loops and parallel edges but all results have straightforward generalizations to q-multigraphs. Our aim was to give a self-contained treatment of results. For further background on graphs and polytopes we refer to the books of Diestel [6] and Ziegler [21], respectively.

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2. Undirected graphs

Erdős and Gallai [8] gave a complete characterization of degree sequences of simple graphs: A vector $\mathbf{d} = (d_1, ..., d_n) \in \mathbb{Z}^n_{\geq 0}$ is the degree sequence of an undirected graph G on vertices $1, \dots, n$ if and only if

$$d_1 + d_2 + \cdots + d_n \text{ is even, and} \tag{2}$$

$$\sum_{i \in S} d_i - \sum_{i \in T} d_i \ \leq |S|(n - |T| - 1) \text{ for all disjoint sets } S, T \subseteq [n]. \tag{3}$$

Geometrically, this result states that degree sequences of undirected graphs on node set $[n]$ are constrained to an intersection of finitely many affine halfspaces corresponding to (3), that is, a convex polyhedron. Peled and Srinivasan [13] pursued this geometric perspective and investigated the relation to the polytope obtained by taking the convex hull of all the finitely many degree sequences in $\mathbb{R}^n$. The drawback of their construction was that not all integer points in this polytope corresponded to degree sequences. This was rectified by Stanley [16] with the following modification. For a vector $\mathbf{d} \in \mathbb{Z}^n$, let us write $\widehat{\mathbf{d}} = (\mathbf{d}, \frac{1}{2}(d_1 + \cdots + d_n))$. The (extended) polytope of degree sequences is

$$\mathcal{O}_n \ := \ \operatorname{conv}\Big\{\widehat{\mathbf{d}}(G) : G = ([n], E) \text{ simple graph}\Big\} \subset \mathbb{R}^{n+1}.$$

Write $[p, q]$ for the line segment connecting the points $p, q \in \mathbb{R}^n$. A polytope $P \subset \mathbb{R}^n$ is a **zonotope** if it is the pointwise vector sum of finitely many line segments. Stanley's main insight was that $\mathcal{O}_n$ is in fact a zonotope. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$.

Lemma 2.1 ([16]). *For $n \geq 0$,*

$$\mathcal{O}_n \ = \ \sum_{1 \leq i < j \leq n} [0, \widehat{e}_i + \widehat{e}_j].$$

These particular zonotopes (and their associated hyperplane arrangements) occur in the study of root systems and signed graphs (see [19]) which yields that $\mathcal{O}_n$ is given by all points $p \in \mathbb{R}^n$ satisfying the Erdös-Gallai conditions. This line of thought can be completed to a proof of the Erdös–Gallai theorem by showing that for every point $\widehat{\mathbf{d}} \in \mathcal{O}_n \cap \mathbb{Z}^{n+1}$ there is a set $E \subset \binom{[n]}{2}$ such that

$$\widehat{\mathbf{d}} \ = \ \sum_{ij \in E} \widehat{e}_i + \widehat{e}_j,$$

which then shows that $\mathbf{d}$ is the degree sequence of $G = ([n], E)$.

This makes the treatment of degree sequences amendable to the geometric combinatorics of zonotopes. In particular, the number of degree sequences of simple graphs on n vertices is exactly $|\mathcal{O}_n \cap \mathbb{Z}^{n+1}|$. This also furnishes a generalization of the Erdős–Gallai conditions to multigraphs as obtained by Chungphaisan [5] by purely combinatorial means. From the geometric perspective, the generalization states that the lattice points in $q\mathcal{O}_n$ bijectively correspond to degree sequences of loopless q-multigraphs. For this, the key observation is that lattice zonotopes are *normal*, that is, every $\mathbf{d} \in q\mathcal{O}_n \cap \mathbb{Z}^{n+1}$ is the sum of q lattice points in $\mathcal{O}_n$.

Using the enumerative theory of lattice points in lattice polytopes, i.e. *Ehrhart theory* (see [2]), it is possible to get exact counting formulas. A graph $H = ([n], E)$ is a **quasi-forest** if every component of H has at most one cycle and the cycle is odd. Define

$$h(n,i) \ := \ \sum_H \max(1, 2^{c(H)-1}),$$

where the sum ranges over all quasi-forests H on $[n]$ with i edges. For the special zonotope $\mathcal{O}_n$, Stanley used Ehrhart theory to show the following.

