COMPUTATION OF DILATED KRONECKER COEFFICIENTS

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Abstract. The computation of Kronecker coefficients is a challenging problem. In this paper we present an approach to it based on methods from symplectic geometry and residue calculus. We outline a general algorithm for the problem and then illustrate its effectiveness in several interesting examples. As a byproduct of our algorithm, we are also able to compute several Hilbert series.

Contents

Introduction

Let $N = n_1 n_2 \cdots n_s$ where $n_1, n_2, \ldots, n_s$ are positive integers, and write $\mathbb{C}^N = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_s}$. Endow each $\mathbb{C}^{n_i}$ with the usual Hermitian inner product. Then the group $U(n_1) \times \cdots \times U(n_s)$ acts unitarily on the complex vector space $\mathbb{C}^N$ via the exterior tensor product, i.e. $(k_1, \ldots, k_s)(v_1 \otimes \cdots \otimes v_s) = k_1 v_1 \otimes \cdots \otimes k_s v_s$. By means of this action we obtain a representation of $U(n_1) \times \cdots \times U(n_s)$ on $Sym(\mathbb{C}^N)$, the full symmetric algebra of $\mathbb{C}^N$. The aim of this article is to give an algorithm to compute the multiplicity of an irreducible representation of $U(n_1) \times \cdots \times U(n_s)$ in $Sym(\mathbb{C}^N)$. As a corollary, we obtain an algorithm to compute the corresponding Hilbert series.

The algebra $Sym(\mathbb{C}^N)$ has a natural grading by degree. For notational convenience we let $Sym^c(\mathbb{C}^N)$ be the space of symmetric tensors of degree c, so we have

$$Sym(\mathbb{C}^N) = \oplus_{c=0}^{\infty} Sym^c(\mathbb{C}^N).$$

The action of $U(n_1) \times \cdots \times U(n_s)$ provides a refinement of this, namely,

$$Sym(\mathbb{C}^N) = \oplus g(\mu_1, \mu_2, \ldots, \mu_s) V_{\mu_1}^{U(n_1)} \otimes \cdots \otimes V_{\mu_s}^{U(n_s)}.$$

Here μ_j are polynomial representations of $U(n_j)$ indexed by Young diagrams with n_j rows. The content of a Young diagram is the number of its boxes, and we denote the content of the Young diagram μ by $|\mu|$. If one considers the action of the center of $U(n_j)$, one sees that all diagrams μ_j, $j = 1, \ldots, s$ such that $V_{\mu_1}^{U(n_1)} \otimes \cdots \otimes V_{\mu_s}^{U(n_s)}$ occurs in $Sym^c(\mathbb{C}^N)$ have the same content c. Hence the diagrams also index irreducible representations π_{μ_j} of the symmetric group $\mathfrak{S}_c$. By Schur duality, $g(\mu_1, \mu_2, \ldots, \mu_s)$ is then the multiplicity of the trivial representation of the symmetric group $\mathfrak{S}_c$ in $\pi_{\mu_1} \otimes \cdots \otimes \pi_{\mu_s}$.

The numbers $g(\mu_1, \mu_2, \ldots, \mu_s)$ are called *Kronecker coefficients,* while the function $k \mapsto g(k\mu_1, k\mu_2, \ldots, k\mu_s)$, $k \in \{0, 1, 2, \ldots\}$, gives the *dilated Kronecker coefficients*. It follows from the $[Q, R] = 0$ theorem, obtained by Meinrenken-Sjamaar [28], that the function $k \mapsto g(k\mu_1, k\mu_2, \ldots, k\mu_s)$ is given by a quasi-polynomial formula for $k \geq 1$.

In this paper we shall present an algorithm that computes the function $g(\mu_1, \mu_2, \ldots, \mu_s)$ locally. More precisely, given as input a fixed s-tuple $(\mu_1^0, \mu_2^0, \ldots, \mu_s^0)$, our algorithm produces a symbolic function which coincides with the function $g(\mu_1, \mu_2, \ldots, \mu_s)$ in a conic neighborhood of this fixed s-tuple. In particular since the algorithm is valid in a conic neighborhood, we can compute the *dilated Kronecker coefficients* $k \mapsto g(k\mu_1, k\mu_2, \ldots, k\mu_s)$ as a quasi-polynomial function of k. The Maple implementation of this algorithm is available at [38].

We now describe briefly our approach. For $s = 2$, the decomposition of $Sym(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$ is given by the Cauchy formula (see Ex. 10). For $s > 2$, we use the Cauchy formula to reduce the number s of factors. First we observe (see Ex. 5) that it suffices to consider the case where $n_1 = max_i(n_i)$, and when $n_1 \leq M = n_2 n_3 \cdots n_s$. Consider the obvious homomorphism of $K = U(n_2) \times \cdots \times U(n_s)$ into $G = U(M)$.

Using the Cauchy formula, we can write

$$Sym(\mathbb{C}^N) = Sym(\mathbb{C}^{n_1} \otimes \mathbb{C}^M) = \oplus_{\mu_1 \in P\Lambda_{U(n_1), \geq 0}} V_{\mu_1}^{U(n_1)} \otimes V_{\tilde{\mu}_1}^{U(M)}$$

where $P\Lambda_{U(n_1),\geq 0}$ indexes the finite dimensional irreducible polynomial representations of $U(n_1)$ and $\tilde{\mu_1}$ is the polynomial representation of $G = U(M)$ obtained from μ_1 by adding more zeros on the right (see Ex.4). Now restrict $V_{\tilde{\mu}_1}^{U(M)}$ to K and obtain

$$V_{\tilde{\mu}_1}^{U(M)} = \oplus_{\mu_2\in\hat{U}(n_2),\dots,\mu_s\in\hat{U}(n_s)} m_{G,K}(\tilde{\mu_1}, \mu_2\otimes\cdots\otimes\mu_s) V_{\mu_2}^{U(n_2)}\otimes\cdots\otimes V_{\mu_s}^{U(n_s)}$$

where $m_{G,K}(\tilde{\mu_1}, \mu_2 \otimes \cdots \otimes \mu_s)$ is the branching coefficient computing the multiplicity of the representation $\mu_2 \otimes \cdots \otimes \mu_s$ in the restriction of $V^G_{\tilde{\mu_1}}$ to $K = U(n_2) \times U(n_3) \times \cdots \times U(n_s)$ via the specified action.

Then we have

Corollary 1.

$$g(\mu_1, \dots, \mu_s) = \begin{cases} m_{G,K}(\tilde{\mu_1}, \mu_2 \otimes \cdots \otimes \mu_s) & \textit{if } |\mu_1| = |\mu_2| = \cdots = |\mu_s|, \\ 0 \quad \textit{otherwise} \end{cases}$$

Thus the objective of the paper is to give a formula for the branching coefficient and an algorithm to implement it.

Our expression for $m_{G,K}(\lambda, \mu)$, in a general context of branching rules, is the content of Theorem 31. The starting point for the computation is the notion of Jeffrey-Kirwan residue that allows us to produce a symbolic function of (λ, μ) coinciding with $m_{G,K}(\lambda, \mu)$ in a conic neignborhood of a given couple (λ^0, μ^0). This problem is part of the more general problem of computing (symbolically) multiplicities. Our algorithm is based on a variation of the Kostant multiplicity formula for a weight [23], together with Szenes-Vergne [35] iterated residues formula to compute partition functions. However, we use the results of Meinrenken-Sjamaar on piecewise quasi-polynomial behavior of multiplicity functions in order to justify our method.

Some of the results herein were presented at the conference Quantum Marginals (2013) (recorded on video) at the Isaac Newton Institute, Oxford. The preprint [1] is an extended version of the talk of the second author at this conference, and is not intended to be published. However, it may be interesting to the reader to consult this survey paper, since various aspects of the theory that come into play (Hamiltonian geometry, convexity, quasi-polynomial behavior, Jeffrey-Kirwan residues) are presented there, together with an extended bibliography. In this article, our focus is the application of our general methods for computing branching coefficients to the particularly challenging case of Kronecker coefficients.

Introductory examples

Throughout the text we have included many examples. Some are known and are included to show the consistency of our results with other techniques of computation, and several new examples are included to show the power, and the limits, of our computational approach. For example, we consider the case of $n_1 = 6, n_2 = 3, n_3 = 2$. Then given as input a point $(\alpha^0, \beta^0, \gamma^0)$, where α^0 is a Young diagram with 6 rows, β^0 with 3 rows and γ^0 with 2 rows, we can effectively compute symbolically the Kronecker coefficient $g(\alpha, \beta, \gamma)$ in a conic neighborhood of $(\alpha^0, \beta^0, \gamma^0)$.

We also can compute the dilated Kronecker coefficient $g(k\alpha, k\beta, k\gamma)$ with α, β, γ Young diagrams with 3 rows. For general α, β, γ with 3 rows, we obtain a quasi-polynomial of degree 11 (with coefficients periodic functions of period at most 12, see Ex. 38). When α, β, γ are special, the degree might be much smaller.

The dilated rectangular Kronecker coefficients, i.e. those involving partitions of rectangular shape, are of special interest because of their direct relation with invariant theory. If we consider the group $U(n_1) \times U(n_2) \times \cdots \times U(n_s)$ acting on $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_s}$ and characters $\chi_u = [p_u, \ldots, p_u]$, $1 \leq u \leq s$, with equal content $c := p_u n_u$, then $g(k\chi_1, k\chi_2, \ldots, k\chi_s)$ is equal to

$$\dim \left[Sym^{kc} \left(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_s} \right)^{SL(\mathbb{C}^{n_1}) \times SL(\mathbb{C}^{n_2}) \times \cdots \times SL(\mathbb{C}^{n_s})} \right].$$

Thus the series $R(t) = \sum_{k=0}^{\infty} g(k\chi_1, k\chi_2, \ldots, k\chi_s) t^k$ is the Hilbert series of the ring of invariant polynomials under the action considered. This is presented in Ex. 6.

Particularly challenging examples are the Hilbert series for 5-qubits given by [25] and the Hilbert series for entanglement of 4-qubits given by [41]. We compute these (and correct a misprint in [25]) with our residue techniques in Section 5.2, especially Ex. 39.

We conclude this introduction with two examples, first a classical one, (see [19], [40], [20]), then a new one.

Example 2.

The dilated Kronecker coefficient $m(k) \stackrel{def}{=} g(k[1,1,1], k[1,1,1], k[1,1,1])$, for $\chi = [1,1,1]$, corresponds to the Hilbert series of the ring of invariants of $SL(\mathbb{C}^3) \times SL(\mathbb{C}^3) \times SL(\mathbb{C}^3)$ in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. An efficient way to represent periodic functions is by using step-polynomials (see [37]). In this representation, $m(k)$ is given by the following quasi-polynomial

$$1 - \frac{3}{2}\left\{\frac{1}{3}k\right\} + \frac{3}{2}\left\{\frac{1}{3}k\right\}^2 - \frac{3}{2}\left\{\frac{1}{2}k\right\} - \left\{\frac{3}{4}k\right\}^2 + \left\{\frac{3}{4}k\right\}\left\{\frac{1}{2}k\right\} + \left\{\frac{1}{2}k\right\}^2 +$$

$$\left(\frac{1}{4} - \frac{1}{4}\left\{\frac{1}{2}k\right\}\right) k + \frac{1}{48}k^2$$

Here for $s \in \mathbb{R}$, $\{s\} = s - \lfloor s \rfloor \in [0,1)$ where $\lfloor s \rfloor$ denotes the largest integer smaller or equal to s. It is easy to check that (fortunately) the result of our algorithm agrees with the determination of the ring of invariants, $[\oplus_{k=0}^{\infty} Sym^{3k}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)]^{SL(\mathbb{C}^3)\times SL(\mathbb{C}^3)\times SL(\mathbb{C}^3)}$, which is freely generated with generators in degree $2,3,4$. Indeed

$$\sum_{k=0}^{\infty} m(k)t^k = \frac{1}{(1-t^2)(1-t^3)(1-t^4)}.$$

□

Example 3.

Another example is

$$m(k) = g(k[3,3,3,3], k[4,4,4], k[4,4,4])$$
$$= \dim \left[Sym^{12k}(\mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)^{SL(\mathbb{C}^4)\times SL(\mathbb{C}^3)\times SL(\mathbb{C}^3)} \right].$$

Our algorithm gives the Hilbert series

$$\sum_{k=0}^{\infty} m(k)t^k = \frac{1+t^9}{(1-t)\left(1-t^2\right)^2\left(1-t^3\right)\left(1-t^4\right)}.$$

□

Outline of the article

In Section 1, we consider the action of a connected compact Lie group K in an Hermitian finite dimensional space $\mathcal{H}$. Then

$$Sym(\mathcal{H}) = \sum_{\lambda \in \hat{K}} m_K^{\mathcal{H}}(\lambda) V_\lambda^K.$$

We introduce the Kirwan cone, which is the asymptotic support of the multiplicity function $\lambda \mapsto m_K^{\mathcal{H}}(\lambda)$. The Kirwan cone is a polyhedral cone. A facet (that is a face of codimension one) of this cone generates an hyperplane that we call a wall of $C_K(\mathcal{H})$.

In Section 2, we define topes and Orlik-Solomon bases. We discuss quasi-polynomial functions and recall the Szenes-Vergne iterated residue formula for the piecewise quasi-polynomial function $m_K^{\mathcal{H}}(\lambda)$, when K is a torus. We discuss the Meinrenken-Sjamaar theorem on the piecewise quasi-polynomial behavior of $m_K^{\mathcal{H}}(\lambda)$, when K is a compact connected Lie group. This theorem is important for our present work since it allows us to compute $m_{\mathcal{H}}^K(\lambda)$ by a deformation argument. We state some corollaries on the degrees of the function $k \mapsto m_K^{\mathcal{H}}(k\lambda)$,on the interior of the Kirwan cone, as well as on faces.

In Section 3, we consider the general problem of branching rules: given a homomorphism $K \to G$ and an irreducible representation

V_λ^G of G, decompose it into irreducible representations of K, $V_\lambda^G = \oplus m_{G,K}(\lambda,\mu) V_\mu^K$. In Theorem 23, we recall some of the qualitative properties of this function, as follows from Meinrenken-Sjamaar. This again allows us to compute $m_{G,K}(\lambda,\mu)$ by deformation arguments. We discuss the regions where $m_{G,K}(\lambda,\mu)$ is given by a quasi-polynomial function. Our main result is Theorem 23. As we already said, essentially we use the Kostant multiplicity formula, together with iterated residues method to compute $m_{G,K}(\lambda,\mu)$. However, in some examples the stabilizer of λ in the Weyl group of G is large, and we take advantage of this situation. When some roots of G restrict to 0 on Cartan subalgebras of K, again we use residue techniques.

In Section 4, we review the algorithm that we use for computing Kronecker coefficients.

In Section 5, we list many examples. As for the computations in [12] of dilated Littlewood-Richardson coefficients $c(k\lambda, k\mu, k\nu)$, our method to compute $g(k\lambda_1, k\lambda_2, \cdots, k\lambda_s)$ is efficient for small $n_1, n_2, \ldots, n_s$, but does not depend on the size of the λ_i. More precisely, fixing the number n_i of rows, the algorithm is polynomial in terms of $\lambda_1, \lambda_2, \ldots, \lambda_s$ (and relatively quick).

1. Towards a multiplicity formula: the tools

We use a general setting of notations. Let K be a compact connected Lie group and T_K be a maximal torus of K. We denote by $\mathfrak{k}$ and $\mathfrak{t}_\mathfrak{k}$ the corresponding Lie algebras. We use the upper index $*$ to denote dual space, the lower index $\mathbb{C}$ to denote complexification, for example we write $\mathfrak{t}_\mathfrak{k}^*$ and $\mathfrak{k}_\mathbb{C}$.

Denote by $\mathcal{W}_\mathfrak{k}$ the Weyl group, and by $w \mapsto \epsilon(w) = \det_{\mathfrak{t}_\mathfrak{k}} w$ its sign representation.

The weight lattice Λ_K of T_K is a lattice in $i\mathfrak{t}_\mathfrak{k}^*$, $i = \sqrt{-1}$. If $\lambda \in \Lambda_K \subset i\mathfrak{t}_\mathfrak{k}^*$, it determines a one dimensional representation of T_K by $t \mapsto e^{\langle \lambda, X\rangle}$, with $t = \exp X$, $X \in \mathfrak{t}_\mathfrak{k}$. As λ takes imaginary values on $\mathfrak{t}_\mathfrak{k}$, $e^{\langle \lambda, X\rangle}$ is of modulus 1.

We denote by $\Gamma_K \subset i\mathfrak{t}_\mathfrak{k}$ the dual lattice of Γ_K: if $\lambda \in \Lambda_K$, $\gamma \in \Gamma_K$, then $\langle \lambda, \gamma\rangle$ is an integer.

