

# Total positivity of Riordan arrays

Xi Chen, Huyile Liang and Yi Wang \*

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

## Abstract

We present sufficient conditions for total positivity of Riordan arrays. As applications we show that many well-known combinatorial triangles are totally positive and many famous combinatorial numbers are log-convex in a unified approach.

MSC: 05A20; 15B36; 15A45

Keywords: Riordan array; Totally positive matrix; Log-convex sequence; Log-concave sequence

## 1 Introduction

Riordan arrays play an important unifying role in enumerative combinatorics [18]. A (*proper*) Riordan array, denoted by  $(g(x), f(x))$ , is an infinite lower triangular matrix whose generating function of the  $k$ th column is  $x^k f^k(x)g(x)$  for  $k = 0, 1, 2, \dots$ , where  $g(0) = 1$  and  $f(0) \neq 0$ . A Riordan array  $R = [r_{n,k}]_{n,k \geq 0}$  can also be characterized by two sequences  $(a_n)_{n \geq 0}$  and  $(z_n)_{n \geq 0}$  such that

$$r_{0,0} = 1, \quad r_{n+1,0} = \sum_{j \geq 0} z_j r_{n,j}, \quad r_{n+1,k+1} = \sum_{j \geq 0} a_j r_{n,k+j} \quad (1.1)$$

for  $n, k \geq 0$  (see [6, 10, 14, 20] for instance). Call  $(a_n)_{n \geq 0}$  and  $(z_n)_{n \geq 0}$  the  $A$ - and  $Z$ -sequences of  $R$  respectively.

Many triangles in combinatorics are Riordan arrays with simple  $A$ - and  $Z$ -sequences. For example, the Pascal triangle with  $Z = (1, 0, \dots)$  and  $A = (1, 1, 0, \dots)$ , the Catalan triangle with  $Z = (2, 1, 0, \dots)$  and  $A = (1, 2, 1, 0, \dots)$  [17], the Motzkin triangle with  $Z = (1, 1, 0, \dots)$  and  $A = (1, 1, 1, 0, \dots)$  [1], the ballot table with  $Z = A = (1, 1, 1, \dots)$  [2], the large Schröder triangle with  $Z = (2, 2, 2, \dots)$  and  $A = (1, 2, 2, \dots)$  [6], and the little Schröder triangle with  $Z = A = (1, 2, 2, \dots)$  [6]. Such triangles arise often in the enumeration of lattice paths, e.g., the Dyck paths, the Motzkin paths, and the Schröder paths and so on [6, 14, 18]. The 0th column of such an array counts the corresponding lattice paths, including the Catalan numbers, the Motzkin numbers, the large and little Schröder numbers. There have been quite a few papers concerned with combinatorics of Riordan arrays (see [6, 10, 14, 16, 18, 20] for instance). Our concern in the present paper is total positivity of Riordan arrays.

---

\*Email address: wangyi@dlut.edu.cn (Y. Wang)

Following Karlin [11], an infinite matrix is called *totally positive of order  $r$*  (or shortly,  $TP_r$ ), if its minors of all orders  $\leq r$  are nonnegative. The matrix is called  $TP$  if its minors of all orders are nonnegative. Let  $(a_n)_{n \geq 0}$  be an infinite sequence of nonnegative numbers. It is called a *Pólya frequency sequence of order  $r$*  (or shortly, a  $PF_r$  sequence), if its Toeplitz matrix

$$[a_{i-j}]_{i,j \geq 0} = \begin{bmatrix} a_0 & & & & & \\ a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ a_3 & a_2 & a_1 & a_0 & & \\ \vdots & & \dots & & \ddots & \end{bmatrix}$$