Corollary 2.2. *The number of degree sequences of simple graphs with node set* $[n]$ *is* $\sum_i h(n,i)$.

A degree sequence $\mathbf{d}$ is called **tight** if $\mathbf{d}$ attains equality in one of the Erdős-Gallai conditions, i.e., if $\widehat{\mathbf{d}}$ is in the boundary of $\mathcal{O}_n$. A graph $G = ([n], E)$ is **split** if there is a partition $V = V_c \uplus V_i$ such that V_c is a clique and V_i is independent. We call G **weakly split** if there is a partition $V = V_c \uplus V_i \uplus V_o$ such that the restriction to $V_c \cup V_i$ is a non-empty split graph, every node of V_o is adjacent to every node of V_c but to no node of V_i.

Theorem 2.3. *A degree sequence* $\mathbf{d}$ *is tight if and only if it is the degree sequence of a weakly split graph. The number of tight degree sequences is*

$$2 \sum_{n-i \text{ odd}} h(n,i).$$

Proof. Let $\mathbf{d} \in \mathcal{O}_n$ be a degree sequence and let $G = ([n], E)$ be one of its realizations. It follows from the Erdős–Gallai condition (3) that $\mathbf{d}$ is tight if and only if there is a partition $V_c \uplus V_i \uplus V_o$ of the node set such that

$$|V_c|(|V_c|-1) + |V_c||V_o| \ = \ \sum_{i \in V_c} d_i - \sum_{i \in V_i} d_i. \tag{4}$$

Let us define N_{ab} for the number of edges between V_a and V_b for $a, b \in \{i, c, o\}$. Then the right hand side can be written as

$$2N_{cc} + N_{ci} + N_{co} - (2N_{ii} + N_{ic} + N_{io}) \ = \ 2N_{cc} + N_{co} - 2N_{ii} - N_{io}.$$

Hence, G satisfies (4) if only if N_{cc} and N_{co} are maximal and N_{ii} as well as N_{io} are minimal. That is, $2N_{cc} = |V_c|(|V_c|-1)$, $N_{co} = |V_c||V_o|$, and $N_{ii} = N_{io} = 0$. This, however, is the case if and only if G is weakly split.

Geometrically, a sequence $\mathbf{d}$ is weakly split if and only if $\mathbf{d} \in \partial\mathcal{O}_n \cap \mathbb{Z}^n$. It follows from Ehrhart theory (see [2]) that

$$|\operatorname{relint}(\mathcal{O}_n) \cap \mathbb{Z}^n| \ = \ h(n,n) - h(n,n-1) + \cdots + (-1)^n h(n,0).$$

Hence, together with Corollary 2.2, we get

$$|\partial\mathcal{O}_n \cap \mathbb{Z}^n| \ = \ |\mathcal{O}_n \cap \mathbb{Z}^n| - |\operatorname{relint}(\mathcal{O}_n) \cap \mathbb{Z}^n| \ = \ 2 \sum_{n-i \text{ odd}} h(n,i). \qquad \square$$

Havel [10] and Hakimi [9] showed that any two simple graphs on the same node set with the same degree sequence can be transformed into each other via a finite sequence of 2-switches: In a graph $G = (V, E)$ containing four vertices $u, v, w, x \in V$ with $uv, wx \in E$ and $uw, vx \notin E$, the operation replacing uv and wx by uw and vx is called a **2-switch** or **Σ-operation** (see Figure 2).

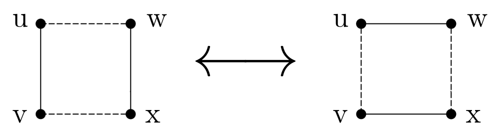


FIGURE 2. Example of a 2-switch or Σ-operation.

A degree sequence $\mathbf{d}$ is called **uniquely realizable** if there is a unique graph G with $\mathbf{d} = \mathbf{d}(G)$, that is, if no 2-switch can be applied to G. From the results of Havel–Hakimi, it follows that the uniquely realizable degree sequences are exactly those coming from threshold graphs. A graph $G = ([n], E)$ is a **threshold graph** if G has a single node or there is a node $v \in [n]$ such that either v is isolated or connected to all remaining vertices and $G - v$ is a threshold graph.