Let $\Delta_\mathfrak{k} \subset i\mathfrak{t}_\mathfrak{k}^*$ be the root system for $\mathfrak{k}$ with respect to $\mathfrak{t}_\mathfrak{k}$. If $\alpha \in \Delta_\mathfrak{k}$, its coroot H_α is in $i\mathfrak{t}_\mathfrak{k}$, and $\langle \alpha, H_\alpha\rangle = 2$. Fix $\Delta_\mathfrak{k}^+$, a positive system for $\Delta_\mathfrak{k}$, and let

$$i\mathfrak{t}_{\mathfrak{k},\geq 0}^* = \{\xi, \langle \xi, H_\alpha\rangle \geq 0, \alpha \in \Delta_\mathfrak{k}^+\}$$

be the corresponding positive Weyl chamber. Let $\rho_\mathfrak{k} = \frac{1}{2}\sum_{\alpha\in\Delta_\mathfrak{k}^+} \alpha$.

We denote by $\Lambda_{K,\geq 0}$ the "cone" of dominant weights, that is the set $\Lambda_K \cap i\mathfrak{t}_{\mathfrak{k},\geq 0}^*$. We parameterize $\hat{K}$, the set of classes of irreducible finite dimensional representations of K, by $\Lambda_{K,\geq 0}$: given $\lambda \in \Lambda_{K,\geq 0}$, we denote by V_λ^K the corresponding irreducible representation of K with highest weight λ.

When the group K is understood, we denote Λ_K simply by Λ, T_K by T, $\mathfrak{t}_\mathfrak{k}$ by $\mathfrak{t}$, etc.

Finally, when dealing with different groups $K, G, ..$ etc... as in Section 3, then we will use corresponding German letters for the Lie algebras, $\mathfrak{k}, \mathfrak{g}, \ldots$, and subscripts to distinguish the objects: $\Lambda_K, \Lambda_G, \mathfrak{t}_\mathfrak{k}, \mathfrak{t}_\mathfrak{g}, \ldots$.

Example 4.

We consider the case $K = U(n)$ and $T \subset K$ the torus consisting of the diagonal matrices. Then the Lie algebra $\mathfrak{k}$ consists of the $n \times n$ anti Hermitian matrices and $i\mathfrak{k}$ is the space of Hermitian matrices. If we identify $\mathfrak{k}$ and $\mathfrak{k}^*$ via the bilinear form $\mathrm{Tr}(AB)$, then $\mathfrak{t} = \mathfrak{t}^*$ is the set of diagonal anti Hermitian matrices. The positive Weyl chamber is $i\mathfrak{t}^*_{\geq 0} = \{\xi = [\xi_1, \xi_2, \ldots, \xi_n]\}$ with $\xi_j \in \mathbb{R}$ and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ where ξ represents the Hermitian matrix with diagonal entries ξ_j. Denote by $\Lambda_{U(n)} = \{\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]\}$ with $\lambda_j \in \mathbb{Z}$ the weight lattice of T and by $\Gamma_{U(n)} \subset i\mathfrak{t}$ the dual lattice.

The "cone" of dominant weights is $\Lambda_{U(n),\geq 0} = \{\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]\}$ with $\lambda_j \in \mathbb{Z}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

If $\lambda \in \Lambda_{U(n),\geq 0}$ is such that $\lambda_n \geq 0$, then λ indexes a finite dimensional irreducible polynomial representation of $GL(n, \mathbb{C})$. The corresponding subset of $\Lambda_{U(n),\geq 0}$ will be denoted by $P\Lambda_{U(n),\geq 0}$ (P for polynomial). If $\lambda \in P\Lambda_{U(n),\geq 0}$, we also identify λ to a Young diagram with n rows. Recall that the content $|\lambda|$ of the corresponding diagram is the number of its boxes, that is $|\lambda| = \sum_i \lambda_i$. The dominant weight $[k, k, \ldots, k]$ corresponds to a rectangular Young diagram with k columns and n rows and indexes the one-dimensional representation $\det(g)^k$ of $U(n)$.

Assume now $N \geq n$, then there is a natural injection from $P\Lambda_{U(n),\geq 0}$ to $P\Lambda_{U(N),\geq 0}$ obtained just by adding more zeros on the right of the sequence λ. We denote by $\tilde{\lambda}$ the new sequence so obtained.

Define similarly $Pi\mathfrak{t}^*_{\mathfrak{u}(n)} = \{\xi \in i\mathfrak{t}^*_{\geq 0}$ with $\xi_n \geq 0\}$ and $\tilde{\xi}$.

□

Let $\mathcal{H}$ be a finite dimensional Hermitian vector space provided with a representation of K by unitary transformations. Assume (temporarily) that K contains the subgroup of homotheties $\{e^{i\theta} Id_\mathcal{H}\}$. Consider $Sym(\mathcal{H})$, the space of symmetric tensors, so we have

$$Sym(\mathcal{H}) = \oplus_{\lambda \in \hat{K}} m_K^\mathcal{H}(\lambda) V_\lambda^K$$

where $m_K^{\mathcal{H}}(\lambda)$, the multiplicity of V_λ^K in $Sym(\mathcal{H})$, is finite. We also write $Sym(\mathcal{H}) = \oplus_\mu m_{T_K}^{\mathcal{H}}(\mu)e^\mu$, where $m_{T_K}^{\mathcal{H}}(\mu)$ is the multiplicity of the weight μ. The relation between the K and the T_K multiplicities is given by the following formula:

$$m_K^{\mathcal{H}}(\lambda) = \sum_{w\in\mathcal{W}_{\mathfrak{k}}} \epsilon(w) m_{T_K}^{\mathcal{H}}(\lambda + \rho_{\mathfrak{k}} - w\rho_{\mathfrak{k}}). \tag{1}$$

Before going on, we give two examples. The first one is the main example we will be interested in this paper and the second is related to the computation of Hilbert series.

Example 5. *(Kronecker example)*

Consider $\mathcal{H} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_s}$ with action of $K = U(n_1) \times \cdots \times U(n_s)$ on $\mathcal{H}$. Denote by $M = n_2 n_3 \cdots n_s$. In computing Kronecker coefficients, we may assume $n_1 \leq M$, and that n_1 is the maximum of the n_i. Indeed, if $n_1 \geq M$, by Cauchy formula, the multiplicities $g(\mu_1, \mu_2, \ldots, \mu_s)$ stabilize in the sense that $g(\mu_1, \mu_2, \ldots, \mu_s)$ is non zero only if $\mu_1 = \tilde{\nu}_1$ is obtained from an element ν_1 in $P\Lambda_{U(M),\geq 0}$, by adding more zeroes on its right, and

$$g(\mu_1, \mu_2, \ldots, \mu_s) = g(\nu_1, \mu_2, \ldots, \mu_s).$$

Thus it is sufficient to study Kronecker coefficients in the case where $n_1 \leq M = n_2 n_3 \cdots n_s$ (n_i is the number of rows of the Young diagrams corresponding to μ_i).
$\square$

Example 6. *Hilbert series*

Assume that $\mathfrak{k} = \mathfrak{z} \oplus [\mathfrak{k}, \mathfrak{k}]$, and assume that the center $\mathfrak{z} = \mathbb{R}J$ of $\mathfrak{k}$ acts by the homothety on $\mathcal{H}$. Consider $\chi \in \Lambda_K$ such that $\chi(iJ) = 1$ and $\chi = 0$ on $i(\mathfrak{t}_{\mathfrak{k}} \cap [\mathfrak{k}, \mathfrak{k}])$. Then $m_K^{\mathcal{H}}(k\chi) = \dim[S^k(\mathcal{H})]^{[K,K]}$. So the series $R(t) = \sum_{k=0}^{\infty} m_K^{\mathcal{H}}(k\chi)t^k$ is the Hilbert series of the ring of invariant polynomials under the action of $[K_{\mathbb{C}}, K_{\mathbb{C}}]$. This is a Gorenstein ring, so its Hilbert series is of the form $\frac{P(t)}{\prod_{j=1}^{N}(1-t^{a_j})}$, where $P(t)$ is a palindromic polynomial, ([30]). Furthermore, it follows from [28] that the degree of $P(t)$ is strictly less than $\sum_j a_j$.
$\square$

Equip $\mathcal{H}$ with a K invariant Hermitian form $\langle -, - \rangle$ so that $\langle Xv, v\rangle$ is purely imaginary when $X \in \mathfrak{k}$. The K action on $\mathcal{H}$ admits a *moment map,* (in the sense of symplectic geometry [14]) $\Phi_K : \mathcal{H} \to i\mathfrak{k}^*$ given by

$$\Phi_K(v)(X) = \langle Xv, v\rangle, v \in \mathcal{H}, X \in \mathfrak{k}.$$

We consider $i\mathfrak{t}^*_{\mathfrak{k},\geq 0}$ as a subset of $i\mathfrak{k}^*$. The *Kirwan cone* is the intersection of the image of the moment map with the positive Weyl chamber, $C_K(\mathcal{H}) = \Phi_K(\mathcal{H}) \cap i\mathfrak{t}^*_{\mathfrak{k},\geq 0}$.

Kirwan convexity theorem implies that $C_K(\mathcal{H})$ is a rational polyhedral cone. The cone $C_K(\mathcal{H})$ is related to the multiplicities through the following basic result, which is a particular case of Mumford theorem [29](a proof of Mumford theorem, following closely Mumford argument, can be found in [6]).

Proposition 7. *We have $m_K^{\mathcal{H}}(\lambda) = 0$ if $\lambda \notin C_K(\mathcal{H})$. Conversely, if λ is a dominant weight belonging to $C_K(\mathcal{H})$, there exists an integer $k > 0$ such that $m_K^{\mathcal{H}}(k\lambda)$ is non zero.*

Thus the support of the function $m_K^{\mathcal{H}}(\lambda)$ is contained in the Kirwan cone $C_K(\mathcal{H})$ and its asymptotic support is exactly $C_K(\mathcal{H})$.

Remark 8.

As $C_K(\mathcal{H})$ is a rational polyhedral cone, it can be described by inequations determined by a finite number of elements $X_a \in \Gamma_K$ as: $C_K(\mathcal{H}) = \{\xi \in i\mathfrak{t}^*_{\mathfrak{k},\geq 0} | \langle X_a, \xi\rangle \geq 0, \forall a\}$. It is in general quite difficult to determine the explicit inequations of the cone $C_K(\mathcal{H})$. An algorithm to describe the inequations of this cone, based on Ressayre's notion of dominant pairs [33], is given in Vergne-Walter [39].

□

Define $\mathcal{H}_{pure} = \{v \in \mathcal{H}, \langle v, v\rangle = 1\}$, the set of elements of $\mathcal{H}$ of norm 1. Such an element is called a pure state. A pure state in the space $\mathbb{C}^2$ is called a qubit, and a pure state in $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ is called a N-qubit. The *Kirwan polytope* is the rational polytope defined by: $\Delta_K(\mathcal{H}) = \Phi_K(\mathcal{H}_{pure})$ and $C_K(\mathcal{H}) = \mathbb{R}_{\geq 0}\Delta_K(\mathcal{H})$ is the cone over the Kirwan polytope.
Remark that it is not always true that the cone $C_K(\mathcal{H})$ has not empty interior in $i\mathfrak{t}^*_{\mathfrak{k},\geq 0}$, (see [1], Ex. 14).

Example 9.

Let us write explicitly Higuchi-Sudbery-Szulc [17] description of the Kirwan cone for $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ and $K = (U(2))^{\times N}$.

Consider $\lambda_1 = [\lambda_1^1, \lambda_2^2], \ldots, \lambda_N = [\lambda_1^N, \lambda_2^N]$ a sequence of N elements of $Pi\mathfrak{t}^*_{\mathfrak{u}(2),\geq 0}$ (that is $\lambda_j^1 \geq \lambda_j^2 \geq 0$). Then $(\lambda_1, \ldots, \lambda_N) \in C_K(\mathcal{H})$ if and only if, for any $j = 1, 2, \ldots, N$, $\lambda_j^2 \leq \sum_{k\neq j} \lambda_k^2$ and $\lambda_1^1 + \lambda_1^2 = \lambda_2^1 + \lambda_2^2 = \cdots = \lambda_N^1 + \lambda_N^2$.

□

We conclude this section with one more example of the connection between multiplicities and Kirwan cone.

Example 10. *(The Cauchy formula)*

Let N, n be positive integers, and assume $N \geq n$. Let $\mathcal{H} = \mathbb{C}^n \otimes \mathbb{C}^N$ under the action of $K = U(n) \times U(N)$. As we already discussed, the decomposition of $Sym(\mathbb{C}^n \otimes \mathbb{C}^N)$ with respect to $\mathrm{U}(n) \times U(N)$ (see [24], page 63) is given by *Cauchy formula*:

$$Sym(\mathbb{C}^n \otimes \mathbb{C}^N) = \oplus_{\lambda \in P\Lambda_{U(n),\geq 0}} V_\lambda^{U(n)} \otimes V_{\tilde{\lambda}}^{U(N)}. \tag{2}$$

Using the Hermitian inner product, we identify $A \in \mathcal{H}$ to a matrix $A : \mathbb{C}^n \to \mathbb{C}^N$. Then the moment map $\Phi_K : \mathcal{H} \to i\mathfrak{k}^*$ is given by $\Phi_K(A) = [AA^*, A^*A]$ with value Hermitian matrices of size n and size N respectively. We can identify the Kirwan cone $C_K(\mathcal{H})$ as the "diagonal" $(\xi, \tilde{\xi})$ with $\xi \in Pi\mathfrak{t}^*_{\mathfrak{u}(n),\geq 0}$, thus seeing that the multiplicity function determined in Cauchy formula (2) is supported exactly on the set $\Lambda_K \cap C_K(\mathcal{H})$ (and with value 1).

□

2. Multiplicities and Partitions functions

2.1. **Topes, Iterated residues and Orlik-Solomon bases.** Let E be a real vector space.

First, recall the notion of iterated residue. If f is a meromorphic function in one variable z, consider its Laurent series $\sum_n a_n z^n$ at $z = 0$. The coefficient of z^{-1} is denoted by $\mathrm{Res}_{z=0} f$.

Let $r = \dim E$. Consider a basis of E and an order on it: $\overrightarrow{\sigma} = [\alpha_1, \alpha_2, \ldots, \alpha_r]$. For $z \in E^*_{\mathbb{C}}$, let $z_j = \langle z, \alpha_j \rangle$. Then $(z_1, z_2, \ldots, z_r)$ are coordinates for $E^*_{\mathbb{C}} \sim \mathbb{C}^r$. Express a meromorphic function $f(z)$ on $E^*_{\mathbb{C}}$ with poles on a union of hyperplanes as a function $f(z) = f(z_1, z_2, \ldots, z_r)$, (in particular f may have poles on $z_j = 0$). Define the **iterated residue** functional associated to $\overrightarrow{\sigma}$ by:

$$Res_{\overrightarrow{\sigma}}(f(z)) := \mathrm{Res}_{z_1=0}(\mathrm{Res}_{z_2=0} \cdots (\mathrm{Res}_{z_r=0} f(z_1, z_2, \ldots, z_r)) \cdots). \tag{3}$$

Let $\Psi = [\psi_1, \ldots, \psi_N]$ be a finite list of vectors of E. We say that a hyperplane $H \subset E$ is Ψ-admissible if H is generated by elements of Ψ. We denote by $\mathcal{A}(\Psi)$ the set of admissible hyperplanes. We say that $\xi \in E$ is Ψ-regular if ξ doesn't belong to any hyperplane in the set of admissible hyperplanes $\mathcal{A}(\Psi)$ and we say that τ is a Ψ-tope if τ is a connected component of the complement of the union of the Ψ-admissible hyperplanes.

Let σ be a sublist of Ψ such that elements of σ form a basis of E. We say that σ is an *ordered basis of* Ψ . The underlying set to the list σ will be called simply a basis of Ψ. We denote by $Cone(\sigma)$ the cone generated by elements of σ.

Example 11.

Consider $E = \mathbb{R}\epsilon_1 \oplus \mathbb{R}\epsilon_2$. Let $\Psi = [\psi_1, \psi_2, \psi_3] = [\epsilon_1, \epsilon_2, \epsilon_1 + \epsilon_2]$, then $\{[\psi_1, \psi_2], [\psi_1, \psi_3], [\psi_2, \psi_3]\}$ is the set of ordered bases of Ψ.
$\square$

Assume now that the elements in Ψ **generates a lattice** L in E. For σ a basis of Ψ, let d_σ be the smallest integer such that $d_\sigma L$ is contained in the lattice $\mathbb{Z}\sigma$ generated by σ. Then we define the *index* of Ψ (with respect to L) as $q(\Psi) :=$ least common multiple$\{d_\sigma, \sigma$ basis of $\Psi\}$.

Let $\overrightarrow{\sigma} = [\psi_{i_1}, \psi_{i_2}, \dots, \psi_{i_r}]$ be an **ordered basis** of Ψ. Then $\overrightarrow{\sigma}$ is an **Orlik Solomon basis**, OS in short, if for each $1 \leq l \leq r$, there is no $j < i_l$ such that the elements $\psi_j, \psi_{i_l}, \dots, \psi_{i_r}$ are linearly dependent. Denote by $\mathcal{OS}(\Psi)$ the set of OS bases. Note that the notion of Orlik Solomon basis depends on the order on Ψ.

Example 12.