is  $TP_r$ . It is called  $PF$  if its Toeplitz matrix is  $TP$ . We say that a finite sequence  $a_0, a_1, \dots, a_n$  is  $PF_r$  ( $PF$ , resp.) if the corresponding infinite sequence  $a_0, a_1, \dots, a_n, 0, \dots$  is  $PF_r$  ( $PF$ , resp.). A fundamental result of Aissen, Schoenberg and Whitney states that a finite sequence of nonnegative numbers is  $PF$  if and only if its generating function has only real zeros (see [11, p. 399] for instance). For example, the sequence  $(r, s, t)$  of nonnegative numbers is  $PF$  if and only if  $s^2 \geq 4rt$ . We say that a nonnegative sequence  $(a_n)$  is *log-convex* (*log-concave*, resp.) if  $a_i a_{j+1} \geq a_{i+1} a_j$  ( $a_i a_{j+1} \leq a_{i+1} a_j$ , resp.) for  $0 \leq i < j$ . Clearly, the sequence  $(a_n)$  is log-concave if and only if it is  $PF_2$ , i.e., its Toeplitz matrix  $[a_{i-j}]_{i,j \geq 0}$  is  $TP_2$ , and the sequence is log-convex if and only if its Hankel matrix  $[a_{i+j}]_{i,j \geq 0}$  is  $TP_2$  [4].

There are often various total positivity properties in a Riordan array. For example, the Pascal matrix is  $TP$  [11, p. 137] and each row of it is log-concave (see [22] for more information), the Catalan numbers, the Motzkin numbers, the large and little Schröder numbers form a log-convex sequence respectively [13]. However, there is no systematic study of total positivity of Riordan arrays. The object of this paper is to study various positivity properties of Riordan arrays, including the total positivity of such a matrix, the log-convexity of the 0th column and the log-concavity of each row. The paper is organized as follows. In the next section, we present sufficient conditions for total positivity of Riordan arrays. As applications, we show that many well-known combinatorial triangles are totally positive and many famous combinatorial numbers are log-convex in a unified approach. In Section 3, we propose some problems for further work.

## 2 Main results and applications

We first present a basic result about total positivity of Riordan arrays. Let  $R = [r_{n,k}]_{n,k \geq 0}$  be a Riordan array defined by the recursive system (1.1). Call

$$J(R) = \begin{bmatrix} z_0 & a_0 & & & & \\ z_1 & a_1 & a_0 & & & \\ z_2 & a_2 & a_1 & a_0 & & \\ z_3 & a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & \dots & & \ddots \end{bmatrix}.$$

the *coefficient matrix* of the Riordan array  $R$ .

**Theorem 2.1.** *Let  $R$  be a Riordan array defined by (1.1).*

- (i) *If the coefficient matrix  $J(R)$  is  $TP_r$  ( $TP$ , resp.), then so is  $R$ .*
- (ii) *If  $R$  is  $TP_2$  and all  $z_n \geq 0$ , then the 0th column  $(r_{n,0})_{n \geq 0}$  of  $R$  is log-convex.*

To prove Theorem 2.1, we need two lemmas. The first is direct by definition and the second follows from the classic Cauchy-Binet formula.

**Lemma 2.2.** *A matrix is  $TP_r$  ( $TP$ , resp.) if and only if its leading principal submatrices are all  $TP_r$  ( $TP$ , resp.).*

**Lemma 2.3.** *If two matrices are  $TP_r$  ( $TP$ , resp.), then so is their product.*

*Proof of Theorem 2.1.* (i) It suffices to show that  $J(R)$  is  $TP_r$  implies  $R$  is  $TP_r$ . Let

$$R_n = \begin{bmatrix} r_{0,0} & & & & \\ r_{1,0} & r_{1,1} & & & \\ r_{2,0} & r_{2,1} & r_{2,2} & & \\ \vdots & & & \ddots & \\ r_{n,0} & r_{n,1} & r_{n,2} & \cdots & r_{n,n} \end{bmatrix}$$

be the  $n$ th leading principal submatrix of  $R$ . Then by Lemma 2.2, it suffices to show that  $R_n$  is  $TP_r$  for  $n \geq 1$ . We proceed by induction on  $n$ . Assume that  $R_n$  is  $TP_r$ . By (1.1), we have

$$\begin{bmatrix} r_{0,0} & & & & \\ r_{1,0} & r_{1,1} & & & \\ \vdots & & \ddots & & \\ r_{n+1,0} & r_{n+1,1} & \cdots & r_{n+1,n+1} & \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 0 & r_{0,0} & & & \\ \vdots & \vdots & \ddots & & \\ 0 & r_{n,0} & \cdots & r_{n,n} & \end{bmatrix} \begin{bmatrix} 1 & & & & \\ z_0 & a_0 & & & \\ \vdots & \vdots & \ddots & & \\ z_n & a_n & \cdots & a_0 & \end{bmatrix},$$