The vertices of the $\binom{n}{2}$-dimensional cube $[0,1]^{\binom{n}{2}}$ are the characteristic vectors of subsets of $\binom{[n]}{2}$ and hence are in bijection with simple graphs on the node set $V = [n]$. If $\chi_E \in \{0,1\}^{\binom{[n]}{2}}$ represents $G = ([n], E)$, then its degree sequence is the image under the linear projection π that takes e_{ij} to $\widehat{e}_i + \widehat{e}_j$ and $\mathcal{O}_n = \pi([0,1]^{\binom{n}{2}})$. Thus, $\mathbf{d}$ is uniquely realizable if $\pi^{-1}(\mathbf{d}) \cap [0,1]^{\binom{n}{2}}$ contains a unique lattice point. It is easy to see that this happens when $\mathbf{d}$ is a vertex of $\mathcal{O}_n$. Vertices of $\mathcal{O}_n$ are easy to describe. The sequence $\mathbf{d}(G)$ is a vertex if and only if there exists a vector $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ such that ij is an edge of G if and only if $c_i + c_j > 0$ for every $1 \leq i < j \leq n$ (see e.g. [13, Section 2]). For undirected graphs, it turns out that $\mathbf{d}$ is uniquely realizable if and only if $\mathbf{d}$ is a vertex of $\mathcal{O}_n$, as was shown by Koren [11]. As we will see in the next section, this is special to the undirected situation. The number of vertices (and hence the number of threshold graphs) was determined by Beissinger and Peled [3], the number of all faces is given in [16].

3. DIRECTED GRAPHS

Let D, D' be two digraphs on the node set $[n]$ that differ only by a single directed edge connecting nodes i and j. Then the difference of their degree sequences will be $\pm(e_i - e_j)$. Following the geometric approach laid out in the previous section, we are led to consider the zonotope

$$\mathcal{D}_n \; := \sum_{1 \leq i < j \leq n} [e_i - e_j, e_j - e_i].$$

This is a polytope that is full-dimensional in the $n-1$ dimensional linear subspace given by all $\mathbf{d} \in \mathbb{R}^n$ with $d_1 + \cdots + d_n = 0$. It is clear from the definition that it contains all degree sequences of directed graphs $D = ([n], A)$. The polytope is invariant under permuting coordinates and, in fact, it is the convex hull of all permutations of the point

$$(-n+1, -n+3, -n+5, \ldots, n-5, n-3, n-1). \tag{5}$$

This shows that $\mathcal{D}_n = 2\Pi_{n-1} - (n+1)\mathbf{1}$ where Π_{n-1} is the well-known permutahedron; see [21]. In particular, it is straightforward to show that for every $\mathbf{d} \in \mathcal{D}_n \cap \mathbb{Z}^n$, there is $A \subset [n] \times [n]$ such that

$$\mathbf{d} \; = \sum_{(i,j) \in A} e_i - e_j$$

and A gives rise to a simple digraph D with $\mathbf{d} = \mathbf{d}(D)$. The vertices of $\mathcal{D}_n$ are in bijection with the net-degree sequences of tournaments on n vertices.

An inequality description of Π_{n-1} is well-known and we conclude an Erdős–Gallai-type result for directed graphs.

Theorem 3.1. *For $\mathbf{d} \in \mathbb{Z}^n$ the following conditions are equivalent:*

(i) *There is a (simple) digraph $D = ([n], A)$ with net-degree sequence $\mathbf{d} = \mathbf{d}(D)$;*
(ii) $\mathbf{d} \in \mathcal{D}_n$*;*
(iii) $\mathbf{d}$ *satisfies $d_1 + \cdots + d_n = 0$ and*

$$\sum_{i \in I} d_i \ \leq \ |I|(n - |I|)$$

for all $\emptyset \neq I \subsetneq [n]$.

For $\mathbf{d}$ in non-increasing order, Theorem 3.1(iii) yields the conditions found by Pirzada et al. [14].

A simple digraph $D = (V, A)$ is **weakly split** if there is a partition $V = V_s \uplus V_t$ such that $V_s \times V_t \subseteq A$. This class of digraphs gives us a characterization of the tight net-degree sequences.

Proposition 3.2. *A vector $\mathbf{d} \in \mathbb{Z}^n$ is contained in the boundary of $\mathcal{D}_n$ if and only if it is the degree sequence of a weakly split digraph.*

Proof. By Theorem 3.1(iii), $\mathbf{d} \in \partial\mathcal{D}_n$ if and only if

$$\sum_{i \in V_t} d_i \ = \ |V_t|(n - |V_t|)$$

for some nonempty subset $V_t \subset [n]$. Equality holds for a realization $D = ([n], A)$ if and only if $V_s \times V_t \subseteq A$ for $V_s = [n] \setminus V_t$. □

Using Exercise 4.67 in [17] together with Theorem 3.1, we obtain the following enumerative results regarding net-degree sequences. We write $f(n, i)$ for the number of forests on n labelled vertices with i edges.