We continue with Ex. 2.1. We compute $\mathcal{OS}(\Psi) = \{[\psi_1, \psi_2], [\psi_1, \psi_3]\}$.
$\square$

Definition 13. *If τ is a Ψ-tope, we denote by $\mathcal{OS}(\Psi, \tau) = \{\overrightarrow{\sigma} \in \mathcal{OS}(\Psi),\ \tau \subset Cone(\sigma)\}$.*

The set $\mathcal{OS}(\Psi, \tau)$ is called the set of **OS adapted bases to** τ.

Example 14.

We continue with Ex. 2.1. Then $\tau = \mathbb{R}_{>0}\epsilon_2 \oplus \mathbb{R}_{>0}(\epsilon_1 + \epsilon_2)$ is a Ψ-tope, and $\mathcal{OS}(\Psi, \tau) = \{[\psi_1, \psi_2]\}$. $\square$

For an algorithmic method to compute $\mathcal{OS}(\Psi, \tau(v))$, see [1] Section 4.9.6. The method is based on the notion of Maximal Nested Sets (MNS) of De Concini-Procesi, [13], and developed in [3]. Here v is a Ψ-regular element, and $\tau(v)$ the unique tope which contains v.

2.2. **Quasi-polynomial function.** We now describe the nature of the function $m_K^{\mathcal{H}}(\lambda)$ on $C_K(\mathcal{H})$. In particular the results will apply to Kronecker coefficients.

Let L be a lattice in a real vector space E. Given an integer q, a function c on L will be called a periodic function on L of period q, if $c(\lambda + q\nu) = c(\lambda)$ for all λ, μ in L. We say that c is periodic if there exists a q such that c is periodic of period q. If $q = 1$, then c is a constant function on L.

Let $u \in L$ and q an integer. The restriction to a coset $u + qL$ of a polynomial function on E will be called a polynomial function on the coset $u + qL$.

A **quasi-polynomial function** on L is a function on L which is a linear combination of products of polynomial functions on L with periodic functions. In other words a quasi polynomial function on L can be written as $p(\lambda) = \sum_i c_i(\lambda)p_i(\lambda)$ where p_i are polynomial and the functions c_i are periodic. If all functions $c_i(\lambda)$ are periodic of period q, we say that p is quasi-polynomial of period q. In this case, for any $\lambda_0 \in L$, the function $\lambda \mapsto p(\lambda_0 + q\lambda)$ is a polynomial function on L. So we can represent a periodic polynomial function of period q as a family of polynomials indexed by L/qL. If q is very large, the above description is not efficient (the numbers of cosets being quite large). In this present work, we will only have to consider relatively small periods q.

The space of quasi-polynomial functions is graded: we say that p is homogeneous of degree k if the polynomials p_j are homogeneous of degree k. As for polynomials, we say that p is of degree k if p is a sum of homogeneous terms of degree less or equal to k, and the term of degree k is non zero.

Example 15.

$$m(k) = \frac{1}{2}k^2 + k + \frac{3}{4} + \frac{1}{4}(-1)^k$$

is a quasi-polynomial function of $k \in \mathbb{Z}$, of degree 2 and period 2. On each of the 2 cosets the quasi-polynomial function $m(k)$ coincides with a polynomial $m^{[i]}(k)$ defined by

$$m(k) = \begin{cases} m_0(k) = \frac{1}{2}k^2 + k + \frac{3}{4} + \frac{1}{4} & \text{if } k = 0 \pmod 2 \\ m_1(k) = \frac{1}{2}k^2 + k + \frac{3}{4} - \frac{1}{4} & \text{if } k = 1 \pmod 2 \end{cases}$$

□

In practice, the quasi-polynomial p will be naturally obtained as a sum of quasi-polynomial functions $p_1, p_2, \ldots, p_u$ of periods $q_1, q_2, \ldots, q_u$. So p is of period q where q is the least common multiple of $q_1, q_2, \ldots, q_u$. Furthermore, in our examples, when the period q_i is large, the degree of the corresponding quasi polynomial p_i is small. So it is already more efficient to keep p as represented as $\sum p_i$, the number of cosets needed to describe each p_i being q_i, since $\sum q_i$ is usually much smaller that q. We thus will say that the set of periods of the quasi-polynomial function p is the set $\{q_1, q_2, \ldots, q_u\}$. For example

$$m(k) = \frac{1}{2}k^2 + k + \frac{3}{4} + \frac{1}{4}(-1)^k$$

is the sum of $p_1(k) = \frac{1}{2}k^2 + k + \frac{3}{4}$ of period 1 and degree 2 and of $p_2(k) = \frac{1}{4}(-1)^k$ of period 2 and degree 0. So the set of periods of m is $\{1, 2\}$.

In the Kronecker case, we do not know the set of periods of the dilated Kronecker coefficient. Here are some examples.

Example 16.

(See Section 5). For the case $n_1 = n_2 = n_3 = 3$, the function $g(k\lambda, k\mu, k\nu)$ is a quasi polynomial function of k with set of periods included in $\{1, 2, 3, 4\}$ leading to polynomial behavior on cosets $f + 12\mathbb{Z}$.

For the 4-qubits case $n_1 = n_2 = n_3 = n_4 = 2$, the function $g(k\lambda, k\mu, k\nu)$ is a quasi-polynomial with set of periods included in $\{1, 2, 3\}$ leading to polynomial behavior on cosets $f + 6\mathbb{Z}$.

For the 5-qubits case $n_1 = n_2 = n_3 = n_4 = n_5 = 2$, the function $g(k\lambda, k\mu, k\nu)$ is a quasi-polynomial with set of periods included in $\{1, 2, 3, 4, 5\}$ leading to polynomial behavior on cosets $f + 60\mathbb{Z}$.

$\square$

In the case of one variable, we can give the following characterization of quasi-polynomial functions $p(k)$. If the function $p(k)$ is quasi-polynomial, its generating series $\sum_{k=0}^{\infty} p(k)t^k$ is the Taylor expansion at $t = 0$ of a rational function $R(t) = \frac{P(t)}{\prod_{i=1}^{s}(1-t^{a_i})}$, where the a_i are integers, and $P(t)$ a polynomial in t of degree strictly less than $\sum_i a_i$. The correspondence is as follows. Consider a quasi-polynomial $p(k)$ of period q, equal to 0 on all cosets except the coset $f + q\mathbb{Z}$, with $0 \leq f < q$. Write the polynomial function $j \mapsto p(f + qj)$ of degree R in terms of binomials: $p(f + qj) = \sum_{n=0}^{R} a(n)\binom{j+n}{n}$. Then

$$\sum_{j=0}^{\infty} p(f + qj)t^{f+qj} = t^f \sum_{n=0}^{R} a(n)\frac{1}{(1 - t^q)^{n+1}}.$$

In our examples, the degree of the quasi-polynomial function p, as well as its period, will not be very large, so there is no computational difficulty to write the rational function $R(t)$ starting from $p(k)$, and conversely. We give a striking example of the function R, giving the Hilbert series of entanglement of 4-qubits, and the corresponding p in Ex. 39.

Let us return to the setting of K acting on a N-dimensional complex Hermitian space $\mathcal{H}$ and let $E = i\mathfrak{t}_{\mathfrak{k}}^*$. We choose an order on the weights for the action of T_K and write

$$\Psi = [\psi_1, \psi_2, \ldots, \psi_N]$$

with $\psi_i \in \Lambda_K \subset E$

Now, we do not necessarily assume that the action of K contains the homotheties $e^{i\theta}\mathrm{Id}_{\mathcal{H}}$. Instead, we assume that the cone $Cone(\Psi)$

generated by Ψ is a pointed cone: $Cone(\Psi) \cap -Cone(\Psi) = \{0\}$. This condition insures that the multiplicity for the action of T_K are finite. For convenience we also assume that the lattice of weights Λ_K is generated by Ψ.

Let $\mathcal{P}_\Psi$ be the function on Λ_K that computes the number of ways we can write $\mu \in \Lambda_K$ as $\sum x_i\psi_i$ with x_i nonnegative integers. The function $\mathcal{P}_\Psi(\mu)$ is called the Kostant *partition function* (with respect to Ψ).

It is thus immediate to see that

$$m_{T_K}^{\mathcal{H}}(\mu) = \mathcal{P}_\Psi(\mu). \tag{4}$$

The cone $C_{T_K}(\mathcal{H})$ is just the cone $Cone(\Psi)$ generated by the list Ψ of weights.

For $z \in (\mathfrak{t}_\mathfrak{k})_\mathbb{C}$ such that $\Re(\langle\psi, z\rangle) < 0$ for all $\psi \in \Psi$, we have the equality

$$\frac{1}{\prod_{\psi\in\Psi}(1 - e^{\langle\psi,z\rangle})} = \sum_{\mu\in Cone(\Psi)} \mathcal{P}_\Psi(\mu)e^{\langle\mu,z\rangle}. \tag{5}$$

Let $y \in i\mathfrak{t}_\mathfrak{k}^*$. Define the polytope $\Pi_\Psi(y) = \{[x_1, \ldots, x_N] \in \mathbb{R}^N, x_i \geq 0, \sum_{a=1}^N x_a\psi_a = y\}$, then $m_{T_K}^{\mathcal{H}}(\mu)$ $(\mu \in \Lambda_K)$ is the number of integral points in the polytope $\Pi_\Psi(\mu)$. Remembering that $Sym(\mathcal{H}) = \oplus_{\lambda\in\hat{K}} m_K^{\mathcal{H}}(\lambda)V_\lambda^K$ and $Sym(\mathcal{H}) = \oplus_{\mu\in\hat{T}_K} m_{T_K}^{\mathcal{H}}(\mu)e^\mu$, then for $\lambda \in \Lambda_{K,\geq 0}$, we have

$$m_K^{\mathcal{H}}(\lambda) = \sum_{w\in\mathcal{W}_\mathfrak{k}} \epsilon(w)\mathcal{P}_\Psi(\lambda+\rho_\mathfrak{k}-w\rho_\mathfrak{k}) = \sum_{w\in\mathcal{W}_\mathfrak{k}} \epsilon(w)m_{T_K}^{\mathcal{H}}(\lambda+\rho_\mathfrak{k}-w\rho_\mathfrak{k}). \tag{6}$$

Remark 17.

Christandl-Doran-Walter [10] compute the multiplicity $m_K^{\mathcal{H}}(\lambda)$ for the Kronecker coefficients, when $\mathcal{H} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$. They use the fact that the multiplicity for T_K is the number of points in the polytope $\Pi_\Psi(\lambda)$ and employ the "Barvinok algorithm", as implemented in [37]. Their method is of polynomial complexity, when the number of rows is fixed.

□

We now describe our own approach to compute multiplicities based on iterated residues. For $z \in (\mathfrak{t}_\mathfrak{k})_\mathbb{C}$, define :

$$S_{T_K}^\Psi(\mu, z) = e^{\langle\mu,z\rangle}\frac{1}{\prod_{\psi\in\Psi}(1 - e^{-\langle\psi,z\rangle})}.$$

Let Γ_K be the dual lattice to Λ_K. Let $q := q(\Psi)$ be the index of Ψ with respect to Λ_K. So, if σ is a basis of Ψ, then $q\Lambda_K \subset \sum_{\psi\in\sigma} \mathbb{Z}\psi$.

If $\gamma \in \Gamma_K$, and if we apply an iterated residue to the function $z \mapsto S_{T_K}^\Psi(\mu, z + \frac{2i\pi}{q}\gamma)$, we obtain a quasi-polynomial function of μ of period

q. Indeed, the residue depends on μ through the Taylor series at $z = 0$ of $e^{\langle \mu, z+\frac{2i\pi}{q}\gamma\rangle} = e^{\langle \mu, \frac{2i\pi}{q}\gamma\rangle} e^{\langle \mu, z\rangle}$, and $e^{\langle \mu, \frac{2i\pi}{q}\gamma\rangle}$ is a periodic function of μ of period q.

Definition 18. *Let τ be a Ψ-tope. Define*

$$p_\tau^\Psi(\mu) = \sum_{\overrightarrow{\sigma} \in \mathcal{OS}(\Psi,\tau)} \sum_{\gamma \in \Gamma/q\Gamma} Res_{\overrightarrow{\sigma}} S_{T_K}^\Psi(\mu, z + \frac{2i\pi}{q}\gamma).$$

Remark 19.

The role of τ here is to select the set $\mathcal{OS}(\Psi, \tau)$, that is the paths along which to calculate the iterated residue in z of the function $z \mapsto S_{T_K}^\Psi(\mu, z + \frac{2i\pi}{q}\gamma)$.
$\square$

Theorem 20. *([35]) Let $\tau \subset i\mathfrak{t}_\mathfrak{k}^*$ be a Ψ-tope and $\mu \in \overline{\tau}$. If τ is contained in $Cone(\Psi)$, then for any $\mu \in \overline{\tau} \cap \Lambda_K$,*

$$m_{T_K}^{\mathcal{H}}(\mu) = p_\tau^\Psi(\mu).$$

We now recall a consequence of Meinrenken-Sjamaar theorem ([28], see also [32] for a different proof). This theorem is important in our computational applications to justify the correctness of our algorithm.

A cone decomposition of a rational polyhedral cone C is a set $\{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_m\}$ of (closed) rational polyhedral cones such that

i) $C = \cup_{i=1}^m \mathfrak{c}_i$,

ii) $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_m$ have all the same dimension $\dim C$

iii) $\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_m$ intersect along faces, that is $\mathfrak{c}_i \cap \mathfrak{c}_j$ is a face of both $\mathfrak{c}_i$ and $\mathfrak{c}_j$.

We say that a cone $\mathfrak{c} \subset i\mathfrak{t}_\mathfrak{k}^*$ is solid if its interior is non empty. Define $d = \dim_{\mathbb{C}}(\mathcal{H}) - |\Delta_\mathfrak{k}^+| - \dim C_K(\mathcal{H})$. Then, if the cone $C_K(\mathcal{H})$ is solid, $d = \dim_{\mathbb{C}}(\mathcal{H}) - |\Delta_\mathfrak{k}^+| - \dim \mathfrak{t}_\mathfrak{k}$.

Theorem 21. *(Meinrenken-Sjamaar) There exists a cone decomposition $C_K(\mathcal{H}) = \cup_a \mathfrak{c}_a$, in cones $\mathfrak{c}_a$, and for each a, there exists a quasi-polynomial function $p_{K,a}^{\mathcal{H}}$ of degree d on the lattice Λ_K such that, if $\lambda \in \mathfrak{c}_a \cap \Lambda_K$,*

$$m_K^{\mathcal{H}}(\lambda) = p_{K,a}^{\mathcal{H}}(\lambda).$$

Remark 22.

Theorem 21 and Theorem 23 are two particular cases of the $[Q, R] = 0$ theorem, [28]. This theorem gives a geometric formula for the quantization of a K-Hamiltonian manifold M. In Theorem 21, M is $\mathcal{H}$, and in Theorem 23, M is T^*G. For more details, see [1].

□

Thus the multiplicity function $\lambda \mapsto m_K^{\mathcal{H}}(\lambda)$ is a piecewise quasi-polynomial function supported on the Kirwan cone. However it is quite difficult to cover explicitly the cone $C_K(\mathcal{H})$ by a finite number of polyhedral cones $\mathfrak{c}_a$ where the function $m_K^{\mathcal{H}}$ is quasi-polynomial on $\mathfrak{c}_a$. Informally we will say that such decomposition is a decomposition in cones of "quasi-polynomiality". Already when K is a torus, this is the difficult problem of describing the decomposition of the cone $C_K(\mathcal{H})$ in chambers (see [2]).

Our iterated residues algorithm produces, for a given input λ^0, a quasi-polynomial function $\lambda \mapsto w_K^{\mathcal{H}}(\lambda, \lambda^0)$ coinciding with $m_K^{\mathcal{H}}$ in a conic neighborhood of λ^0 in $C_K(\mathcal{H})$. Indeed, given $v \in C_K(\mathcal{H})$ close to λ^0 and not belonging to any admissible hyperplane, we can compute $\cap_\sigma \mathfrak{c}(\sigma)$ over the OS basis adapted to v. Then, on $\cap_\sigma \mathfrak{c}(\sigma) \cap C_K(\mathcal{H})$, $m_K^{\mathcal{H}}$ is quasi-polynomial.

In particular, for any dominant weight λ belonging to $C_K(\mathcal{H})$, the function $k \to m_K^{\mathcal{H}}(k\lambda)$ is of the form $m_K^{\mathcal{H}}(k\lambda) = \sum_{i=0}^{N} c_i(k) k^i$ where $c_i(k)$ are periodic functions of k. This formula is valid **for any** $k \geq 0$ (so $c_0(0) = 1$). The highest degree term for which this function is non zero will be called the **degree** of the quasi-polynomial function $m_K^{\mathcal{H}}(k\lambda)$. For any λ contained in the relative interior of $C_K(\mathcal{H})$, this degree is d. Consider now λ in the boundary of $C_K(\mathcal{H})$. It is clear that the degree of the quasi-polynomial function $k \to m_K^{\mathcal{H}}(k\lambda)$ is less or equal to d, as this is the restriction of a quasi-polynomial of degree d. In particular the dilated Kronecker multiplicity function $k \mapsto g(k\mu_1, \ldots, k\mu_s)$ is a quasi-polynomial function of degree at most $d = \prod_{j=1}^{s} n_j - \sum_{j=1}^{s} \frac{n_j(n_j-1)}{2} - \sum_{j=1}^{s} n_j + s - 1$. If the degree is strictly smaller than d, the corresponding point is in the boundary of the Kirwan cone.