or briefly,

$$R_{n+1} = \begin{bmatrix} 1 & O \\ O & R_n \end{bmatrix} \begin{bmatrix} 1 & O \\ \zeta_n & \mathcal{A}_n \end{bmatrix}, \quad (2.1)$$

where  $\zeta_n = [z_0, z_1, \dots, z_n]'$  and  $\mathcal{A}_n = [a_{i-j}]_{0 \leq i, j \leq n}$ . By the induction hypothesis,  $R_n$  is  $TP_r$ , so is the first matrix on the right hand side of (2.1). On the other hand,  $[\zeta_n, \mathcal{A}_n]$  is  $TP_r$  since it is a submatrix of the  $TP_r$  matrix  $J(R)$ , so is the second matrix on the right hand side of (2.1). It follows from Lemma 2.3 that the product  $R_{n+1}$  is  $TP_r$ . Thus the matrix  $R$  is  $TP_r$ .

(ii) Note that

$$\begin{bmatrix} r_{0,0} & r_{1,0} \\ r_{1,0} & r_{2,0} \\ r_{2,0} & r_{3,0} \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} r_{0,0} & & & \\ r_{1,0} & r_{1,1} & & \\ r_{2,0} & r_{2,1} & r_{2,2} & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & z_0 \\ 0 & z_1 \\ 0 & z_2 \\ \vdots & \vdots \end{bmatrix}. \quad (2.2)$$

Clearly, the second matrix on the right hand side of (2.2) is  $TP_2$  since all  $z_n$  are nonnegative. Now  $R$  is  $TP_2$  by the assumption. Hence the matrix on the left hand side of (2.2) is  $TP_2$  by Lemma 2.3. In other words, the sequence  $(r_{n,0})_{n \geq 0}$  is log-convex.  $\square$

In the sequel we consider applications of Theorem 2.1 to two classes of special interesting Riordan arrays. The first are recursive matrices introduced by Aigner [1, 2]. Let  $a, b, s, t$  be four nonnegative numbers. Define a Riordan array  $R(a, b; s, t) = [r_{n,k}]_{n,k \geq 0}$  by

$$r_{0,0} = 1, \quad r_{n+1,0} = ar_{n,0} + br_{n,1}, \quad r_{n+1,k} = r_{n,k-1} + sr_{n,k} + tr_{n,k+1}.$$

Following Aigner [1, 2], the numbers  $C_n(a, b; s, t) = r_{n,0}$  are called the *Catalan-like numbers*. Many well-known triangles are recursive matrices. For example, the Pascal triangle, the Catalan triangle and the Motzkin triangle are  $R(1, 0; 1, 0)$ ,  $R(2, 1; 2, 1)$  and  $R(1, 1; 1, 1)$  respectively. Also, the Catalan-like numbers unify many famous counting coefficients, such as the Catalan numbers  $C_n(2, 1; 2, 1)$ , the Motzkin numbers  $C_n(1, 1; 1, 1)$ , the central binomial coefficients  $C_n(2, 2; 2, 1)$ , and the large Schröder numbers  $C_n(2, 2; 3, 2)$ . See [2] for details.

**Theorem 2.4.** *Let  $a, b, s, t$  be four nonnegative numbers.*

- (i) *If  $as \geq b$  and  $s^2 \geq t$ , then the sequence  $(r_{n,0})_{n \geq 0}$  is log-convex.*
- (ii) *If  $s^2 \geq 4t$  and  $a \frac{s + \sqrt{s^2 - 4t}}{2} \geq b$ , then the matrix  $R(a, b; s, t)$  is totally positive.*

**Remark 2.5.** From Theorem 2.4 it follows immediately that the Pascal triangle and the Catalan triangle are totally positive, and that the Catalan numbers, the Motzkin numbers, the central binomial coefficients, and the large Schröder numbers are log-convex respectively.