Corollary 3.3. *The number of net-degree sequences of digraphs on n labelled nodes is*

$$\sum_{i=0}^{n} 2^i f(n, i).$$

The number of weakly split digraphs on n vertices equals

$$\sum_{i=1}^{\lceil \frac{n}{2} \rceil} 2^{2i-1} f(n, 2i - 1).$$

Two obvious operations that leave the net-degree sequence of a digraph $D = (V, A)$ unchanged are the following: A **$\boldsymbol{\Delta}$-operation** either adds a directed triangle to three pairwise non-adjacent nodes or removes a directed triangle. Likewise, we may replace an oriented edge (u, w) with two edges $(u, v), (v, w)$ where $v \in V$ is a node not adjacent to either u or w. This or its inverse is called a **$\boldsymbol{\Lambda}$-operation**. See Figure 3 for an illustration of both operations. The following result is taken from [14].

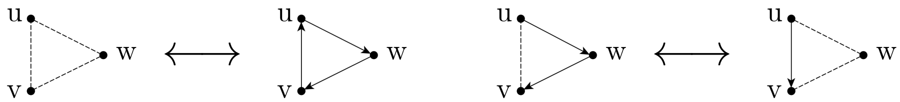
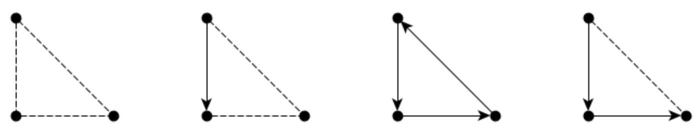


FIGURE 3. The two net-degree preserving operations for digraphs.

Theorem 3.4. *Two digraphs $D_1 = (V, A_1)$ and $D_2 = (V, A_2)$ have the same net-degree sequence if and only D_1 can be obtained from D_2 by a sequence of Δ- and Λ-operations.*

From this result, we obtain a simple characterization of those digraphs corresponding to uniquely realizable net-degree sequences.

Corollary 3.5. *The degree sequence of a simple digraph D is uniquely realizable if and only if D does not contain one of the following four graphs as induced subgraph:*



Proof. A digraph does not permit a Δ- or Λ-operation if and only if it avoids the given four induced subgraphs. The claim now follows from Theorem 3.4. □

A digraph $D = (V, A)$ is **transitive** if whenever a node v can be reached from $u \in V$ by a directed path, then (u, v) is a directed edge in D. Transitive and acyclic digraphs $D = (V, A)$ correspond exactly to partial order relations on V and we simply call D a **poset**. A poset D is of **width 2** if every antichain, that is every collection of nodes not reachable from one another, is of size at most 2.

Theorem 3.6. *A digraph $D = (V, A)$ has a uniquely realizable net-degree sequence if and only if $|V| \leq 2$ or D is a connected, width-2 poset.*

Proof. If $D = (V, E)$ contains at most 2 nodes it is clearly uniquely realizable and for a connected, width-2 poset it is easy to see that it satisfies the conditions of Corollary 3.5.

Conversely, let $D = (V, A)$ be a uniquely realizable digraph. By using (D4) from Corollary 3.5 and induction on the length of the path, it follows that D is transitive. Combining (D3) and (D4) shows that D cannot contain cycles and hence is a poset. If $|V| > 2$, then (D2) excludes distinct components. Finally, (D1) assures us every antichain of D is of size at most 2. Hence D is a connected, width-2 poset. □

Using this bijection, one can enumerate uniquely realizable degree sequences:

Theorem 3.7. *Let $U(n)$ be the number of uniquely realizable net-degree sequences of simple digraphs on $n \geq 0$ labelled vertices. Then*

$$U(n) \;=\; \frac{n!}{\sqrt{3}}\Big((\sqrt{3}-1)^{-n-1} - (-\sqrt{3}-1)^{-n-1}\Big),$$

with exponential generating function

$$\sum_{n\geq 0} \frac{U(n)x^n}{n!} \;=\; \frac{1}{1-x-\frac{x^2}{2}}.$$

Proof. From Theorem 3.6, we infer that $U(0) = U(1) = 1$ and $U(2) = 3$. For $n \geq 3$, $D = (V, A)$ is a connected, width-2 poset. Let $M \subseteq V$ be the set of maximal elements. It follows from the definition of width-2 posets that M contains either one or two elements. Removing M from D yields a uniquely realizable digraph on the node set $V \setminus M$. Hence, we infer that

$$U(n) \; = \; n \cdot U(n-1) + \binom{n}{2} \cdot U(n-2)$$

for all $n \geq 2$. Now, define $\bar{U}(n) := \frac{1}{n!}U(n)$. Then $\bar{U}(n)$ satisfies the linear recurrence relation $\bar{U}(n) = U(n-1) + \frac{1}{2}U(n-2)$. The generating function as well as the formula can now be inferred using [17, Theorem 4.1.1]. □

The first values of $U(n)$ are $1, 1, 3, 12, 66, 450, 3690$. Consistent with our interpretation, $U(n)$ occurs in connection with ordered partitions of $[n]$ with parts of size at most 2 and Boolean intervals in the weak Bruhat order of S_n; see [1] and Proposition 3.8.

In terms of geometry, it is clear every vertex of $\mathcal{D}_n$ corresponds to a uniquely realizable degree sequence. Indeed, every tournament is uniquely determined by its net-degree sequence. In the language of posets, tournaments correspond to total (or linear) orders and consequently have no incomparable elements. Taking into account the number of pairs of incomparable elements, we can locate the uniquely realizable net-degree sequences in $\mathcal{D}_n$.

Proposition 3.8. *Let $\mathbf{d} \in \mathcal{D}_n \cap \mathbb{Z}^n$ be a net-degree sequence and let $F \subseteq \mathcal{D}_n$ be the inclusion-minimal face that contains $\mathbf{d}$ in its relative interior. Then the following statements are equivalent:*

(i) $\mathbf{d}$ *is uniquely realizable. Its corresponding poset has k pairs of incomparable elements.*
(ii) *F is a proper face lattice-isomorphic to the k-dimensional cube $[-1, 1]^k$.*

Proof. The polytope $\mathcal{D}_n$ is lattice-isomorphic to the permutahedron $2\Pi_{n-1}$. The k-dimensional faces of Π_{n-1} are in bijection to ordered partitions of $[n]$ into $n-k$ non-empty parts; [21, Ch. 0]. For an ordered partition $B = (B_1, \dots, B_{n-k})$ with sizes $a_i = |B_i|$, the corresponding face is lattice-isomorphic to $\Pi_{a_1-1} \times \cdots \times \Pi_{a_{n-k}-1}$. Hence, with Theorem 3.6, it suffices to prove that width-2 posets with k pairs of incomparable elements are in bijection with ordered partitions of $[n]$ into $n - k$ parts. The corresponding poset P is graded and the partition corresponds exactly to the rank partition of P. □

4. Bidirected graphs

Let $B = (G, \tau)$ be a bidirected graph as defined in the introduction. As before, we will assume throughout that $G = (V, E)$ is an undirected graph without loops or parallel edges on the node set $V = \{1, \dots, n\}$. The bidirection function $\tau : V \times E \to \{-1, 0, +1\}$ gives every node-edge-incidence an orientation. Let us write $\mathbf{d}(B)_i^+ := |\{e \in E : \tau(i, e) = +1\}|$ for the number of edges locally oriented towards i and define $\mathbf{d}(B)_i^-$ likewise. The **net-degree sequence** $\mathbf{d}(B)$ of B then satisfies $\mathbf{d}(B)_i = \mathbf{d}(B)_i^+ - \mathbf{d}(B)_i^-$ for all $i = 1, \dots, n$.

