On an application of multidimensional arrays

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Abstract

This article discusses some difficulties in the implementation of combinatorial algorithms associated with the choice of all elements with certain properties among the elements of a set with great cardinality.The problem has been resolved by using multidimensional arrays. Illustration of the method is a solution of the problem of obtaining one representative from each equivalence class with respect to the described in the article equivalence relation in the set of all $m \sim n$ binary matrices. This equivalence relation has an application in the mathematical modeling in the textile industry.

Keywords: binary matrix; equivalence relation; factor-set; cardinality; multidimensional array

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1 Introduction and task formulation

The following problem often occurs in computer science:

Problem 1.1. *Let M be a finite set and let $\sim$ be an equivalence relation in M. Describe and implement an algorithm that receives exactly one representative from each equivalence class with respect to $\sim$.*

As a consequence of this problem follows the combinatorial problem of finding the cardinality of the factor set $\widetilde{M} = M_{/\sim}$ consisting of all equivalence classes of M with respect of $\sim$.

We assume that for every $x \in M$, there is a procedure $K(x)$ which receives all elements of M, which are equivalent to x.

Since M is a finite set, then there exists bijective mapping

$$b \ : \leftrightarrow \{1, 2, \dots, |M|\} \,,$$

which will call *numbering function*. Thus, each element of M uniquely corresponds to an element of Boolean array $H[\ ]$ with size equal to the cardinality $|M|$ of the set M. Moreover, the element $x \in M$ is *selected* if $H[b(x)] = 1$ and x is *not selected* if $H[b(x)] = 0$.

The next algorithm is a modification of the well-known method, known as "Sieve of Eratosthenes" [Reingold, Nievergeld and Deo (1977); Yordzhev and Markovska (2007)] solves Problem 1.1.

Algorithm 1.2. *Receives exactly one representative of each equivalence class of the factor-set $\widetilde{M} = M_{/\sim}$.*

***Input**: Finite set M*

***Output**: Set $N \subseteq M$*

1. $N := \emptyset$;

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2. *Declare a Boolean array $H[\ ]$ with size equal to the cardinality $|M|$ of the set M and put $H[b(x)] := 0$ for all $x \in M$;*
3. *For every $x \in M$ such that $H[b(x)] = 0$ **do***

   $\{$ ***Begin** of loop 1*
4. $N := N \cup \{x\}$;
5. $H[b(x)] := 1$;
6. *Using the procedure $K(x)$ obtain the set $P_x = \{y \in M \mid y \sim x\}$;*
7. *For every $y \in P_x$ obtained in step 6 **do***

   $\{$ ***Begin** of loop 2*
8. $H[b(y)] := 1$;

   ***End** of loop 2* $\}$

   ***End** of loop 1* $\}$
9. *End of the algorithm.*

Algorithm 1.2 has a number of disadvantages, the main of which is that it is practically inapplicable for programs when a sufficiently great number of elements is present in the base set M. This limitation comes from the maximum integer, which can be used in the corresponding programming environment. For example, by standard in the C++ language the biggest number of the type **unsigned long int** is equal to $2^{32} - 1$, which in a number of cases is insufficient for the previously defined array $H[\,]$ to be completely addressed. The purpose of this article is to avoid this problem by using a multidimensional Boolean array, the elements of which have a one-to-one correspondence to the elements of the base set, with a much smaller range of the indices. There are many publications related to multidimensional arrays, for example [Mishra (2014)], but they are not used for our specific goals and objectives. Another solution to the problem is the use of dynamic data structures or other special programming techniques [Collins (2002); Sutter (2002); Tan, Steeb and Hardy (2001)] but it is not the subject of consideration in this article.

Binary (or Boolean, or (0,1)-matrix) is a matrix whose elements are equal to 0 or 1.

Let $\mathcal{B}_{m\times n}$ be the set of all $m \times n$ binary matrices. It is well known that

$$|\mathcal{B}_{m\times n}| = 2^{mn} \tag{1.1}$$

In this work, we will consider and solve the following special case of Problem 1.1:

Problem 1.3. *Let $\mathcal{B}_{m\times n}$ be the set of all $m \times n$ binary matrices and let $X, Y \in \mathcal{B}_{m\times n}$. We define an equivalence relation ρ as follows: $X\rho Y$ if and only if we can obtain X from Y by a sequential moving of the last row or column to the first place. Find the cardinality $|\mathcal{B}_{m\times n/\rho}|$ of the factor-set $\widetilde{M} = \mathcal{B}_{m\times n/\rho}$ and receive a single representative of each equivalence class.*

The proof that ρ is an equivalence relation is trivial and we will omit it here.