**Remark 2.6.** Let

$$H(a, b; s, t) = [r_{n+m,0}]_{n,m \geq 0} = \begin{bmatrix} r_{0,0} & r_{1,0} & r_{2,0} & \cdots \\ r_{1,0} & r_{2,0} & r_{3,0} & \cdots \\ r_{2,0} & r_{3,0} & r_{4,0} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

be the Hankel matrix of the Catalan-like numbers  $r_{n,0}$ . Aigner [1, 2] computed the determinants of the leading principal submatrices of  $H$ . Aigner's Fundamental Theorem in [2] gives  $H = RTR'$ , where  $T = \text{diag}(1, t, t^2, t^3, \dots)$ . So the total positivity of  $R$  implies that of the Hankel matrix  $H$ . In particular, if  $s^2 \geq 4t$  and  $a \frac{s + \sqrt{s^2 - 4t}}{2} \geq b$ , then the Hankel matrix  $H(a, b; s, t)$  is totally positive.

By Theorem 2.1, to prove Theorem 2.4, it suffices to prove that the coefficient matrix of  $R(a, b; s, t)$

$$\begin{bmatrix} a & 1 & & & \\ b & s & 1 & & \\ & t & s & 1 & \\ & & t & s & 1 \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

is  $\text{TP}_2$  and  $\text{TP}$  under the conditions respectively. We do this by establishing the following stronger result.

**Proposition 2.7.** *Let  $a, b, r, s, t$  be five nonnegative numbers and the Jacobi matrix*

$$J = \begin{bmatrix} a & r & & & \\ b & s & r & & \\ & t & s & r & \\ & & t & s & r \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then

(i)  $J$  is  $TP_2$  if and only if  $as \geq br$  and  $s^2 \geq rt$ .

(ii)  $J$  is  $TP$  if and only if  $s^2 \geq 4rt$  and  $a(s + \sqrt{s^2 - 4rt})/2 \geq br$ .

*Proof.* (i) is obvious, and it remains to prove (ii).

Clearly, the tridiagonal matrix  $J$  is  $TP$  if and only if all its principal minors containing consecutive rows and columns are nonnegative (see, e.g., [15, Theorem 4.3]), i.e., all determinants of forms

$$d_n = \det \begin{bmatrix} s & r & & & \\ t & s & r & & \\ & t & s & \ddots & \\ & & \ddots & \ddots & r \\ & & & t & s \end{bmatrix}_{n \times n}$$

and

$$D_n = \det \begin{bmatrix} a & r & & & \\ b & s & r & & \\ & t & s & \ddots & \\ & & \ddots & \ddots & r \\ & & & t & s \end{bmatrix}_{(n+1) \times (n+1)}$$

are nonnegative. Note that all  $d_n$  are nonnegative if and only if the Toeplitz matrix

$$\begin{bmatrix} r & & & \\ s & r & & \\ t & s & r & \\ & t & s & r \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

is  $TP$ , i.e., the sequence  $(r, s, t)$  is  $PF$ . Hence all  $d_n \geq 0$  if and only if  $s^2 \geq 4rt$ . We complete the proof of (ii) by showing that all  $D_n \geq 0$  if and only if  $(s + \sqrt{s^2 - 4rt})/2 \geq br/a$ .

Note that  $D_0 = a$  and  $D_n = ad_n - brd_{n-1}$  by expanding the determinant along the first column, where  $d_0 = 1$  and  $d_1 = s$ . Hence all  $D_n \geq 0$  if and only if all  $d_n/d_{n-1} \geq br/a$ . We next show that the sequence  $d_n/d_{n-1}$  is nonincreasing and convergent to  $(s + \sqrt{s^2 - 4rt})/2$ , which means that all  $d_n/d_{n-1} \geq br/a$  is equivalent to  $(s + \sqrt{s^2 - 4rt})/2 \geq br/a$ , and so all  $D_n \geq 0$  if and only if  $(s + \sqrt{s^2 - 4rt})/2 \geq br/a$ .