Defining orientations of edges locally renders net-degree sequences of bigraphs far less restricted as in the undirected or directed setting. A characterization is nevertheless not vacuous. Indeed, a key observation is that the underlying undirected graph $G = |B|$ has ordinary degree sequence $\mathbf{d}(G) = \mathbf{d}(B)^+ + \mathbf{d}(B)^-$ and hence

$$\mathbf{d}(G)_i \; \equiv \; \mathbf{d}(B)_i \mod 2 \quad \text{and} \quad \mathbf{d}(G)_i \; \geq \; |\mathbf{d}(B)_i|, \tag{6}$$

for all $i = 1, \ldots, n$. Borrowing from Section 2, we write $\widehat{\mathbf{d}} = (\mathbf{d}, \frac{1}{2}(d_1 + \cdots + d_n)) \in \mathbb{R}^{n+1}$. Towards a geometric perspective on degree sequences of bigraphs, we define the polytope

$$\mathcal{B}_n \;:=\; \operatorname{conv}\left\{ \sum_{i<j} \tau_{ij}\widehat{e}_i + \tau_{ji}\widehat{e}_j : \tau \in \{-1,+1\}^{n\times n} \right\}.$$

Theorem 4.1. *For a vector $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ the following are equivalent:*

(i) *There is a bigraph B with $\mathbf{d} = \mathbf{d}(B)$;*
(ii) *$\widehat{\mathbf{d}} \in \mathcal{B}_n$;*
(iii) *$d_1 + \cdots + d_n$ is even and $-(n-1) \le d_i \le n-1$ for $i = 1, \ldots, n$.*

Proof. Let $\mathbf{d} = \mathbf{d}(B)$ be the net-degree sequence of a bigraph B. The net-degree sequence is given by

$$\mathbf{d}(B) \;=\; \sum_{i\in V, e\in E} \tau(i,e)e_i$$

and (6) together with (2) shows that $\widehat{\mathbf{d}} \in \mathcal{B}_n$. This proves (i) $\Longrightarrow$ (ii). The implication (ii) $\Longrightarrow$ (iii) is obvious.

For (iii) $\Longrightarrow$ (i), we first observe that $T = \{i \in [n] : d_i \not\equiv n-1 \bmod 2\}$ has an even number of elements which follows from the fact that $d_1 + \cdots + d_n \equiv n(n-1) - |T|$ modulo 2. Let $G = ([n], E)$ be the graph obtained from the complete graph K_n by removing a matching of the elements of T. Hence, G is a simple undirected graph with ordinary degree sequence $\mathbf{d}' := \mathbf{d}(G)$. The degrees satisfy $\mathbf{d}(G)_i = n-2$ if $i \in T$ and $= n-1$, otherwise. We can now turn G into a bigraph B by bidirecting its edges. For every node $i \in [n]$, we let $|\mathbf{d}_i|$ edges point to i or away from i depending on the sign of $\mathbf{d}_i$. By construction, the number of unoriented edges incident to i is even and we can orient them so that their contribution to $\mathbf{d}(B)_i$ cancels. See Figure 4 for an illustration. $\square$

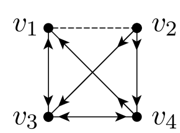


FIGURE 4. Realizing $\mathbf{d} = (2, -2, 3, 1)$ by a bigraph.

Theorem 4.1 gives us a means to count degree sequences.

Corollary 4.2. *The number of net-degree sequences of bigraphs on n labelled vertices is $\frac{(2n-1)^n+1}{2}$.*

Proof. The number of lattice points in the cube $[-(n-1), (n-1)]^n$ is $(2(n-1)+1)^n$ and according to Theorem 3.1(iii) those with even coordinate sum are in bijection with degree sequences of bigraphs. Now if $u \in [-(n-1), (n-1)]^n$ has even coordinate sum, then so has $-u$ and $u \neq -u$ unless $u = 0$. Hence, there are $\frac{(2(n-1)+1)^n-1}{2} + 1$ lattice points with even coordinate sum. $\square$

Bigraphs simultaneously generalize undirected graphs and digraphs. Hence, it is natural to assume that there are more degree-preserving operations for bigraphs. The most basic operation that is not present in the directed or undirected setting is the **$\boldsymbol{\Gamma}$-operation**: A

Γ-operation swaps the local orientations of two edges uv and vw incident to a node v with $\tau(v, uv) \neq \tau(v, vw)$; see Figure 5. In particular, a Γ-operation leaves the underlying graph $|B|$ untouched and only changes the bidirection function τ.

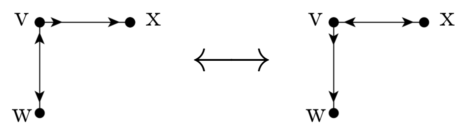


Figure 5. The Γ-operation.

For two bigraphs with the same degree sequence and the same underlying graph, the Γ-operation is sufficient to transform one into the other.