2.3. **Faces and degree.** Consider a decomposition $C_K(\mathcal{H}) = \cup_a \mathfrak{c}_a$ in cones of quasi-polynomiality. We already remarked that the degree d of the quasi polynomial $p_{K,a}^{\mathcal{H}}$ is the same for each $\mathfrak{c}_a$. Let us however remark that the periods of the quasi polynomial $p_{K,a}^{\mathcal{H}}$ can be different in different cones $\mathfrak{c}_a$. On the example when $\mathcal{H} = \mathbb{C}^6 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$ (see 37), we produce a cone $\mathfrak{c}_a$ where $p_{K,a}^{\mathcal{H}}$ is of degree 8 and periods $\{1, 2, 3\}$, and a cone $\mathfrak{c}_b$ where $p_{K,b}^{\mathcal{H}}$ is of degree 8 and polynomial, that is with period $\{1\}$.

When F is a face of $C_K(\mathcal{H})$, then $F = \cup_a F \cap \mathfrak{c}_a$, where we restrict the decomposition to cones $\mathfrak{c}_a$ such that $\dim(\mathfrak{c}_a \cap F) = \dim F$. In other words, the closed cone $\mathfrak{c}_a$ contains a point v_F in the relative interior of F. We will say that such a cone $\mathfrak{c}_a$ is adjacent to the face F.

The restriction to $F \cap \mathfrak{c}_a$ of the function $m_K^{\mathcal{H}}$ is the restriction of the quasi-polynomial $p_{K,a}^{\mathcal{H}}$. Thus the function $m_K^{\mathcal{H}}$ restricted to F is again a piecewise quasi-polynomial function. We can consider the degree of this quasi polynomial function restricted to $F \cap \mathfrak{c}_a$. The degree drops, but if F is a regular face (that is a face intersecting the interior of the Weyl chamber), then the degree is the same on each cone $\mathfrak{c}_a \cap F$ and can be computed with a formula analogous to the Formula in Theorem 21. Indeed, Meinrenken-Sjamaar geometric formula for multiplicities implies a reduction principle of multiplicities on regular faces: the function $m_K^{\mathcal{H}}$ restricted to a regular face F coincides with a multiplicity function $m_{K_0}^{\mathcal{H}_0}$, for smaller datas (see [1]). An example is given in Section 5, Ex. 37. When K is a torus, and $\mathfrak{c}_a$ adjacent to a facet F, then the quasi polynomial $p_{K,a}^{\mathcal{H}}$ vanishes on a certain number of affine hyperplanes parallel to the hyperplane generated by the facet F, leading to divisibility properties. We believe this is also the case for the functions $p_{K,a}^{\mathcal{H}}$, however we do not have a precise guess. We give a striking example of this divisibility property in Ex. 37.

We recall that a point λ is called a *stable point*, if the function $k \mapsto m_K^{\mathcal{H}}(k\lambda)$ is a bounded function of k. Then it takes value zero or one [31]. For a given non zero $\lambda \in C_K(\mathcal{H}) \cap \Lambda_{K,\geq 0}$, the *saturation factor* is the smallest positive k such that $m_K^{\mathcal{H}}(k\lambda) \neq 0$.

3. The Branching Rules

3.1. **Branching cone.** Consider a pair $K \subset G$ of two compact connected Lie groups, with Lie algebras $\mathfrak{k}, \mathfrak{g}$ respectively. Let $\pi : \mathfrak{g}^* \to \mathfrak{k}^*$ be the projection. Let T_G, T_K be maximal tori of G,K. We may assume, and we do so, that $T_K \subset T_G$. Let $\mathfrak{t}_\mathfrak{g}$, $\mathfrak{t}_\mathfrak{k}$ be the corresponding Cartan subalgebras. Given $\xi \in i\mathfrak{t}_\mathfrak{g}^*$, denote by $\overline{\xi}$ the restriction $\xi_{|i\mathfrak{t}_\mathfrak{k}}$. We choose Weyl chambers $i\mathfrak{t}^*_{\mathfrak{g},\geq 0}$, $i\mathfrak{t}^*_{\mathfrak{k},\geq 0}$, and we denote the corresponding cones of dominant weights by $\Lambda_{G,\geq 0}$, $\Lambda_{K,\geq 0}$. We denote by $\Lambda_{G,K,\geq 0}$ the sum $\Lambda_{G,\geq 0} \oplus \Lambda_{K,\geq 0}$, by $i\mathfrak{t}^*_{\mathfrak{g},\mathfrak{k},\geq 0}$ the sum $i\mathfrak{t}^*_{\mathfrak{g},\geq 0} \oplus i\mathfrak{t}^*_{\mathfrak{k},\geq 0}$ of the closed positive Weyl chambers relatives to G, K, and by $i\mathfrak{t}^*_{\mathfrak{g},\mathfrak{k},>0}$ its interior. We may also choose compatible positive root systems on K, G: if λ is dominant for G, then the restriction of λ to $i\mathfrak{t}_\mathfrak{k}$ is dominant.

For $\lambda \in \Lambda_{G,\geq 0}$ (resp. $\mu \in \Lambda_{K,\geq 0}$), denote by V_λ^G (resp. V_μ^K) the irreducible representation of G (resp. K) of highest weight λ (resp. μ). Denote by λ^*, μ^*, etc. the contragradient representations.

Define

$$V = \oplus_{\lambda \in \Lambda_{G,\geq 0}} V_\lambda^G \otimes V_{\lambda^*}^G.$$

So, under the action of $G \times K$,

$$V = \oplus_{\lambda,\mu} m_{G,K}(\lambda,\mu) V_\lambda^G \otimes V_{\mu^*}^K$$

(λ varies in $\Lambda_{G,\geq 0}$, and μ in $\Lambda_{K,\geq 0}$). Here $m_{G,K}(\lambda,\mu)$ is the multiplicity of the representation μ in the restriction of V_λ^G to K.

Define

$$C_{G,K} = \left\{(\xi,\eta) \in i\mathfrak{t}^*_{\mathfrak{g},\geq 0} \times i\mathfrak{t}^*_{\mathfrak{k},\geq 0}; \eta \in \pi(G \cdot \xi)\right\}.$$

It is a particular case of Kirwan convexity theorem that $C_{G,K}$ is a polyhedral cone, that the support of the function $m_{G,K}(\lambda,\mu)$ is contained in $C_{G,K}$ and that its asymptotic support is exactly the cone $C_{G,K}$.

Remark that if $G = K$, the cone $C_{G,K}$ is just the diagonal $\{(\xi,\xi), \xi \in i\mathfrak{t}^*_{\mathfrak{g},\geq 0}\}$ in $i\mathfrak{t}^*_{\mathfrak{g},\mathfrak{g},\geq 0}$. However, **we assume from now on** that no nonzero ideal of $\mathfrak{k}$ is an ideal of $\mathfrak{g}$ (this condition excludes the preceding case). It implies that the polytope $C_{G,K}$ is solid (Duflo, private communication). As we already pointed out, the $[Q,R] = 0$ theorem of Meinrenken-Sjamaar implies the following theorem.

Theorem 23. *There exists a cone decomposition $C_{G,K} = \cup_a \mathfrak{c}_a$, in* **solid polyhedral cones** *$\mathfrak{c}_a$ such that $m_{G,K}(\lambda,\mu), (\lambda,\mu) \in \mathfrak{c}_a \cap (\Lambda_G \oplus \Lambda_K)$, is given by a non zero quasi-polynomial function on each cone $\mathfrak{c}_a$.*

In particular, for any pair (λ,μ) of dominant weights contained in $C_{G,K}$, the function $k \mapsto m_{G,K}(k\lambda,k\mu)$ is of the form: $m_{G,K}(k\lambda,k\mu) = \sum_{i=0}^N E_i(k)k^i$ where $E_i(k)$ are periodic functions of k. This formula is valid **for any** $k \geq 0$, and in particular $E_0(0) = 1$.

To describe the cone $C_{G,K}$ is difficult, and has been the object of numerous works, notably Berenstein-Sjamaar, Belkale-Kumar, Kumar, Ressayre. We refer to the survey article [9]. The complete description of the multiplicity function $m_{G,K}$, in particular the decomposition of $C_{G,K}$ in $\cup_a \mathfrak{c}_a$ is even more so. However, we will give an algorithm where, given as input (λ,μ), the output is the dilated function $k \mapsto m_{G,K}(k\lambda,k\mu)$. In particular, we can test if the point (λ,μ) is in the cone $C_{G,K}$ or not, according if the output is not zero or zero.

We implicitly choose an order and consider the list Ψ of non zero restrictions of the roots $\Delta^+_\mathfrak{g}$ to $i\mathfrak{t}_\mathfrak{k}$. We say that Ψ is the list of restricted roots (for the pair $\mathfrak{g},\mathfrak{k}$).

Recall that an hyperplane in $i\mathfrak{t}^*_\mathfrak{k}$ is Ψ-admissible if it is spanned by elements of Ψ. Let $\mathcal{A}$ be the set of Ψ-admissible hyperplanes. For $H \in \mathcal{A}$, consider $X \in \mathfrak{t}_\mathfrak{k}$ such that $H = X^\perp$. Let $\mathcal{W}_\mathfrak{g}$ be the Weyl group of G. If $X \in i\mathfrak{t}_\mathfrak{k} \subset i\mathfrak{t}_\mathfrak{g}$, consider $w^{-1}X \in \mathfrak{t}_\mathfrak{g}$ and the hyperplane

$$H(w) = \{(\xi,\nu) \in i\mathfrak{t}^*_\mathfrak{g} \oplus i\mathfrak{t}^*_\mathfrak{k}; \langle \xi, w^{-1}X\rangle - \langle \nu, X\rangle = 0\}.$$

When H varies in $\mathcal{A}$, and w in the Weyl group of G, we obtain a finite set $\mathcal{F}$ of hyperplanes in $i\mathfrak{t}_{\mathfrak{g}}^* \oplus i\mathfrak{t}_{\mathfrak{k}}^*$.

Consider a connected component τ of the complement of the union of the hyperplanes belonging to $\mathcal{F}$. We say that τ is a *tope* for the system of hyperplanes $\mathcal{F}$. So τ is an open conic subset of $i\mathfrak{t}_{\mathfrak{g}}^* \oplus i\mathfrak{t}_{\mathfrak{k}}^*$. Thus, if $(\xi,\nu) \in \tau$, for any Ψ-admissible hyperplane H with equation X, and any $w \in \mathcal{W}_{\mathfrak{g}}$, we have

$$\langle \xi, w^{-1}X\rangle - \langle \nu, X\rangle \neq 0. \tag{7}$$

Then for each $w \in \mathcal{W}_{\mathfrak{g}}$, the element $\overline{w(\xi)} - \nu$ is Ψ*-regular*, that is, is not on any hyperplane of $\mathcal{A}$. Given $w \in \mathcal{W}_{\mathfrak{g}}$, we denote by $\mathfrak{a}(\overline{w(\xi)} - \nu)$ the unique tope for Ψ which contains the element $\overline{w(\xi)} - \nu$. The tope $\mathfrak{a}(\overline{w(\xi)} - \nu)$ depends only on w and τ, so we denote it by $\mathfrak{a}_w^\tau$.

The facets of the cones $\mathfrak{c}_a$ generates hyperplanes belonging to the family $\mathcal{F}$, as follows from the description of the Duistermaat-Heckman measure [16]. Thus given a cone $\mathfrak{c}_a$ and a tope τ, then $\tau \cap i\mathfrak{t}_{\mathfrak{g},\mathfrak{k},\geq 0}^*$ is either contained in $\mathfrak{c}_a$, or is disjoint from $\mathfrak{c}_a$. The closed cone $\mathfrak{c}_a$ is the union of the closures of the sets $\tau \cap i\mathfrak{t}_{\mathfrak{g},\mathfrak{k},\geq 0}^*$ over the τ such that $\tau \cap \mathfrak{c}_a$ is non empty. Remark that there might be several topes τ needed to obtain $\mathfrak{c}_a$.

Let $\lambda \in \Lambda_{G,\geq 0}$. We say that λ is regular if $\langle \lambda, H_\alpha\rangle \neq 0$ for all roots $\alpha \in \Delta_{\mathfrak{g}}$. Otherwise, we say that λ is singular. The function $m_{G,K}(\lambda,\mu)$ can in principle by computed by Heckman formula [16]. However, in the case we are interested in, Heckman formula is of formidable complexity, but on the other hand the parameter λ is quite singular. We will obtain formulae for $m_{G,K}(\lambda,\mu)$, maybe less beautiful, but of much smaller complexity, taking advantage of the fact that λ vanishes on a large number of H_α. We will comment in Remark 33 over the advantages of writing a specific formula for the singular case instead of using the Kostant-Heckman branching theorem.

Thus fix a subset Σ of the simple roots of $\Delta_{\mathfrak{g}}^+$. Let $i\mathfrak{t}_{\mathfrak{g},\Sigma}^*$ be the set of the elements $\xi \in i\mathfrak{t}_{\mathfrak{g}}^*$ such that $\langle \xi, H_\alpha\rangle = 0$ for all $\alpha \in \Sigma$. We define consistently $\mathfrak{t}_{\mathfrak{g},\mathfrak{k},\Sigma,\geq 0}^* = i\mathfrak{t}_{\mathfrak{g},\Sigma}^* \oplus i\mathfrak{t}_{\mathfrak{k}}^*$, $\Lambda_G^\Sigma = \Lambda_G \cap i\mathfrak{t}_{\mathfrak{g},\Sigma}^*$, a lattice in $i\mathfrak{t}_{\mathfrak{g},\Sigma}^*$, $\Lambda_{G,K,\geq 0}^\Sigma = \Lambda_{G,K,\geq 0} \cap i\mathfrak{t}_{\mathfrak{g},\mathfrak{k},\Sigma}^*$, and similarly. Define

$$C_{G,K}^\Sigma = \{(\xi,\nu) \in C_{G,K}; \xi \in i\mathfrak{t}_{\mathfrak{g},\Sigma}^*\}.$$

If Σ is empty, then $C_{G,K}^\Sigma = C_{G,K}$. Otherwise, $(\xi,\nu) \in C_{G,K}^\Sigma$ if ν belongs to the projection on $i\mathfrak{k}^*$ of the singular orbit $G\xi$. Thus the cone $C_{G,K}^\Sigma$ is contained in the cone $C_{G,K}$ and is in its boundary if Σ is not empty, and our aim is to describe a specific formula for the function $m_{G,K}$ on this cone $C^{\Sigma}_{G,K}$.

The cone $C^{\Sigma}_{G,K}$ is solid in $i\mathfrak{t}^*_{\mathfrak{g},\Sigma}$ if and only if there exists $\xi \in i\mathfrak{t}^*_{\mathfrak{g},\Sigma}$ such that the projection on $i\mathfrak{k}^*$ of the singular orbit $G\xi$ has a non zero interior in $i\mathfrak{k}^*$. In other words, if and only if the Kirwan polytope $\pi(G\xi) \cap i\mathfrak{t}^*_{\mathfrak{k}}$ is solid.

Example 24. *([39])*

Consider the embedding of $K = (U(n_2) \times U(n_3))/Z$ in $G = U(n_2 n_3)$. Here $n_2, n_3 \geq 2$ and Z is the subgroup $\{t_2 Id, t_3 Id\}$ of the center of $U(n_2) \times U(n_3)$ with $t_2 t_3 = 1$. We take λ a dominant weight of G with more than two non zero coordinates. Then $\pi(G\lambda)$ has interior in $i\mathfrak{k}^*$.
$\square$

We consider the system of hyperplanes $\mathcal{F}_\Sigma$ in $i\mathfrak{t}^*_{\mathfrak{g},\Sigma} \oplus i\mathfrak{t}^*_{\mathfrak{k}}$ defined by the equations $\langle \xi, w^{-1}X\rangle - \langle \nu, X\rangle = 0$, where X is an equation for a Ψ-admissible hyperplane, and $w \in \mathcal{W}_{\mathfrak{g}}$. That is, an hyperplane in $\mathcal{F}_\Sigma$ is the intersection of an hyperplane belonging to $\mathcal{F}$ with $i\mathfrak{t}^*_{\mathfrak{g},\Sigma} \oplus i\mathfrak{t}^*_{\mathfrak{k}}$. If $(\xi, \nu) \in i\mathfrak{t}^*_{\mathfrak{g},\Sigma} \oplus i\mathfrak{t}^*_{\mathfrak{k}}$ is in a tope τ_Σ for $\mathcal{F}_\Sigma$, then (ξ, ν) is in a unique tope τ for $\mathcal{F}$.

Remark that if τ_Σ is a tope for $\mathcal{F}_\Sigma$, then $\tau_\Sigma \cap C^{\Sigma}_{G,K}$ is empty if $C^{\Sigma}_{G,K}$ is not solid. If the cone $C^{\Sigma}_{G,K}$ is solid, it is the union of the closures of the sets $\tau_\Sigma \cap \mathfrak{t}^*_{\mathfrak{g},\mathfrak{k},\Sigma,\geq 0}$ contained in $C^{\Sigma}_{G,K}$.