The equivalence classes of $\mathcal{B}_{m\times n}$ by the equivalence relation ρ are a particular kind of *double coset* [Bogopolski (2008); Curtis and Rainer (1962); De Vos (2010)]. They make use of substitutions group theory and linear representation of finite group theory [Curtis and Rainer (1962); De Vos (2010)].

When $m = n$, the elements of the factor-set $\widetilde{M} = \mathcal{B}_{n\times n/\rho}$ put carry into practice in the textile technology [Borzunov (1983); Yordzhev and Kostadinova (2012)].

In [Yordzhev (2005)] an algorithm is shown, which utilizes theoretical graphical methods for finding the factor set $\widetilde{S} = S_{n/\rho}$, where $S_n \subset \mathcal{B}_{n\times n}$ is a set of all permutation matrices, i.e. binary matrices having exactly one 1 on each row and each column. In [Yordzhev (2014)] we extended this problem in the case when ρ is an arbitrary permutation.

The author of this paper is not familiar with an existing a general formula expressed as a function of m and n for finding $|\mathcal{B}_{m\times n/\rho}|$. The goal of this paper is to describe an effective algorithm for finding the number of elements of the factor set $\widetilde{M} = \mathcal{B}_{m\times n/\rho}$, as well as finding a single representative of each equivalence class. Here we will describe an algorithm, which overcomes some difficulties, which would inevitably arise with sufficiently great m and n if we apply the classical algorithm (Algorithm 1.2). The main difficulty arises from the great number of elements of $\widetilde{M} = \mathcal{B}_{m\times n/\rho}$ with comparatively small integers m and n, according to formula (1.1).

For undefined notions and definitions, we refer to [Aigner (1979); Sachkov and. Tarakanov (2002)].

2 Description of an algorithm with the use of a multidimensional array

Theorem 2.1. *Let us denote by $\mathcal{P}_n$ the set*

$$\mathcal{P}_n = \{0, 1, \dots, 2^n - 1\} \tag{2.1}$$

Then a one-to-one correspondence (bijection) between the elements of the Cartesian product $\mathcal{P}_n^m = \underbrace{\mathcal{P}_n \times \mathcal{P}_n \times \cdots \times \mathcal{P}_n}_{m}$ and the elements of the set $B_{m\times n}$ of all $m \times n$ binary matrices exists.

Proof. We consider the mapping $\alpha : \mathcal{P}_n^m \to \mathcal{B}_{m\times n}$, defined in the following way: If $\pi \in \mathcal{P}_n^m$ and $\pi =< p_1, p_2, \dots, p_m >$ then let us denote by z_i, $i = 1, 2, \dots, m$, the representation of the integer p_i in a binary notation, and if less than n digits 0 or 1 are necessary, we fill z_i from the left with insignificant zeros, so that z_i will be written with exactly n digits. Since by definition, $p_i \in \mathcal{P}_n$, i.e. $0 \le p_i \le 2^n - 1$, this will always be possible. Then we form an $m \times n$ binary matrix, so that the i-th row is z_i, $i = 1, 2, \dots m$. Apparently this is a correctly defined mapping of $\mathcal{P}_n^m$ to $\mathcal{B}_{m\times n}$. It is clear that for different n-tuples from $\mathcal{P}_n^m$ with the help of α we will obtain different matrices from $\mathcal{B}_{m\times n}$, i.e. α is an injection. Conversely, rows of each binary matrix can be considered as natural numbers, written in binary system by using exactly n digits 0 or 1, eventually with insignificant zeros in the beginning, that is, these numbers belong to the set $\mathcal{P}_n = \{0, 1, \dots, 2^n - 1\}$. Therefore each $m \times n$ Binary matrix corresponds to an m-tuple of numbers $< p_1, p_2, \dots, p_m > \in \mathcal{P}_m^n$, that is, α is a surjection. Hence α is a bijection. ☐

It is easy to see the validity of the following statement, which in fact shows the meaning of our considerations.