Lemma 4.3. *Let B and B' be two bigraphs with $|B| = |B'|$ and $\mathbf{d}(B) = \mathbf{d}(B')$. Then there is a finite sequence of Γ-operations that transforms B into B'.*

Proof. Let $B = (G, \tau)$ and $B' = (G, \tau')$ with underlying undirected graph $G = (V, E)$. For a fixed node $v \in V$, the number of edges $e \in E$ with $\tau(v, e) \neq \tau'(v, e)$ is even. Hence, we can pair up these edges and apply a Γ-operation to B for every pair. Since the changes only affect the local orientations at v, we may repeat this process for all the remaining nodes. $\square$

The Σ- or 2-switch operation on undirected graphs has a natural extension to bidirected graphs: A Σ-operation locally replaces an edge e incident to a node v with a new incident edge e'. If G is the graph underlying a bidirected graph B, then a Σ-operation applied to B assigns a local orientation to e' by $\tau(v, e') := \tau(v, e)$; see Figure 6 for a depiction. Note that a Σ-operation on B induces a Σ-operation on the underlying undirected graph $|B|$. Combining this observation with the Havel–Hakimi theorem [10, 9] yields the following corollary.

Corollary 4.4. *Let B and B' be bigraphs such that their underlying graphs $|B|$ and $|B'|$ have identical degree sequences. Then B can be transformed using Σ-operations into a bigraph with underlying graph $|B'|$.*

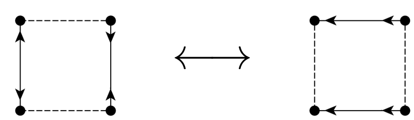


Figure 6. The Σ-operation.

As for directed graphs, an edge uw in a bigraph B may be replaced by edges uv, vw where $v \in V$ is a node not adjacent to either u or w. This procedure is net-degree preserving if we set $\tau(v, uv) = -\tau(v, vw) = 1$. This operation and its inverse is called a **$\boldsymbol{\Lambda}$-operation**; Figure 7 illustrates the operation.

Finally, we extend our repertoire of operations by a **$\boldsymbol{\Delta}$-operation** that corresponds to adding or deleting a bidirected triangle: If u_1, u_2, u_3 are vertices of a bigraph B with edges u_1u_2, u_2u_3, u_3u_1 such that $\tau(u_i, u_iu_j) \neq \tau(u_j, u_iu_j)$ for $i \neq j$, then we may delete the three edges without changing the net-degree of B.

More generally, a **bidirected $\boldsymbol{k}$-cycle** in a bidirected graph $B = (G, \tau)$ is an undirected cycle $u_1, \ldots, u_k$ in G such that $\tau(u_i, u_{i-1}u_i) \neq \tau(u_i, u_iu_{i+1})$ for $i = 2, \ldots, k$ (and the convention

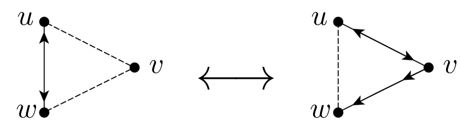


FIGURE 7. The Λ-operation.

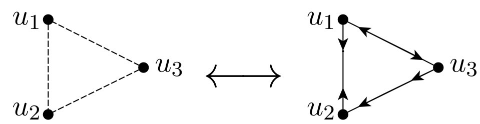


FIGURE 8. The Δ-operation.

that $u_{k+1} := u_1$). Clearly, the net-degree of bidirected cycles is identically zero and we could extend Δ-operations to all bidirected cycles. The next lemma, however, shows that this can always be achieved essentially by Δ- and Λ-operations.

Lemma 4.5. *Let B be a bidirected graph. Then adding a bidirected cycle can be obtained by a sequence of Δ-, Λ-, and Γ-operations.*

Proof. Let us assume that we want to add a cycle of length m on the nodes $1, 2, \dots, m$. To ease notation, we identify $m+1$ with 1. We prove the claim by induction on m with $m = 3$ as the base case. If $i, i+2$ is not an edge for some $i = 1, \dots, m-1$, then we can add the cycle $0, \dots, i, i+2, \dots, m$ by induction and apply a Λ-operation to replace the edge $i, i+2$ by $i, i+1$ and $i+1, i+2$. Hence, we have to assume that all edges $i, i+2$ for $i = 1, \dots, m-1$ are present. If $m = 4$, we replace the edge between 1 and 3 by the edges $1, 2$ and $2, 3$. Then, we add the triangle between $1, 3$ and 4. If $m \geq 5$, then for all vertices $2j$ with $j = 1, \dots, \lfloor \frac{m}{2} \rfloor$, we use a Λ-operation to replace the edge $2j-1, 2j+1$ with the path $2j-1, 2j, 2j+1$ without changing the local orientation at $2j \pm 1$. We may now add the cycle on the vertices $1, 3, \dots, 2\lceil \frac{m}{2} \rceil - 1$. The resulting bigraph B' has already the correct underlying undirected graph and degree sequence $\mathbf{d}(B') = \mathbf{d}(B)$ and appealing to Lemma 4.3 completes the proof. $\square$