3.2. **Branching theorem: a piecewise quasi-polynomial formula.**

When τ_Σ is a tope for $\mathcal{F}_\Sigma$, we now give a specific quasi-polynomial formulae for the function $m_{G,K}$ on $\tau_\Sigma \cap C^{\Sigma}_{G,K}$.

Fix a subset Σ of the simple roots of $\Delta^+_{\mathfrak{g}}$, and let $\mathfrak{l}$ be the Levi subalgebra of $\mathfrak{g}$, with simple root system Σ. Let $\Delta^+_{\mathfrak{l}}$ be its positive root system. Let $\Delta_{\mathfrak{u}} = \Delta^+_{\mathfrak{g}} \setminus \Delta^+_{\mathfrak{l}}$ and denote by $\mathcal{W}_{\mathfrak{l}}$ the Weyl group of $\mathfrak{l}$. For any $\lambda \in \Lambda_{G,\geq 0} \cap i\mathfrak{t}^*_{\mathfrak{g},\Sigma}$, we can write the character formula on T_G as:

$$\chi_{\lambda|_{T_G}} = \sum_{w \in \mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} \frac{e^{w(\lambda)}}{\prod_{\alpha \in \Delta_{\mathfrak{u}}} (1 - e^{-w(\alpha)})}.$$

Remark 25.

This formula, a special case of the Atiyah-Bott fixed point formula, is easily obtained by the Weyl character formula. When λ is regular, $\Delta_{\mathfrak{u}} = \Delta^+_{\mathfrak{g}}$, it is Weyl character formula. When λ is singular, this sum is over an eventually much smaller set of elements w, and of simpler functions. The extreme case is $\lambda = 0$, with just one term equal to 1.

3.2.1. We start by considering the case where all elements of $\Delta_{\mathfrak{g}}$ have a non zero restriction to $i\mathfrak{t}_{\mathfrak{k}}$. The general situation is treated in Subsection 3.2.2.

The space $i\mathfrak{t}_{\mathfrak{k}}$ contains a regular (with respect to $\Delta_{\mathfrak{k}}$) element X which is regular also for $\Delta_{\mathfrak{g}}$. We use this element to define positive compatible root systems $\Delta_{\mathfrak{g}}^+$ and $\Delta_{\mathfrak{k}}^+$ as follows: $\Delta_{\mathfrak{g}}^+ := \{\alpha \in \Delta_{\mathfrak{g}}, \alpha(X) > 0\}$ and $\Delta_{\mathfrak{k}}^+ = \{\alpha \in \Delta_{\mathfrak{k}}, \alpha(X) > 0\}$. Thus the list Ψ of elements of $i\mathfrak{t}_{\mathfrak{k}}^*$ consists on the restrictions of $\Delta_{\mathfrak{g}}^+$ repeated with multiplicities (we implicitly choose an order):

$$\Psi = [\overline{\alpha}, \alpha \in \Delta_{\mathfrak{g}}^+].$$

The list Ψ contains $\Delta_{\mathfrak{k}}^+$. By our construction, all elements ψ in Ψ satisfy $\langle \psi, X\rangle > 0$.

Example 26.

Let $G = SU(n)$ and $K = SU(n_1) \times SU(n_2)$ with $n = n_1 n_2$. We consider $\mathfrak{t}$ the Cartan subalgebra of $\mathfrak{g}$ given by the diagonal matrices of trace zero and $\mathfrak{t}_{\mathfrak{k}} = \mathfrak{t}_1 \times \mathfrak{t}_2$ the Cartan subalgebras of $\mathfrak{k}$ given by the corresponding diagonal matrices. The embedding of K in G leads to the embedding of $i\mathfrak{t}_1 \times i\mathfrak{t}_2 \to i\mathfrak{t}$ given by

$$diag(a_1, \ldots, a_{n_1}) \times diag(b_1, \ldots, b_{n_2}) \to$$

$$diag(a_1{+}b_1, a_2{+}b_1, \ldots, a_{n_1}{+}b_1, a_1{+}b_2, \ldots, a_{n_1}{+}b_2, \ldots, a_1{+}b_{n_2}, \ldots, a_{n_1}{+}b_{n_2}).$$

We take the lexicographic order. The list of restricted roots is thus the list

$$\Psi = [(a_i - a_j + b_k - b_\ell)]. \tag{8}$$

There i, j varies between 1 and n_1, and k, ℓ varies between 1 and n_2. The couple (i, k) being different from (j, ℓ), so all restricted roots are non zero. This lexicographic order is compatible. Explicitly, we can associate it to the element $X = diag(n_1, n_1 - 1, \ldots, 1) \times diag((n_2 - 1)n_1 + 1, (n_2 - 2)n_1 + 1, \ldots, 1)$. For example, for $n_1 = 2, n_2 = 3$, $X = diag(2, 1) \times diag(5, 3, 1)$ with embedded element $diag(7, 6, 5, 4, 3, 2)$. □

We can write the restriction of χ_λ on T_K as a sum, indexed by $w \in \mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}$, of meromorphic functions:

$$\chi_{\lambda|_{T_K}} = \sum_{w \in \mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} \frac{e^{\overline{w(\lambda)}}}{\prod_{\alpha \in \Delta_{\mathfrak{u}}} (1 - e^{-\overline{w(\alpha)}})}.$$

To compute $m_{G,K}(\lambda, \mu)$ for $\Lambda_{G,\geq 0} \cap i\mathfrak{t}^*_{\mathfrak{g},\Sigma}$ by iterated residues, we consider the function of $z \in (\mathfrak{t}_{\mathfrak{k}})_{\mathbb{C}}$:

$$S_{\lambda,\mu}^{\Sigma,w}(z)=\prod_{\beta\in\Delta_{\mathfrak{k}}^+}(1-e^{-\langle\beta,z\rangle})\frac{e^{\langle\overline{w(\lambda)}-\mu,z\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle\overline{w(\alpha)},z\rangle})}. \tag{9}$$

If τ is a tope for $\mathcal{F}$, we have defined the tope $\mathfrak{a}_w^\tau$ for Ψ and the set $\mathcal{OS}(\Psi,\mathfrak{a}_w^\tau)$ of adapted basis to the tope a_w^τ (Def. 13). Let $q=q(\Psi)$ be the index of Ψ. The following proposition is clear.

Proposition 27. *Let*

$$p_\tau^\Sigma(\lambda,\mu)=\sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}}\ \sum_{\gamma\in\Gamma_K/q\Gamma_K}\ \sum_{\overrightarrow{\sigma}\in\mathcal{OS}(\Psi,\mathfrak{a}_w^\tau)} Res_{\overrightarrow{\sigma}} S_{\lambda,\mu}^{\Sigma,w}(z+\frac{2i\pi\gamma}{q}).$$

Then $p_\tau^\Sigma(\lambda,\mu)$ is a quasi-polynomial function on $\Lambda_G^\Sigma\oplus\Lambda_K$.

We can state our formula for the branching coefficients.

Theorem 28. *Let τ_Σ be a tope in $i\mathfrak{t}_{\mathfrak{g},\Sigma}^*\oplus i\mathfrak{t}_{\mathfrak{k}}^*$ for $\mathcal{F}_\Sigma$, and let τ be the tope for $\mathcal{F}$ containing τ_Σ.*

Let $(\lambda,\mu)\in\overline{\tau}_\Sigma\cap\Lambda_{G,K,\geq 0}^\Sigma$. Then

(1) *if $(\lambda,\mu)\notin C_{G,K}^\Sigma$ then*

$$m_{G,K}(\lambda,\mu)=p_\tau^\Sigma(\lambda,\mu)=0.$$

(2) *if $(\lambda,\mu)\in C_{G,K}^\Sigma$,* **and** *the tope τ_Σ intersect $C_{G,K}^\Sigma$, then*

$$m_{G,K}(\lambda,\mu)=p_\tau^\Sigma(\lambda,\mu).$$

Proof. The set $\tau_\Sigma\cap C_{G,K}^\Sigma$ is contained in $\tau\cap C_{G,K}$, so we know from Theorem 23 that on $\overline{\tau}\cap\Lambda_{G,K,\geq 0}$, the function $m_{G,K}$ is given by a quasi-polynomial formula, so a fortiori its restriction to $\overline{\tau_\Sigma}\cap\Lambda_{G,K,\geq 0}^\Sigma$. So it is sufficient to prove (see the uniqueness result in Lemma 30) that when $(\lambda,\mu)\in\tau_\Sigma\cap\Lambda_{G,K,\geq 0}^\Sigma$ is sufficiently far away from all walls belonging to $\mathcal{F}_\Sigma$, then $m_{G,K}(\lambda,\mu)$ coincide with $p_\tau^\Sigma(\lambda,\mu)$.

Let $w\in\mathcal{W}_{\mathfrak{g}}$. Using our regular element X, we can rewrite the formula for $\chi_\lambda|_{T_K}$ polarizing the linear form $\overline{w(\alpha)}$: if $\langle\overline{w(\alpha)},X\rangle<0$, we replace $\overline{w(\alpha)}$ by its opposite; we then make use of the identity $\frac{1}{1-e^{-\beta}}=-\frac{e^\beta}{1-e^\beta}$. Precisely we can define $\Psi_{w,\mathfrak{u}}=\Psi_{w,\mathfrak{u}}^1\cup\Psi_{w,\mathfrak{u}}^2$ with $\Psi_{w,\mathfrak{u}}^1=\{\overline{w\alpha},\alpha\in\Delta_{\mathfrak{u}},\ \langle\overline{w(\alpha)},X\rangle>0\}$, $\Psi_{w,\mathfrak{u}}^2=\{-\overline{w(\alpha)},\alpha\in\Delta_{\mathfrak{u}},\ \langle\overline{w(\alpha)},X\rangle<0\}$. Elements in $\Psi_{w,\mathfrak{u}}$ are positive on X, so $\Psi_{w,\mathfrak{u}}$ is contained in Ψ. In contrast to the regular case, $\Psi_{w,\mathfrak{u}}$ depends on w and may not contain $\Delta_{\mathfrak{k}}^+$.

Let $s^{\Sigma}_w = |\Psi^2_{w,\mathfrak{u}}|$ and $e^{g^{\Sigma}_w} = \prod_{\overline{w(\alpha)},\ \langle \overline{w(a)}, X\rangle < 0} e^{\overline{w(\alpha)}}$, then we obtain that $\chi_{\lambda|_{T_K}}$ is equal to

$$\sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} \left(\frac{e^{\overline{w(\lambda)}}}{\prod_{\psi\in\Psi^1_{w,\mathfrak{u}}}(1-e^{-\psi})} \frac{(-1)^{s^{\Sigma}_w} e^{g^{\Sigma}_w}}{\prod_{\psi\in\Psi^2_{w,\mathfrak{u}}}(1-e^{-\psi})} \right) = \sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} \left(\frac{e^{\overline{w(\lambda)}}(-1)^{s^{\Sigma}_w} e^{g^{\Sigma}_w}}{\prod_{\psi\in\Psi_{w,\mathfrak{u}}}(1-e^{-\psi})} \right).$$

Lemma 29. *The following equality holds on T_K:*

$$\chi_\lambda|_{T_K} = \sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} \sum_{\mu\in\Lambda_K} (-1)^{s^{\Sigma}_w} \mathcal{P}_{\Psi_{w,\mathfrak{u}}}(\overline{w(\lambda)} + g^{\Sigma}_w - \mu) e^{\mu}$$

where $\mathcal{P}_{\Psi_{w,\mathfrak{u}}}$ is the partition function determined by the restricted roots $\Psi_{w,\mathfrak{u}}$. So

$$m_{G,T_K}(\lambda,\mu) = \sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} (-1)^{s^{\Sigma}_w} \mathcal{P}_{\Psi_{w,\mathfrak{u}}}(\overline{w(\lambda)} + g^{\Sigma}_w - \mu). \tag{10}$$

When K is the maximal torus T_G and λ is regular, Formula 10. is Kostant multiplicity formula for a weight [23]. The formula above is obtained by the same method. Let us now use Formula 6

$$m_{G,K}(\lambda,\mu) = \sum_{\tilde{w}\in\mathcal{W}_{\mathfrak{k}}} \epsilon(\tilde{w}) m_{G,T_K}(\lambda, \mu - \tilde{w}(\rho_{\mathfrak{k}}) + \rho_{\mathfrak{k}}).$$

We obtain for $(\lambda,\mu) \in \Lambda_{G,K,\geq 0}$

$$m_{G,K}(\lambda,\mu) = \sum_{\tilde{w}\in\mathcal{W}_{\mathfrak{k}}} \epsilon(\tilde{w}) \sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} (-1)^{s^{\Sigma}_w} \mathcal{P}_{\Psi_{w,\mathfrak{u}}}(\overline{w(\lambda)} + g^{\Sigma}_w - (\mu - \tilde{w}(\rho_{\mathfrak{k}}) + \rho_{\mathfrak{k}})). \tag{11}$$

Observe that, if λ is regular, then we may rewrite this expression as a sum of partitions functions for $\Psi\backslash\Delta^+_{\mathfrak{k}}$ obtaining Heckman formula, [16], but we will not use this fact.

The point (λ,μ) being in τ_Σ, the point $\overline{w(\lambda)} - \mu$ is in $\mathfrak{a}^{\tau}_w$. We can assume that (λ,μ) is sufficiently far away from all walls, so that $\overline{w(\lambda)} + g^{\Sigma}_w - (\mu - \tilde{w}(\rho_{\mathfrak{k}}) + \rho_{\mathfrak{k}}))$ is also in $\mathfrak{a}^{\tau}_w$. Now use Theorem 20 to express

$$\mathcal{P}_{\Psi_{w,\mathfrak{u}}}(\overline{w(\lambda)} + g^{\Sigma}_w - (\mu - \tilde{w}(\rho_{\mathfrak{k}}) + \rho_{\mathfrak{k}})).$$

We obtain that $m_{G,K}(\lambda,\mu)$ is equal to

$$\sum_{\tilde{w}\in\mathcal{W}_{\mathfrak{k}}} \epsilon(\tilde{w}) \sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} (-1)^{s^{\Sigma}_w} \sum_{\gamma\in\Gamma_K/q\Gamma_K} \sum_{\overrightarrow{\sigma}\in\mathcal{OS}(\Psi_{w,\mathfrak{u}},\mathfrak{a}^{\tau}_w)} Res_{\overrightarrow{\sigma}} \frac{e^{\langle \overline{w(\lambda)} + g^{\Sigma}_w - (\mu - \tilde{w}(\rho_{\mathfrak{k}}) + \rho_{\mathfrak{k}}), z + \frac{2i\pi}{q}\gamma\rangle}}{\prod_{\psi\in\Psi_{w,\mathfrak{u}}}(1 - e^{-\langle\psi, z + \frac{2i\pi}{q}\gamma\rangle})}.$$

Inverting the polarization process, we rewrite

$$(-1)^{s_w^\Sigma}\frac{e^{\langle\overline{w(\lambda)}+g_w^\Sigma,z+\frac{2i\pi}{q}\gamma\rangle}}{\prod_{\psi\in\Psi_{w,\mathfrak{u}}}(1-e^{-\langle\psi,z+\frac{2i\pi}{q}\gamma\rangle})}=\frac{e^{\langle\overline{w(\lambda)},z+\frac{2i\pi}{q}\gamma\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle\overline{w\alpha},z+\frac{2i\pi}{q}\gamma\rangle})}.$$

So, remembering that $\prod_{\beta\in\Delta_{\mathfrak{k}}^+}(1-e^{-\beta})=\sum_{\tilde{w}}e^{-\rho_{\mathfrak{k}}+\tilde{w}(\rho_{\mathfrak{k}})}$, we obtain

$$m_{G,K}(\lambda,\mu)=p_\tau(\lambda,\mu)$$

when $(\lambda,\mu)\in\tau_\Sigma\cap\Lambda_{G,K,\geq 0}^\Sigma$ is "very" far away from all the walls $H(w)$. Now the proof follows by the following lemma which shows that a quasi-polynomial function is determined by its values on a sufficiently large subset.

Lemma 30. *Let p_1,p_2 be two quasi-polynomial functions on a lattice L. If there exists a open cone τ such that p_1,p_2 agree on a translate $(s+\tau)\cap L$ of τ, then $p_1=p_2$.*

The proof is left to the reader.

3.2.2. We now explain how to deal with the general case. We start with the character formula

$$\chi_{\lambda|_{T_G}}=\sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}}\frac{e^{w(\lambda)}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-w(\alpha)})}.$$

We cannot write directly the restriction of each term $\frac{e^{w(\lambda)}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-w(\alpha)})}$ to T_K since the denominator could vanish identically on T_K. So we compute a limit formula as follows. Choose $X_1\in i\mathfrak{t}_{\mathfrak{g}}$ so that $\langle w(\alpha),X_1\rangle\neq 0,\forall\alpha\in\Delta_{\mathfrak{u}}$ and $w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}$.