Proposition 2.1. *Let us denote by μ the maximum integer, which we use when coding the elements of the set $\mathcal{B}_{m\times n}$ by means of the bijection, defined in Theorem 2.1. Then, for sufficiently great m and n, the following is valid:*

$$\mu = \max(2^n - 1, m) \ll |\mathcal{B}_{m\times n}| = 2^{mn} \tag{2.2}$$

Proof. Trivial. ☐

Let a and b be integers, $b \neq 0$. With a/b we will denote the operation "integer division" of a by b, i.e. if the division has a remainder, then the fractional part is cut, and with $a\%b$ we will denote the remainder when dividing a by b. In other words, if $\dfrac{a}{b} = p + \dfrac{q}{b}$, where p and q are integers, $0 \le q < b$ then by definition $a/b = p$, $a\%b = q$.

We consider the function

$$\xi(a) = (a\%2)\, 2^{n-1} + a/2, \tag{2.3}$$

where $\%$ and $/$ are the defined in the above operations.

Definition 2.1. Let α be the defined in the proof of Theorem 2.1 bijection and let the functions $f_r, f_c : \mathcal{P}_n^m \to \mathcal{P}_n^m$ be defined such that for every $\pi =< p_1, p_2, \ldots, p_m >\in \mathcal{P}_n^m$

$$f_r(\pi) =< p_m, p_1, p_2, \ldots p_{m-1} > \tag{2.4}$$

$$f_c(\pi) =< \xi(p_1), \xi(p_2), \ldots, \xi(p_m) >, \tag{2.5}$$

where the function $\xi(a)$ is the defined with (2.3).

Theorem 2.2. *Let $A \in \mathcal{B}_{m\times n}$ be an arbitrary $m \times n$ binary matrix and let α be the defined in the proof of Theorem 2.1 bijection. Let us to get the matrices*

$$B = \alpha\left(f_r\left(\alpha^{-1}(A)\right)\right) \tag{2.6}$$

and

$$C = \alpha\left(f_c\left(\alpha^{-1}(A)\right)\right) \tag{2.7}$$

Then B is obtained from A by moving the last row to the first place, and C is obtained from A by moving the last column to the first place (respectively the first row or column becomes the second, the second becomes the third respectively etc.).

Proof. Let $\pi =< p_1, p_2, \ldots, p_m >= \alpha^{-1}(A) \in \mathcal{P}_n^m$. Then the integer p_i, $0 \le p_i \le 2^n - 1$, $i = 1, 2, \ldots, m$ will correspond to the i-th row of the matrix A. Then obviously, the matrix $B = \alpha(f_r(< p_1, p_2, \ldots, p_m >)) = \alpha(< p_m, p_1, p_2, \ldots, p_{m-1} >)$ is obtained from A by moving the last row in the place of the first one, and moving the remaining rows one row below.

Let $p_i \in \mathcal{P}_n = \{0, 1, \ldots, 2^n - 1\}$, $i = 1, 2, \ldots, m$. Then $d_i = p_i\%2$ gives the last digit of the binary notation of the integer p_i. If p_i is written in binary notation with precisely n digits, optionally with insignificant zeros in the beginning, then by applying integer division of p_i by 2, we practically remove the last digit d_i and we move it to the first position, in case we multiply by 2^{n-1} and add it to $p_i/2$. This is, by definition, how the function $\xi(p_i)$ works. Hence, the $m \times n$ binary matrix $C = \alpha(f_c(< p_1, p_2, \ldots, p_m >)) = \alpha(< \xi(p_1), \xi(p_2), \ldots, \xi(p_m) >)$ is obtained from the matrix A by moving the last column to the first position, and all the other columns are moved one column to the right. □

From the definitions of the functions f_r, according to (2.4) and f_c, according to (2.5) it is easy to verify the validity of the following