An argument analogous to the above shows that we can add arbitrary bidirected paths using essentially Λ-operations. A **bidirected path** in a bidirected graph $B = (G, \tau)$ is an undirected path $u_1, \dots, u_k$ in G such that $\tau(u_i, u_{i-1}u_i) \neq \tau(u_i, u_iu_{i+1})$ for $i = 2, \dots, k-1$.

Lemma 4.6. *Let B be a bidirected graph. Then adding a bidirected path can be obtained by a sequence of Λ- and Γ-operations.*

The following theorem asserts that these four operations suffice to navigate among bigraphs with the same net-degree sequence. The bigraphs in Figures 5, 7, and 8 show that the Γ-, Λ-, and Δ-operations are necessary. To see that the Σ-operation is necessary it suffices to consider 4-cycles where every edge points into its two end nodes.

Theorem 4.7. *Let B and B' be bigraphs with $\mathbf{d}(B) = \mathbf{d}(B')$. Then there is a finite sequence of Γ-, Σ-, Λ-, and Δ-operations that transforms B into B'.*

Proof. Let $B = (G, \tau)$ and $B' = (G', \tau')$ on n nodes. The strategy of the proof is as follows. We show that we can transform both B and B' into bigraphs B_0 and B_0' such that their underlying undirected graphs G_0 and G'_0 have identical (ordinary) degree sequences. Corollary 4.4 then shows that we can assume that $G_0 = G'_0$ and Lemma 4.3 completes the proof.

Let $T := \{i \in [n] : \mathbf{d}(B)_i \not\equiv n-1 \bmod 2\}$ be the set of odd-degree vertices in $H := K_n \setminus G$. Since T is of even size, we can label its elements $v_1, v_2, \ldots, v_{2m}$. The edge set $E(H)$ can be decomposed into edge-disjoint cycles $C_1, \ldots, C_s$ and paths $P_1, \ldots, P_m$ such that, without loss of generality, P_i has endpoints v_{2i-1} and v_{2i}; see [15, Section 29.1]. For every $j = 1, \ldots, s$, we apply Lemma 4.5 to add a bidirected cycle to B with underlying graph C_j. By the same token, if P_i is of length > 1, then we can replace the bidirected edge between v_{2i-1} and v_{2i} in B by a bidirected path with underlying graph P_i using Lemma 4.6. The resulting bigraph B_0 has net-degree sequence $\mathbf{d}(B)$ and, more importantly, every node v in $|B_0| = G_0$ has ordinary degree $n-2$ if $v \in T$ and $n-1$ otherwise. Repeating this process for B', we obtain our desired bigraphs B_0 and B'_0. $\square$

A complete set of degree-preserving operations as given in Theorem 4.7 also gives insight into the class of uniquely realizable net-degree sequences. A bigraph has a uniquely realizable degree sequence if and only if no degree-preserving operation is applicable. Recall that a node v is a sink or a source if all edges are pointing into v or away from v, respectively.

Corollary 4.8. *The net-degree sequence of a bigraph B is uniquely realizable if and only if every node is a sink or a source and there is at most one pair of nodes not joined by an edge.*

In particular, a net-degree sequence $\mathbf{d} = (d_1, \ldots, d_n) \in \mathcal{B}_n \cap \mathbb{Z}^n$ is uniquely realizable if and only if $n-2 \leq |d_i| \leq n-1$ for all $i = 1, \ldots, n$ and there at most two entries with absolute value $n-2$.

In particular, the set of uniquely realizable net-degree sequences is contained in the boundary of $\mathcal{B}_n$. The vertices of $\mathcal{B}_n$ are uniquely realizable as in the case of ordinary and directed graphs.

Corollary 4.9. *The number of uniquely realizable net-degree sequences of bidirected graphs on n nodes is*

$$2^n \binom{n}{2} + 2^n.$$

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