Let $z\in(\mathfrak{t}_{\mathfrak{k}})_{\mathbb{C}}$ and $w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}$. For ϵ small, consider the expression $\frac{e^{\langle\overline{w(\lambda)},z\rangle}e^{\langle w(\lambda),\epsilon X_1\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle w(\alpha),z+\epsilon X_1\rangle})}$ and define $\Theta_\epsilon(z,X_1)=\frac{e^{\langle w(\lambda),\epsilon X_1\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle w(\alpha),z+\epsilon X_1\rangle})}$. The function $\epsilon\mapsto\Theta_\epsilon(z,X_1)$ has a pole at $\epsilon=0$ of order $p=|\{\alpha\in\Delta_{\mathfrak{u}}|\ \langle w(\alpha),z\rangle=0\}|$. Consider the Laurent expansion $\sum_{i\geq -p}c_i\epsilon^i$ of this function at $\epsilon=0$ and say that c_0 is its constant term. Define $F_w(z,X_1)$ to be the constant term of the Laurent expansion of $\epsilon\mapsto\Theta_\epsilon(z,X_1)$ at $\epsilon=0$ and define

$$G_w(z,X_1)=e^{\langle\overline{w(\lambda)},z\rangle}F_w(z,X_1).$$

When the order p of the pole is 0, $G_w(z,X_1)$ is just equal to

$$\frac{e^{\langle\overline{w(\lambda)},z\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle\overline{w(\alpha)},z\rangle})}.$$

Observe that $\sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}}G_w(z,X_1)$ does not depend on the choice of X_1 and it is equal to the restricted character $\chi_\lambda(\exp(z))$.

The function $G_w(z, X_1)$ is of the form P/Q where P is a sum of exponentials and Q is a product of the form $\frac{1}{\prod_{\psi\in\Psi}(1-e^{-\psi})^{n_\psi}}$, Ψ being the list of (nonzero) restricted roots. Thus the restricted character $\chi_\lambda|_{T_K}$ can be expressed again as a sum of partition functions associated to lists of elements belonging to Ψ (with eventual higher multiplicities). Define

$$S^{\Sigma,w}_{\lambda,\mu}(z) = \prod_{\beta\in\Delta_{\mathfrak{k}}}(1-e^{-\langle\beta,z\rangle})e^{\langle\overline{w(\lambda)}-\mu,z\rangle}F_w(z,X_1)$$
$$= \prod_{\beta\in\Delta_{\mathfrak{k}}}(1-e^{-\langle\beta,z\rangle})e^{-\langle\mu,z\rangle}G_w(z,X_1).$$

Remark that in the case where the restriction to $i\mathfrak{t}_{\mathfrak{k}}$ of any $\alpha\in\Delta_{\mathfrak{g}}$ is non zero, the function $S^{\Sigma,w}_{\lambda,\mu}(z)$ is indeed equal to the function defined by Equation (9). In the general case, the function $S^{\Sigma,w}_{\lambda,\mu}(z)$ depends on our choice of X_1, but we leave this choice implicit.

Given a tope τ for $\mathcal{F}$, we define, similarly to what we did in Proposition 27

$$p^{\Sigma}_{\tau}(\lambda,\mu) = \sum_{w\in\mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}}\ \sum_{\gamma\in\Gamma_K/q\Gamma_K}\ \sum_{\overrightarrow{\sigma}\in\mathcal{OS}(\Psi,\mathfrak{a}^{\tau}_{w})} Res_{\overrightarrow{\sigma}} S^{\Sigma,w}_{\lambda,\mu}(z+\frac{2i\pi\gamma}{q})$$

and, by arguing as in the proof of Lemma 29, we can prove that

Theorem 31. *Let τ_Σ be a tope in $i\mathfrak{t}^*_{\mathfrak{g},\Sigma}\oplus i\mathfrak{t}^*_{\mathfrak{k}}$ for $\mathcal{F}_\Sigma$, and let τ be the tope for $\mathcal{F}$ containing τ_Σ.*

Let $(\lambda,\mu)\in\overline{\tau}_\Sigma\cap\Lambda^{\Sigma}_{G,K,\geq 0}$. Then

(1) *if $(\lambda,\mu)\notin C^{\Sigma}_{G,K}$ then*
$$m_{G,K}(\lambda,\mu) = p^{\Sigma}_{\tau}(\lambda,\mu) = 0.$$
(2) *if $(\lambda,\mu)\in C^{\Sigma}_{G,K}$,* **and** *the tope τ_Σ intersect $C^{\Sigma}_{G,K}$, then*
$$m_{G,K}(\lambda,\mu) = p^{\Sigma}_{\tau}(\lambda,\mu).$$

Example 32.

Let us give two simple examples to illustrate the difference of the computation of $S^{w,\Sigma}_{\lambda,\mu}$ in the case 3.2.1 and in the case 3.2.2.

We consider first the case of $G = U(4)$ and $K = SU(2)\times SU(2)$ embedded in G by considering $\mathbb{C}^4 = \mathbb{C}^2\otimes\mathbb{C}^2$. Then $\mathfrak{t}_{\mathfrak{k}}$ is two dimensional, and none of the roots of $\mathfrak{g}$ vanishes identically in $i\mathfrak{t}_{\mathfrak{k}}$.

Let $\lambda = [k,k,0,0]$ be a highest weight for a representation of G. If $\Sigma = \{e_1-e_2, e_3-e_4\}$, then $\lambda\in i\mathfrak{t}^*_{\mathfrak{g},\Sigma}$.

$\Delta_{\mathfrak{u}}$ restricted to $i\mathfrak{t}_{\mathfrak{k}}$ is $[e_2,-e_2,-e_2-e_1,-e_2+e_1]$ where e_1,e_2 are the simple roots of $SU(2)\times SU(2)$. If $z=[z_1,z_2]$ are the coordinates of

$z \in i\mathfrak{t}_{\mathfrak{k}}$ with respect to the dual basis of $\{e_1, e_2\}$ then the expression for the function $\frac{e^{\langle w(\lambda), z\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle w(\alpha),z\rangle})}$ restricted to $i\mathfrak{t}_{\mathfrak{k}}$ is

$$\frac{u_2^k}{\left(1-\frac{1}{u_2}\right)^2\left(1-\frac{1}{u_1u_2}\right)\left(1-\frac{u_1}{u_2}\right)},$$

where we made the change of variables $e^{z_1} = u_1$ and similarly. To obtain the restricted character, we have to sum over 6 permutations and we obtain that

$$\chi_\lambda(\exp(z)) = \frac{u_2^k}{\left(1-\frac{1}{u_2}\right)^2\left(1-\frac{u_1}{u_2}\right)\left(1-\frac{1}{u_1u_2}\right)}+$$

$$\frac{u_1^k}{\left(1-\frac{1}{u_1}\right)^2\left(1-\frac{1}{u_1u_2}\right)\left(1-\frac{u_2}{u_1}\right)}+\frac{2}{(1-u_2)(1-u_1)\left(1-\frac{1}{u_1}\right)\left(1-\frac{1}{u_2}\right)}+$$

$$\frac{u_1^{-k}}{(1-u_1)^2(1-u_1u_2)\left(1-\frac{u_1}{u_2}\right)}+\frac{u_2^{-k}}{(1-u_2)^2(1-u_1u_2)\left(1-\frac{u_2}{u_1}\right)}.$$

For the permutation $w = 1$, $\mu = [\mu_1, \mu_2]$ (with μ_1, μ_2 integers), the function $S^{w,\Sigma}_{\lambda,\mu}(z_1, z_2)$ is

$$(1-1/u_1)(1-1/u_2)\frac{u_2^k u_1^{-\mu_1}u_2^{-\mu_2}}{\left(1-\frac{1}{u_2}\right)^2\left(1-\frac{u_1}{u_2}\right)\left(1-\frac{1}{u_1u_2}\right)}$$

(with $u_1 = e^{z_1}$, $u_2 = e^{z_2}$).

Consider now $G = U(4)$, $K_1 = SU(2)\times\{1\}$ contained in K. Continuing with the above example, we now would like to have an expression for the restriction of χ_λ to T_{K_1} as a sum of explicit meromorphic functions. Consider for example the permutation $w = 1$. Then the term $\frac{u_2^k}{\left(1-\frac{1}{u_2}\right)^2\left(1-\frac{1}{u_1u_2}\right)\left(1-\frac{u_1}{u_2}\right)}$ cannot be restricted to $K_1 = SU(2)\times\{1\}$, since $u_2 - 1$ vanishes identically on K_1.

Thus for $z = [z, 0]$ and $X_1 = [0, 1]$, $w = id$, we compute with $u = e^z$ that

$$G_w(z, X_1) = -\frac{1}{2}\frac{(k+4)^2 u}{(u-1)^2}+\frac{1}{2}\frac{(k+4)u}{(u-1)^2}+$$

$$\frac{(k+4)u^2}{(u-1)^3}-\frac{(k+4)u}{(u-1)^3}-\frac{u^3}{(u-1)^4}+\frac{u^2}{(u-1)^4}-\frac{u}{(u-1)^4}.$$

For μ an integer, our function $S^{w,\Sigma}_{\lambda,\mu}(z)$ is $(1-1/u)u^{-\mu}G_w(z, X_1)$ (with $u = e^z$)

□

Remark 33.

We have remarked that our method to compute $m_{G,K}(\lambda,\mu)$ is a generalization of the Kostant-Heckman branching theorem. On the other hand if λ is not regular, the formula obtained for $\chi_\lambda|_{T_K}$ is not very explicit, but it has two obvious advantages from an algorithmic point of view:

• there is a smaller number of elements of the Weyl group over which we sum up.

• the function of which we compute the residues has less poles.

We fully take advantages of these points when we compute the example of Hilbert series, Section 5.2.

□

□

4. The algorithm to compute Kronecker coefficients

We now explain our algorithm to compute $g(\lambda_1,\ldots,\lambda_s)$ using Theorem 28. We assume that $s\geq 3$ and $n_i\geq 2$. From Theorem 21, there exists a cone decomposition of $C_{U(n_1)\times\cdots\times U(n_s)}(\mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_s})$ in cones $\mathfrak{c}_a$ (solid inside the vector space determined by $|\lambda_1|=|\lambda_2|=\cdots=|\lambda_s|$) such that the function $g(\lambda_1,\ldots,\lambda_s)$ is given by a quasi-polynomial formula on $\mathfrak{c}_a$. Our program computes symbolically the branching coefficients in a conic neighborhood of a given point. Let us summarize the steps of the algorithm.

We are given a sequence of s strictly positive integers $[n_1,\ldots,n_s]$ and for each integer n_i a sequence ν_i of integers: $\nu_i=[\nu_1^i,\ldots,\nu_{n_i}^i]$ with $\nu_1^i\geq\nu_2^i\geq\cdots\geq\nu_{n_i}^i\geq 0$. Each ν_i parameterizes an irreducible polynomial representation of $U(n_i)$ of highest weight ν_i. Write $N=\prod_{i=1}^s n_i$ and $M=\prod_{i=2}^s n_i$. We want to compute the dilated Kronecker coefficients $g(k\nu_1,\ldots,k\nu_s)$, that is the multiplicity of the tensor product representation $V_{k\nu_1}^{U(n_1)}\otimes\cdots\otimes V_{k\nu_s}^{U(n_s)}$ in $Sym(\mathbb{C}^N)$.

Our approach uses Cauchy formula to reduce the number of factors s. We may assume that $n_1\leq M=n_2\cdots n_s$ and that $|\nu_1|=|\nu_2|=\cdots=|\nu_s|$.

We set $G=U(M)$ and $K=U(n_2)\times\cdots\times U(n_s)$.

The first reduction step is:

• If $|\nu_1|=|\nu_2|=\cdots=|\nu_s|$ then $g(\nu_1\otimes\cdots\otimes\nu_s)=m_{G,K}(\tilde{\nu_1},\nu_2\otimes\cdots\otimes\nu_s)$ where $\tilde{\nu_1}$ is the highest weight representation of $U(M)$ obtained from ν_1 by adding $M-n_1$ zeros and the branching coefficient $m_{G,K}$ is computed in Theorem 28 via the function defined in Proposition 27.

Let us write $\lambda = \tilde{\nu_1}$, $\mu = \nu_2 \otimes \cdots \otimes \nu_s$. If $n_1 < M$, λ is a singular weight for the group $U(M)$. Denote by Σ the set of simple roots $[e_{n_1+2} - e_{n_1+1}, \ldots, e_M - e_{M-1}]$ of $U(M)$. Let $\mathfrak{l} = \mathfrak{u}(M - n_1)$ be the Lie algebra with this simple root system. We have $\langle \lambda, H_\alpha \rangle = 0$ for all $\alpha \in \Sigma$. Then $(\lambda, \mu) \in \Lambda^{\Sigma}_{G,\geq 0} \oplus \Lambda_{K \geq 0}$ with the notations as in Section 3.

Let us review the key steps of the algorithm to compute $m_{G,K}$. See the discussion in [1] outlining the limits of the implementation.

Given as input (λ, μ), we wish to compute the branching coefficients. Recall that :

$$m_{G,K}(\lambda, \mu) = \sum_{w \in \mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}} \sum_{\gamma \in \Gamma_K / q\Gamma_K} \sum_{\overrightarrow{\sigma} \in \mathcal{OS}(\Psi, \mathfrak{a}^{\tau}_{w})} Res_{\overrightarrow{\sigma}} S^{\Sigma,w}_{\lambda,\mu}(z + \frac{2i\pi\gamma}{q}). \quad (12)$$

One of the difficult point in computing the right hand side of equation (12) is to find the index q of the list of restricted roots (for $\mathfrak{g}, \mathfrak{k}$) with respect to the lattice Λ_K. We do it by brute force in our examples.

Another tricky point is to find a $\mathcal{F}_\Sigma$-tope τ_Σ such that $(\lambda, \mu) \in \overline{\tau_\Sigma}$. We do this by computing a regular point inside the Kirwan cone and deform (λ, μ) along the line from (λ, μ) to this interior point. For doing so

(1) We list all the equations X of the Ψ- admissible hyperplanes.
(2) For each such equation given by X and for $w \in \mathcal{W}_{\mathfrak{g}}$, we compute $H(X, w, \lambda, \mu) = \langle \lambda, wX \rangle - \langle \mu, X \rangle$. As (λ, μ) is in the lattice of weights, and X in the dual lattice, $H(X, w, \lambda, \mu)$ is an integer.
   Remember λ, μ are our fixed input.
(3) If $H(X, w, \lambda, \mu) \neq 0 \ \forall w, X$, then (λ, μ) is $\mathcal{F}_\Sigma$-regular and then it is in a tope τ_Σ. A fortiori it is in a unique $\mathcal{F}$ tope τ and therefore $w(\lambda) - \mu$ is in a unique tope $\mathfrak{a}^{\tau}_{w} \subset i\mathfrak{t}^*_{\mathfrak{k}}$.
   In conclusion we can compute $\mathcal{OS}(\Psi, \mathfrak{a}^{\tau}_{w})$.
(4) Else if $H(w, X, \lambda, \mu) = 0$ for some X and w, then we deform as follows:
   (a) We find $\epsilon = (\epsilon_1, \epsilon_2)$ in the interior of $C^{\Sigma}_{G,K}$. We can find this point in the cases we treat because, either we know the equations of the Kirwan cone, either we know some points in the Kirwan cone by directly computing projections. The lists of integers $(n_1, n_2, \ldots, n_s)$ where the inequations of the cone $C_K(\mathcal{H})$ are known can be found in [1].
   (b) We rescale ϵ so that $|\langle wX, \epsilon_1 \rangle - \langle X, \epsilon_2 \rangle| < 1/2$, so $(\lambda, \mu) + t\epsilon$ stays in the same tope τ_Σ for all $0 < t < 1$.
   (c) We define $(\lambda_{def}, \mu_{def})$ as $(\lambda + \epsilon_1, \mu + \epsilon_2)$.

(5) We can now pick the tope τ defined by $(\lambda_{def}, \mu_{def})$ and compute $\mathcal{OS}(\Psi, a_w^\tau)$ as in step 3.
(6) Now for a given $w \in W_{\mathfrak{g}}$, we compute

$$S_{\lambda,\mu}^{\Sigma,w}(z) = \prod_{\beta\in\Delta_{\mathfrak{k}}^+}(1-e^{-\langle\beta,z\rangle})\frac{e^{\langle\overline{w(\lambda)}-\mu,z\rangle}}{\prod_{\alpha\in\Delta_{\mathfrak{u}}}(1-e^{-\langle\overline{w(\alpha)},z\rangle})} \tag{13}$$

and the residue along an $\mathcal{OS}$ basis adapted to τ with an appropriate series expansion.

Remark 34.

For not so many $w's$, the set of $\mathcal{OS}$ basis adapted to τ is non empty. Indeed, w has to be such that $\overline{w\lambda} - \mu$ is in the cone generated by the restricted roots. This is the so called set of valid permutations for (λ, μ) defined by Cochet in [12]. See also the notion of Weyl alternative sets in [15].
□
(7) Finally to compute $m_{G,K}(\lambda, \mu)$, we have to sum the contribution from $w \in \mathcal{W}_{\mathfrak{g}}/\mathcal{W}_{\mathfrak{l}}$, over the set $\gamma \in \Gamma_K/q\Gamma_K$ and $\overrightarrow{\sigma} \in \mathcal{OS}(\Psi, \mathfrak{a}_w^\tau)$. Each individual term of these two sums, that is if we fix γ and $\overrightarrow{\sigma}$, is easy to compute in low rank, but there can be really many of these terms.