Proposition 2.2. *If by definition*

$$f_r^0(\pi) = f_c^0(\pi) = \pi \tag{2.8}$$

$$f_r^k(\pi) = f_r\left(f_r^{k-1}(\pi)\right) \tag{2.9}$$

$$f_c^k(\pi) = f_c\left(f_c^{k-1}(\pi)\right), \tag{2.10}$$

where $\pi \in \mathcal{P}_n^m$ and k is a positive integer, then

$$f_r^m(\pi) = \pi \tag{2.11}$$

and

$$f_c^n(\pi) = \pi. \tag{2.12}$$

Proof. Trivial. □

As a direct consequence of Theorem 2.1, Theorem 2.2, Proposition 2.2 and their constructive proofs, it follows that the following algorithm that finds exactly one representative of each equivalence class with respect to the defined in Problem 1.3 equivalence relation ρ and that calculates the cardinality of the factor set $\mathcal{B}_{m\times n/\rho}$.

Algorithm 2.3. *Receives exactly one representative of each equivalence class of the factor-set* $\widetilde{M} = M_{/\rho}$ *and calculates the cardinality of the factor set* $\widetilde{M} = M_{/\rho}$ *when* m *and* n *are given.*

1. *We declare the* m*-dimensional Boolean arrays* $W1$ *and* $W2$ *which we will be indexed by using the elements of the set* $\mathcal{P}_n^m$*, i.e.* $W1[< p_1, p_2, \ldots, p_m >]$ *will correspond to the element* $< p_1, p_2, \ldots, p_m > \in \mathcal{P}_n^m$*. We proceed analogically with the array* $W2$*.*
2. *Initially we take all elements of* $W1$ *and* $W2$ *to be 0. In* $W1$ *we will remember all elements selected from* $\mathcal{B}_{m\times n}$ *(one for each equivalence class) by changing* $W1[< p_1, p_2, \ldots, p_m >]$ *to 1 if we have selected the element* $\alpha(< p_1, p_2, \ldots, p_m >)$ *for a representative of the respective equivalence class. We will change the elements of* $W2$ *to 1 for each selection of an element from* $\mathcal{B}_{m\times n}$*, i.e. for each* $\pi'' \in \mathcal{P}_n^m$*, for which there exists* $\pi' \in \mathcal{P}_n^m$*, such that* $W1[\pi'] = 1$ *and* $\alpha(\pi'')\rho\alpha(\pi')$*, or in other words,* π' *and* π'' *encode two different matrices of the same equivalence class as we have chosen* $\alpha(\pi')$ *for a representative of this equivalence class.*
3. *We declare the counter* N*, which we initialize by 0. In case of normal ending of the algorithm,* N *will be showing the cardinality of the factor set* $\mathcal{B}_{m\times n}{}_{/\rho}$*.*
4. *While a zero element exists in* $W2$ ***do***

    { ***Begin*** *of loop 1*
5. *We choose the minimal* $\pi = < p_1, \pi_2, \ldots, \pi_m > \in \mathcal{P}_n^m$ *according to the lexicographic order, for which* $W1[\pi] = 0$*.*
6. $W1[\pi] := 1$*;*
7. $N := N + 1$*;*
8. *For* $i = 1, 2, \ldots, m$ ***do***

    { ***Begin*** *of loop 2*
9. $\pi = f_r^i(\pi)$*.*
10. *For* $j = 1, 2, \ldots, , n$ ***do***

    { ***Begin*** *of loop 3*
11. $\pi := f_c^j(\pi)$*;*
12. $W2[\pi] := 1$*;*

    End of loop 3}

    End of loop 2}

    End of loop 1}
13. ***End*** *of the algorithm.*

3 CONCLUSIONS

Applying the above ideas, a computer program that receives a computer program that gets only one representative from each equivalence class of the factor-set $\widetilde{B}_{n\times n} = B_{n\times n/\rho}$. The purpose of these calculations was to describe and classify some textile structures [Yordzhev and Kostadinova (2012)]. The results relate to obtaining quantitative estimation of all kinds of textile fabric.

In fact, the cardinality of the factor-set M coincides with an integer sequence noted in On-Line Encyclopedia of Integer Sequences [Encyclopedia (2015)] as number A179043, namely

A179043={ 2, 7, 64, 4156, 1342208, 1908897152, 11488774559744, 288230376353050816, 29850020237398264483840, 12676506002282327791964489728, 21970710674130840874443091905462272, 154866286100907105149651981766316633972736, ... }

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