Remark that if ν_1 is a *rectangular tableau*, then $\lambda = \tilde{\nu_1}$ is even more singular. This enable us to compute more easily using a larger set Σ (reducing then the number of roots in $\Delta_{\mathfrak{u}}$ and the number of permutations). Indeed in this case Σ consists of all the simple roots minus one. When all ν_i are rectangular tableaux, this corresponds to the case of Hilbert series. We list the corresponding results in the last Subsection 5.2.

It is not more difficult to compute the function $m_{G,K}(\lambda, \mu)$ on a tope τ with symbolic variables (λ, μ). So given as input (λ^0, μ^0), the output is either the numerical value $m_{G,K}(\lambda^0, \mu^0)$, either the dilated coefficient $k \mapsto m_{G,K}(k\lambda^0, k\mu^0)$, or (in low dimensions), a tope τ containing (λ^0, μ^0) in its closure and the quasi-polynomial function in both variables λ, μ coinciding with $m_{G,K}(\lambda, \mu)$ on the closure of the tope τ.

It is clear that for fixed $n_1, n_2, \ldots, n_s$, the algorithm to compute the dilated Kronecker coefficients $g(k\nu_1, \ldots, k\nu_s)$ is of polynomial complexity with respect to the input $\nu_1, \ldots, \nu_s$.

We checked our results on Kronecker coefficients against computations made by different authors with various theoretical or computational aims (Hilbert series, stability, representations of the symmetric group, etc..). Here is a list probably far from being complete, [8],

[10], [19],[21], [25], [26], [27], [34],[36], [41]. In contrast to our method, some of these computations use directly the representation theory of the symmetric group (see for example [21],[34]) and can be made for (relatively) large number of rows, provided the content $c = |\nu_i|$ is small.

5. Examples

All examples were computed by a Maple program executed on a MacBookpro (Intel core i7 processor). The running time is at most 20 minutes for the most difficult cases. One exception is the example of entanglement of the 4-qubits (see Ex. 39). It was the hardest to compute in terms of running time, but on the other hand, we did not try to optimize the program for this particular case.

5.1. Examples of computation of dilated Kronecker coefficients.

Example 35. *The case* $\mathbb{C}^4 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

This example has been studied in complete details by [8]. In particular, a cone decomposition in 74 cones of quasi-polynomiality is given, together with the corresponding quasipolynomial of degree 2 and period 2.

We now list a number of new examples that we computed using our Maple program.

Example 36. *The case of* 4*-qubits* $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

The Kirwan polytope has been described by Higuchi-Sudbery-Szulc, [17]. We have no idea of the number of cones in a cone decomposition of $C_K(\mathcal{H})$ in cones of quasi-polynomiality. Nevertheless, given highest weights $\alpha, \beta, \gamma, \delta$, we can compute $g(k\alpha, k\beta, k\gamma, k\delta)$ as a periodic polynomial in k. It is a quasi-polynomial of degree at most 7 and period 6. More precisely, this function of the form

$$f(k) + (-1)^k g(k) + h(k)$$

where $f(k)$ is a polynomial of k of degree less or equal to 7, $g(k)$ of degree less or equal to 3, and $h(k)$ is a periodic function of $k \bmod 3$.

Here is an example. When $\alpha = \beta = \gamma = \delta = [2, 1]$, then:

$$g(k\alpha, k\beta, k\gamma, k\delta) =$$

$$\frac{23}{241920}k^7 + \frac{13}{5760}k^6 + \frac{155}{6912}k^5 + \frac{139}{1152}k^4 + \left(\frac{81601}{207360} + \frac{1}{1536}(-1)^k\right)k^3 +$$
$$\left(\frac{9799}{11520} + (-1)^k\frac{5}{256}\right)k^2 + \left(\frac{38545}{32256} + (-1)^k\frac{179}{1536}\right)k + P(k)$$

where

$$P(k) = \left(\frac{5}{243} + \frac{1}{243}\,\theta\right)\left(\theta^2\right)^k + \left(\frac{4}{243} - \frac{1}{243}\,\theta\right)\theta^k + \frac{5279}{6912} + \frac{51}{256}\,(-1)^k$$

where θ is a primitive root $\theta^3 = 1$. Thus the constant term $P(k)$ is a sum of a periodic function of period 2 and of a periodic function of period 3, leading to periodic behavior modulo 6.

The values of $P(k)$ on $0, 1, 2, 3, 4, 5$ are

$$\left[1, \frac{5725}{10368}, \frac{76}{81}, \frac{77}{128}, \frac{77}{81}, \frac{5597}{10368}\right]$$

This formula gives the Kronecker coefficients $g(k\alpha, k\beta, k\gamma, k\delta)$ for any k. Starting from $k = 0$, they are

$$1, 3, 13, 39, 110, 264, 588, 1194, 2289, 4134, 7152, 11865, \ldots$$

Example 37. *The case* $\mathbb{C}^6 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$

When $n_2 = 3$, $n_3 = 2$, it is sufficient to consider the case when $n_1 = 6$. In this case the maximum degree of the quasi-polynomial function $g(k\lambda, k\mu, k\nu)$ is computed by the formula $\prod_{j=1}^{s} n_j - \sum_{j=1}^{s} \frac{n_j(n_j-1)}{2} - \sum_{j=1}^{s} n_j + s - 1 = 8$. Then the multiplicity function $k \mapsto g(k\lambda, k\mu, k\nu)$ is a quasi-polynomial function of the form

$$f(k) + (-1)^k g(k) + h(k)$$

where $f(k)$ is a polynomial of k of degree less or equal to 8, $g(k)$ of degree less or equal to 2 and $h(k)$ is a periodic function of $k \bmod 3$.

Here is an example where the degree of the quasi-polynomial is the maximum one.

We fix $\lambda = [15, 10, 9, 4, 3, 2]$, $\mu = [21, 14, 8], \nu = [27, 16]$ and compute:

$$g(k\lambda, k\mu, k\nu) =$$

$$\frac{413587}{967680}\,k^8 + \frac{66773}{17280}\,k^7 + \frac{3072191}{207360}\,k^6 +$$

$$\frac{1091771}{34560}\,k^5 + \frac{710713}{17280}\,k^4 + \frac{871363}{25920}\,k^3 +$$

$$\left((-1)^k \frac{55}{1024} + \frac{1833073}{107520})\right) k^2 + \left((-1)^k \frac{79}{512} + \frac{117661}{23040}\right) k +$$

$$\left(\frac{10}{243} + \frac{4}{243}\,\theta\right)\theta^k + \left(\frac{2}{81} - \frac{4}{243}\,\theta\right)\left(\theta^2\right)^k + \frac{275}{2048}\,(-1)^k + \frac{398071}{497664}$$

where θ is a third primitive root of 1. The term of the degree zero in $g(k\lambda, k\mu, k\nu)$ is thus a periodic function r of k, of period 6, whose values are given by

$$[r(0), r(1), r(2), r(3), r(4), r(5)] = \left[1, \frac{50429}{82944}, \frac{25}{27}, \frac{749}{1024}, \frac{71}{81}, \frac{18175}{27648}\right].$$

Thus, the values of $g(k\lambda, k\mu, k\nu)$ starting from $k = 0$, are

$$1, 148, 3570, 34140, 197331, 829417, 2797696, \ldots.$$

We now illustrate some other particularly interesting examples that connect the behavior of the quasi polynomial function $g(\lambda, \mu, \nu)$ on cones $\mathfrak{c}_a$ adjacent to a facet F of the Kirwan cone.

Recall that a wall of the Kirwan polytope is regular if it intersects the interior of the Weyl chamber. Thus the Kirwan cone is determined by inequations defined by regular walls, and inequations determining the Weyl chamber. The regular walls of the Kirwan cone for the action of $U(6)\times U(3)\times U(2)$ in $\mathbb{C}^6\otimes\mathbb{C}^3\otimes\mathbb{C}^2$ have been described by Klyachko [22]. Given $\lambda = [\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]$, $\mu = [\mu_1, \mu_2, \mu_3]$ and $\nu = [\nu_1, \nu_2]$, the regular walls leads to the following 5 types of inequations in λ, μ, ν (and equations $|\lambda| = |\mu| = |\nu|$).

More precisely, for each of the inequations F below, Table 1, there is a particular subset S of $\mathfrak{S}_6 \times \mathfrak{S}_3 \times \mathfrak{S}_2$ (where $\mathfrak{S}_k$ is the group of permutations of k elements) computed by Klyachko such that the permuted inequations is a irredundant inequation of the corresponding Kirwan cone. Each subset S contains the identity.

type	Inequations
I	$F_I : \nu_1 - \nu_2 - \lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 \leq 0$
II	$F_{II} : \mu_1 + \mu_2 - 2\,\mu_3 - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + 2\,\lambda_5 + 2\,\lambda_6 \leq 0$
III	$F_{III} : 2\,\mu_1 - 2\,\mu_3 + \nu_1 - \nu_2 - 3\,\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4 + \lambda_5 + 3\,\lambda_6 \leq 0$
IV	$F_{IV} : 2\,\mu_1 + 2\,\mu_2 - 4\,\mu_3 + 3\,\nu_1 - 3\,\nu_2 - 5\,\lambda_1 - 5\,\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 7\,\lambda_6 \leq 0$
V	$F_V : 4\,\mu_1 - 2\,\mu_2 - 2\,\mu_3 + 3\,\nu_1 - 3\,\nu_2 - 7\,\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + 5\,\lambda_5 + 5\,\lambda_6 \leq 0$

TABLE 1. Walls type

We will now give a list of elements $v_F = [\lambda_F, \mu_F, \nu_F]$ in the relative interior of each facet $\{F = 0\} \cap C_K(\mathcal{H})$ of the cone. Here $F \in \{F_I, F_{II}, F_{III}, F_{IV}, F_V\}$. Thus the corresponding dilated Kronecker coefficients $g(kv_F)$ is of the maximum degree (as predicted by Lemma 37 in [1], Table 2), among the dilated coefficients $g(k\lambda, k\mu, k\nu)$ when (λ, μ, ν) varies in a facet of the given type F.

Facet	$v_F = [\lambda, \mu, \nu]$	$g(k\lambda, k\mu, k\nu)$
F_I	$[[288, 192, 174, 120, 30, 6], [343, 270, 197], [654, 156]]$	$1 + 17k$
F_{II}	$[[300, 186, 150, 78, 48, 6], [438, 276, 54], [465, 303]]$	$\frac{121077\, k^3}{4} + \frac{21051\, k^2}{8} + \frac{311\, k}{4} + 3/16\ (-1)^k + \frac{13}{16}$
F_{III}	$[[47, 35, 23, 13, 5, 1], [76, 38, 10], [85, 39]]$	1
F_{IV}	$[[276, 204, 120, 66, 30, 6], [351, 273, 78], [552, 150]]$	$1 + 36k$
F_V	$[[276, 198, 126, 66, 48, 6], [406, 201, 113], [536, 184]]$	$1 + 41k$

TABLE 2. Dilated Kronecker coefficients on walls

Let $C_{III} = \{F_{III} = 0\} \cap C_K(\mathcal{H})$. From the reduction principle of multiplicities on regular faces, we know that the restriction of $g(\lambda, \mu, \nu)$ to $C_{III} \cap \Lambda_K$ (or any facet obtained by the Klyacho permutations) is identically 1. Indeed let $X_0 = [[-3, -1, -1, 1, 1, 3], [2, 0, -2], [1, -1]]$, the element of $i\mathfrak{t}_\mathfrak{k}$ perpendicular to the wall associated to C_{III}. Let K_0 the stabilizer of X_0 in K, and $\mathcal{H}^0$ the subspace of $\mathcal{H}$ stable by X_0. Then K_0 is isomorphic to the subgroup $(U(1) \times U(2) \times U(2) \times U(1)) \times (U(1) \times U(1) \times U(1)) \times (U(1) \times U(1))$ of $U(6) \times U(3) \times U(2)$. The multiplicity $m_K^{\mathcal{H}}$ restricted to $C_{III} \cap \Lambda_K$ coincides with $m_{K_0}^{\mathcal{H}_0}$. This multiplicity function is easily computed to be identically 1 on $C_{III} \cap \Lambda_K$. Thus any element of $C_{III} \cap \Lambda_K$ (or any facet obtained by the Klyacho permutations) is stable.

For each of the cases in Table 2 we can also compute symbolically a quasi-polynomial function coinciding with the Kronecker coefficients on a closed solid cone $\mathfrak{c}_{v_F}$ of $C_K(\mathcal{H})$ containing the element v_F. Following the general method, we compute an element v_F^ϵ close to v_F and not on any admissible wall. For example for v_{F_I} and F_I, we can choose $v_{F_I}^\epsilon = [[291, 194, 175, 120, 30, 6], [347, 272, 197], [659, 157]]$. Then the function $g(\lambda, \mu, \nu)$ on the tope $\tau(v_F^\epsilon)$ containing v_F^ϵ is a quasi-polynomial function. The closure $\mathfrak{c}_{v_F}$ of $\tau(v_F^\epsilon)$ contains v_F and $\mathfrak{c}_{v_F}$ is a cone of quasi-polynomiality adjacent to the facet $\{F = 0\} \cap C_K(\mathcal{H})$. The degree of the quasi polynomial function $g(\lambda, \mu, \nu)$ on $\mathfrak{c}_{v_F}$ is 8, as we know. When we restrict this symbolic quasi polynomial to the element kv_F, we do indeed get $g(kv_F)$.

For the element v_{F_I}, we find that the symbolic function on $\mathfrak{c}_{v_{F_I}}$ is polynomial, instead of merely quasi-polynomial. Remark the striking result that this polynomial function is divisible by 7 linear factors with constant value $1, 2, 3, 4, 5, 6, 7$ on the face F_I. Thus the restriction of this function to F_I is indeed linear. The results for the element v_{F_I} are summarized in Table 3. The polynomial on the next to the last line of Table 3 is the symbolic polynomial in a neighborhood of v_{F_I} in the hyperplane$\{F_I = 0\}$. When computed on v_{F_I} it gives indeed $1 + 17k$, that we have computed independently by using the element v_{F_I}.

$[\lambda,\mu,\nu] \in \tau_I$	$g(\lambda,\mu,\nu)$
$[\lambda,\mu,\nu]$	$\frac{1}{5040}\,(\lambda_1+\lambda_2+\lambda_3-\nu_1+7)\,(\lambda_1+\lambda_2+\lambda_3-\nu_1+6)$ $(\lambda_1+\lambda_2+\lambda_3-\nu_1+5)\,(\lambda_1+\lambda_2+\lambda_3-\nu_1+4)$ $(\lambda_1+\lambda_2+\lambda_3-\nu_1+3)\,(\lambda_1+\lambda_2+\lambda_3-\nu_1+2)$ $(\lambda_1+\lambda_2+\lambda_3-\nu_1+1)\,(\lambda_1+\lambda_2+\lambda_3-\mu_1-\mu_2+1)$
$\lambda=[291\ k,194\ k,175\ k,12\ k,30\ k,6\ k]$ $\mu=[347\ k,272\ k,197\ k],\nu=[659\ k,157\ k]$	$\frac{1}{5040}(7+k)(6+k)(k+5)(k+4)$ $(k+3)(k+2)(k+1)(16k+1)$
$[\lambda,\mu,\nu] \in \{F_I=0\}\cap \mathfrak{c}_{v_{F_I}}$	$1+\frac{503}{140}(\lambda_1+\lambda_2)+\lambda_4+\lambda_5+\frac{363}{140}(\lambda_3+\nu_1)$
$k\ v_{F_I}$	$1+17k$

TABLE 3. Results for the wall of type I

We do not obtain such nice expressions in the other cases, (in particular the quasi polynomials obtained are not polynomials), but we can compute nonetheless the symbolic quasi polynomial. The results of the computations are too long to be included here.

Let us observe that we do not know how to compute the degree when (λ,μ,ν) is on a face defined by the Weyl chamber, see Subsection 5.2 for the case of three rectangular tableaux. Here is an example for which the degree is smaller for singular μ. Consider $\lambda=[9,7,5,3,2,1]$, $\mu=[9,9,9]$, $\nu=[14,13]$, then

$g(k\lambda,k\mu,k\nu)$ is given by the following formula

$\left(\frac{13}{64}\,(-1)^k+\frac{67}{64}\right)k+\frac{17}{12}\,k^2+\frac{617}{432}\,k^3+\frac{19}{24}\,k^4+\frac{55}{288}\,k^5+\frac{1}{81}\,\theta^k\,(-2\,\theta+8)+\frac{1}{81}\,\theta^{2\,k}\,(2\,\theta+10)+\frac{85}{144}+\frac{3}{16}\,(-1)^k$

Here θ is again a primitive root $\theta^3=1$. Thus the term of degree zero is a periodic function r of k such that

$$[r(0),r(1),r(2),r(3),r(4),r(5)]=[1,71/216,17/27,5/8,19/27,55/216]$$

Of course, the value of $g(0,0,0)$ is equal to 1. Here $g(\lambda,\mu,\nu)=5$ and for instance the value $g(17\lambda,17\mu,17\nu)=344715$.

Example 38. *The case of 3-qutrits* $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$

The multiplicity function $k\mapsto g(k\lambda,k\mu,k\nu)$ is a quasi-polynomial function of degree at most 11 and with constant term a periodic function of k (mod 12). The actual numerical values are computed in a rather quick time.

Let us give an example of the dilated Kronecker coefficient. We omit the actual formula as it is too long. The periodic term for the coefficient of degree 0 of $g(k\lambda,k\mu,k\nu)$ with $\lambda=\mu=\nu=[4,3,2]$ is given on k (mod 12) by the values:

$$\left[1,\frac{1166651}{5308416},\frac{13403}{20736},\frac{29899}{65536},\frac{59}{81},\frac{1166651}{5308416},\frac{235}{256},\frac{980027}{5308416},\frac{59}{81},\frac{32203}{65536},\frac{13403}{20736},\frac{980027}{5308416}\right]$$

In this case $g(k\lambda, k\mu, k\nu)$ has precisely degree 11.

5.2. **Rectangular tableaux and Hilbert series.** We give a list of the Kronecker coefficients for the following situation of rectangular tableaux. We use the following notations: $(\mathbb{C}^2)^3 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $[[1,1]]^3 = [[1,1],[1,1],[1,1]]$, $\mathbb{C}^{[4,3,3]} = \mathbb{C}^4 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and similarly. In the following table, the second column refers to the choice of the parameters $[\lambda, \mu, \nu]$ and the third column to the value of the Hilbert series $\sum_k m(k)t^k$ where $m(k)$ is the Kronecher coefficient $g(k\lambda, k\mu, k\nu)$.

type	parameters	value
$(\mathbb{C}^2)^3$	$[[1,1]]^3$	$\frac{1}{1-t^2}$
$(\mathbb{C}^2)^4$	$[[1,1]]^4$	$\frac{1}{(1-t)(1-t^2)^2(1-t^3)}$.
$(\mathbb{C}^2)^5$	$[[1,1]]^5$	HS_{22222}
$(\mathbb{C}^3)^3$	$[[1,1,1]]^3$	$\frac{1}{(1-t^2)(1-t^3)(1-t^4)}$.
$\mathbb{C}^{[4,3,3]}$	$[[3,3,3,3],[4,4,4],[4,4,4]]$	$\frac{1+t^9}{(1-t^2)^2(1-t^4)(1-t)(1-t^3)}$

where

$$HS_{22222} = \sum g(k[1,1], k[1,1], k[1,1], k[1,1], k[1,1]])t^k = P(t)/Q(t)$$

where

$$P(t) = t^{52} + 16\,t^{48} + 9\,t^{47} + 82\,t^{46} + 145\,t^{45} + 383\,t^{44} + 770\,t^{43} +$$
$$1659\,t^{42} + 3024\,t^{41} + 5604\,t^{40} + 9664\,t^{39} + 15594\,t^{38} + 24659\,t^{37} + 36611\,t^{36} + 52409\,t^{35} +$$
$$71847\,t^{34} + 95014\,t^{33} + 119947\,t^{32} + 146849\,t^{31} + 172742\,t^{30} + 195358\,t^{29} + 214238\,t^{28} +$$
$$225699\,t^{27} + 229752\,t^{26} + 225699\,t^{25} + 214238\,t^{24} + 195358\,t^{23} + 172742\,t^{22} + 146849\,t^{21} +$$
$$119947\,t^{20} + 95014\,t^{19} + 71847\,t^{18} + 52409\,t^{17} + 36611\,t^{16} + 24659\,t^{15} + 15594\,t^{14} + 9664\,t^{13} +$$
$$5604\,t^{12} + 3024\,t^{11} + 1659\,t^{10} + 770\,t^{9} + 383\,t^{8} + 145\,t^{7} + 82\,t^{6} + 9\,t^{5} + 16\,t^{4} + 1$$

and

$$Q(t) = (1-t^2)^5(1-t^3)(1-t^4)^5(1-t^5)(1-t^6)^5.$$

We remark that for the case $(\mathbb{C}^2)^5$ of 5-qubits the result in [25] correspond to the series $\sum_k m(k)t^{2k}$ and has a misprint on the value of the coefficient a_n for $n = 42$ (corresponding to the coefficient of t^{21} in our formula for P), as the numerator is not palindromic. So the value a_n for $n = 42$ in [25] has to be replaced by 146849.

For completeness we give the value of the Kronecker coefficients in the examples considered, we omit the actual expression for the Kronecker coefficients in the 5-qubits case and the one for $g(k[3,3,3,3], k[4,4,4], k[4,4,4])$ because the formula is too long to be reproduced here.

$$g(k[1,1], k[1,1]) = \frac{1}{2} + \frac{1}{2}(-1)^k$$

$$g(k[1,1], k[1,1], k[1,1], k[1,1]) =$$

$$\frac{23}{36}+\frac{1}{4}\,(-1)^k+\frac{1}{27}\theta^k\,(2+\theta)+\frac{1}{27}\theta^{2\,k}\,(1-\theta)+\left(\frac{29}{48}+\frac{1}{16}\,(-1)^k\right)k+\frac{1}{16}\,k^2+\frac{k^3}{72}$$

$$g(k[1,1,1],k[1,1,1],k[1,1,1])=$$

$$\frac{107}{288}+\frac{9}{32}\,(-1)^k+\left(1+(-1)^k\right)\frac{1}{16}i^k+\left(1+(-1)^{k+1}\right)\frac{1}{16}i^{k+1}+$$

$$\frac{1}{9}\,\theta^{2\,k}+\frac{1}{9}\,\theta^k+\left(\frac{1}{16}\,(-1)^k+\frac{3}{16}\right)k+\frac{1}{48}\,k^2$$

where θ is a third root of unity. For $g(k[1,1,1],k[1,1,1],k[1,1,1])$ we report, as an example, the expressions on cosets. We have twelve cosets and thus a sequence of 12 polynomials given by the following list

$$[1+\frac{1}{4}\,k+\frac{1}{48}\,k^2,-\frac{7}{48}+\frac{1}{8}\,k+\frac{1}{48}\,k^2,\frac{5}{12}+\frac{1}{4}k+\frac{1}{48}\,k^2,\frac{7}{16}+\frac{1}{8}\,k+\frac{1}{48}\,k^2,$$
$$\frac{2}{3}+\frac{1}{4}\,k+\frac{1}{48}\,k^2,-\frac{7}{48}+\frac{1}{8}\,k+\frac{1}{48}\,k^2,\frac{3}{4}+\frac{1}{4}\,k+\frac{1}{48}\,k^2,\frac{5}{48}+\frac{1}{8}\,k+\frac{1}{48}\,k^2,$$
$$\frac{2}{3}+\frac{1}{4}\,k+\frac{1}{48}\,k^2,\frac{3}{16}+\frac{1}{8}\,k+\frac{1}{48}\,k^2,\frac{5}{12}+\frac{1}{4}\,k+\frac{1}{48}\,k^2,\frac{5}{48}+\frac{1}{8}\,k+\frac{1}{48}\,k^2]$$

The following is the list of values of the Kronecker coefficients computed by the above formula for $0\leq k\leq 20:$

$$[1,0,1,1,2,1,3,2,4,3,5,4,7,5,8,7,10,8,12,10,14].$$

The above values are part of what is known as the OEIS, *The on-line encyclopedia of integers sequences*, sequence A005044.

Observe that in this example the saturation factor is 2.

Example 39. *The Hilbert series of entanglement for* 4*-qubits.*

Consider $\mathcal{H}=(\mathbb{C}^2)^4=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ and consider the standard action of $U(2)\times U(2)\times U(2)\times U(2)$ on $\mathcal{H}$. The space $\mathcal{H}$ is the space of 4-qubits.

We now consider the direct sum $\tilde{\mathcal{H}}=\mathcal{H}\oplus\mathcal{H}$ of two copies of $\mathcal{H}$, so $Sym(\tilde{\mathcal{H}})=Sym(\mathcal{H})\otimes Sym(\mathcal{H})$.

The decomposition of the tensor product representation of $U(2)\times U(2)\times U(2)\times U(2)$ in $Sym(\tilde{\mathcal{H}})=Sym(\mathcal{H})\otimes Sym(\mathcal{H})$ is considered in Wallach [41]. The Hilbert series for invariants is called the Hilbert series of entanglement for 4-qubits.

We write thus $\tilde{\mathcal{H}}=\mathcal{H}\oplus\mathcal{H}=\mathcal{H}\otimes\mathbb{C}^2$, and we consider the action of $U(2)\times U(2)\times U(2)\times U(2)\times\{1\}$ on $\tilde{\mathcal{H}}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$.

We first write $\tilde{\mathcal{H}}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ as $\mathbb{C}^2\otimes(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$, and we consider the action of $U(2)\times U(16)$ on $\mathbb{C}^2\otimes(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2)$.

As in the case of 5-qubits, Cauchy formula allows us to compute $Sym(\tilde{\mathcal{H}})$ as a representation of $U(2)\times G$ with $G=U(16)$.

Consider $\lambda = [1,1]$ and $\tilde{\lambda} = [1,1,0,0,\ldots,0]$ a highest weight for $U(16)$. Let $K = U(2)\times U(2)\times U(2)\times U(2)$ embedded in $G = U(16)$ and let $K_1 = SU(2)\times SU(2)\times SU(2)\times\{1\}$ embedded in K.

Thus following the method outlined in Subsection 3.2.2, and Theorem 31, we can compute the branching coefficient $m(k) = m_{G,K_1}(k\tilde{\lambda}, k\mu)$ with $\mu = 0$ indexing the trivial representation of K_1. We obtain a quasi-polynomial of degree 19 and periodic of period 6, that we list at the end by listing 6 polynomials on cosets of $\mathbb{Z}$ mod $6\mathbb{Z}$. Then the generating function relative to this multiplicity is the Hilbert series of measures of entanglement for 4-qubits as computed by Wallach in [41]. We recompute his formula

$$\sum m(k)t^k = \sum \dim\left[Sym^{2k}(\tilde{\mathcal{H}})^{SL(\mathbb{C}^2)\times SL(\mathbb{C}^2)\times SL(\mathbb{C}^2)\times SL(\mathbb{C}^2)}\right]t^k =$$

$$\frac{P(q)}{(1-q^2)(1-q^4)^{11}(1-q^6)^6}$$

where $t^2 = q$ and

$P(q) = q^{54} + 3\,q^{50} + 20\,q^{48} + 76\,q^{46} + 219\,q^{44} + 654\,q^{42} + 1539\,q^{40} + 3119\,q^{38} + 5660\,q^{36} + 9157\,q^{34} + 12876\,q^{32} + 16177\,q^{30} + 18275\,q^{28} + 18275\,q^{26} + 16177\,q^{24} + 12876\,q^{22} + 9157\,q^{20} + 5660\,q^{18} + 3119\,q^{16} + 1539\,q^{14} + 654\,q^{12} + 219\,q^{10} + 76\,q^{8} + 20\,q^{6} + 3\,q^{4} + 1$

We conclude by listing the 6 polynomials $[W_0, W_1, W_2, W_3, W_4, W_5]$ on the cosets $[0,1,2,3,4,5]$. Define

$$p(k) = \frac{353}{472956150389538816000}k^{19} + \frac{353}{3111553620983808000}k^{18} + \frac{271067}{331899052904939520000}k^{17} + \\ + \frac{90331}{244043421253632000}k^{16} + \frac{96329}{81347807084544000}k^{15} + \frac{4335209}{15252713828352000}k^{14} + \frac{299075479}{56317712596992000}k^{13} + \\ \frac{556811179}{7039714074624000}k^{12} + \frac{1343229996}{14079428149248000}k^{11} + \frac{1507096313}{159993501696000}k^{10}$$

and

$$p_{even}(k) = \frac{30016136009\,k^9}{391095226368000} + \frac{8474560763\,k^8}{16295634432000} + \frac{417926105131\,k^7}{141228831744000} + \frac{84164633999\,k^6}{5884534656000}$$

,

$$p_{odd}(k) = \frac{1920961135001\,k^9}{25030094487552000} + \frac{542157180107\,k^8}{1042920603648000} + \frac{6671912967271\,k^7}{2259661307904000} + \frac{1335013209659\,k^6}{94152554496000}$$

then:

$$W_0(k) := p(k) + p_{even}(k) + \frac{38627139511}{653837184000}\,k^5 + \frac{50415619753}{245188944000}\,k^4 + \frac{266225257897}{463134672000}\,k^3 + \frac{4572054901}{3859455600}\,k^2 + \frac{14055407\,k}{8953560} + 1$$

$$W_1(k) := p(k) + p_{odd}(k) + \frac{219573425545427}{3813178457088000}\,k^5 + \frac{276452038823221}{1429941921408000}\,k^4 + \frac{125778224018203934 89}{248924289678704640000}\,k^3 + \frac{572824001947094231}{622310724196761600}\,k^2 + \\ \frac{159318923928183241}{166314250686431232}\,k + \frac{290588607887}{835884417024}$$

$$W_2(k) := p(k) + p_{even}(k) + \frac{38627139511\,k^5}{653837184000} + \frac{36750520335937\,k^4}{178742740176000} + \frac{1745362160646217\,k^3}{3038626582992000} + \frac{89590754414783\,k^2}{75965664574800} + \\ \frac{815186343623}{528698764440}k + \frac{1506571}{1594323}$$

$$W_3(k) := p(k) + p_{odd}(k) + \frac{301217799563}{5230697472000}\, k^5 + \frac{379529711549}{1961511552000}\, k^4 + \frac{1927034414248049}{3793999233024000}\, k^3 + \frac{29795123615357}{31616660275200}\, k^2 + \frac{109432200819\, k}{104316534784} + \frac{261589}{524288}$$

$$W_4(k) := p(k)+p_{even}(k)+\frac{28157390911519}{476647307136000}\, k^5+\frac{36724846687937178742740176000}{}k^4+\frac{1738714367494217}{3038626582992000}\, k^3+\frac{88327521243583}{75965664574800}\, k^2+\frac{2345378642869}{1586096293320}k+\frac{1353103}{1594323}$$

$$W_5(k) := p(k)+p_{odd}(k)+\frac{301217799563}{5230697472000}\, k^5+\frac{276657428007221}{1429941921408000}\, k^4+\frac{12632281123321577489}{24892428967870464000}\, k^3+\frac{583172408085564631}{622310724196761600}\, k^2 +\frac{56607866326977347\, k}{55438083562143744}+\frac{371050038671}{835884417024}$$

The complete polynomial, with θ a third primitive root of one, is given by:

$$\begin{aligned}
&\frac{353}{472956150389538816000}k^{19} + \frac{353}{3111553620983808000}k^{18} + \frac{271067}{331899052904939520000}k^{17}\\
&+\frac{90331}{244043421253632000}k^{16}+\frac{96329}{8134780708454400}k^{15}+\frac{4335209}{15252713828352000}k^{14}+\frac{299075479}{56317712596992000}k^{13}\\
&+\frac{556811179}{7039714074624000}k^{12} + \frac{13432299961}{14079428149248000}k^{11} + \frac{1507096313}{159993501696000}k^{10}+\\
&\left((-1)^k\,\frac{17}{11890851840} + \frac{3841993839577}{50060188975104000}\right) k^9+\left((-1)^k\,\frac{17}{165150720} + \frac{1084529068939}{2085841207296000}\right) k^8+\\
&\left((-1)^k\,\frac{817}{247726080} + \frac{13358730649367}{4519322615808000}\right) k^7 + \left((-1)^k\,\frac{91}{1474560} + \frac{2681647353643}{188305108992000}\right) k^6+\\
&\left(\frac{2\,(\theta+1)}{1594323}\theta^k - \frac{2\,\theta}{1594323}\,(\theta^k)^2 + (-1)^k\,\frac{1649}{2211840} + \frac{1334555059856737}{22879070742528000}\right) k^5+\\
&\left(\frac{(229\,\theta+251)}{4782969}\theta^k + \frac{(-229\,\theta+22)}{4782969}(\theta^k)^2 + (-1)^k\,\frac{559}{92160} + \frac{570537819046717}{2859883842816000}\right) k^4+\\
&\left(\frac{(3488\,\theta+4192)}{4782969}\theta^k + \frac{(-3488\,\theta+704)}{4782969}(\theta^k)^2 + (-1)^k\,\frac{198924917}{5945425920} + \frac{8967103302680393051}{16594952645246976000}\right) k^3+\\
&\left(\frac{(26512\,\theta+34928)}{4782969}\theta^k + \frac{(-26512\,\theta+8416)}{4782969}(\theta^k)^2 + (-1)^k\,\frac{10001959}{82575360} + \frac{437463645838719389}{414873816131174400}\right) k^2+\\
&\left(\frac{(100700\,\theta+145244)}{4782969}\theta^k + \frac{(-100700\,\theta+14848)}{4782969}(\theta^k)^2\,(-1)^k\,\frac{688047337}{2642411520} + \frac{6335305750969416391}{4989427520592936960}\right) k+\\
&\left(\frac{(5684\,\theta+241220)}{177147}\theta^k + \frac{1}{177147}(\theta^k)^2 + (-1)^k\,\frac{262699}{1048576} + \frac{3413873184941}{5015306502144}\right)
\end{aligned}$$

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