Actually, we have by expanding the determinant along the first column

$$d_n = sd_{n-1} - rtd_{n-2}, \quad d_0 = 1, d_1 = s. \quad (2.3)$$

Solving this linear recurrence relation we obtain

$$d_n = \sum_{k=0}^n \lambda^k \mu^{n-k},$$

where

$$\lambda = \frac{s + \sqrt{s^2 - 4rt}}{2}, \quad \mu = \frac{s - \sqrt{s^2 - 4rt}}{2}$$

are two roots of the characteristic equation  $x^2 - sx + rt = 0$  of (2.3).

By (2.3), we have

$$\begin{bmatrix} d_n & d_{n+1} \\ d_{n-1} & d_n \end{bmatrix} = \begin{bmatrix} s & -rt \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_{n-1} & d_n \\ d_{n-2} & d_{n-1} \end{bmatrix}.$$

It follows that  $d_n^2 - d_{n-1}d_{n+1} = rt(d_{n-1}^2 - d_{n-2}d_n) = \dots = (rt)^{n-1}(d_1^2 - d_0d_2) = (rt)^n \geq 0$ . Thus the sequence  $d_n/d_{n-1}$  is nonincreasing, and is therefore convergent. Let  $\alpha$  be the limit. Rewrite (2.3) as

$$\frac{d_n}{d_{n-1}} = s - \frac{rt}{d_{n-1}/d_{n-2}}.$$

Take the limit to obtain  $\alpha = s - rt/\alpha$ , i.e.,  $\alpha^2 - s\alpha + rt = 0$ , so  $\alpha = \lambda$  or  $\mu$ . Now  $d_1/d_0 = s \geq \lambda$ . Assume that  $d_{n-1}/d_{n-2} \geq \lambda$ . Then

$$\frac{d_n}{d_{n-1}} = s - \frac{rt}{d_{n-1}/d_{n-2}} \geq s - \frac{rt}{\lambda} = \lambda.$$

Thus all  $d_n/d_{n-1} \geq \lambda$ . It follows that the limit  $\alpha = \lambda$  since  $\lambda \geq \mu$ , as desired.  $\square$

The second we concern about are Riordan arrays whose  $A$ - and  $Z$ -sequences are identical or nearly so. Following [6], we say that a Riordan array  $R = [r_{n,k}]_{n,k \geq 0}$  is *consistent* if  $A = Z$ . We say that  $R$  is a *quasi-consistent* Riordan array if  $A = (a_0, a_1, a_2, \dots)$  and  $Z = (a_1, a_2, \dots)$ . In this case, we have

$$r_{n+1,k} = a_0 r_{n,k-1} + a_1 r_{n,k} + a_2 r_{n,k+1} + \dots$$

for all  $n, k \geq 0$ , where  $r_{n,j} = 0$  unless  $0 \leq j \leq n$ . For example, the little Schröder triangle is consistent, the Pascal triangle, the Catalan triangle, the Motzkin triangle and the large Schröder triangle are quasi-consistent, and the ballot table is both consistent and quasi-consistent. The following theorem gives a unified settle for the total positivity of these well-known triangles.

**Theorem 2.8.** *Let  $R$  be a consistent or quasi-consistent Riordan array. Suppose that the  $A$ -sequence of  $R$  is  $PF_r$  ( $PF$ , resp.). Then  $R$  is  $TP_r$  ( $TP$ , resp.). In particular, if the  $A$ -sequence of  $R$  is log-concave, then the 0th column  $(r_{n,0})_{n \geq 0}$  of  $R$  is log-convex, and each row  $(r_{n,k})_{0 \leq k \leq n}$  of  $R$  is log-concave.*

*Proof.* For a consistent or quasi-consistent Riordan array  $R$ , its coefficient matrix is

$$J(R) = \begin{bmatrix} a_0 & a_0 & & & \\ a_1 & a_1 & a_0 & & \\ a_2 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & & \ddots \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & \vdots & & & \ddots \end{bmatrix}$$

respectively. Clearly,  $J(R)$  is  $\text{TP}_r$  if and only if the Toeplitz matrix  $[a_{i-j}]_{i,j \geq 0}$  of the sequence  $(a_n)_{n \geq 0}$  is  $\text{TP}_r$ , or equivalently, the sequence  $(a_n)_{n \geq 0}$  is  $\text{PF}_r$ . Thus it follows from Theorem 2.1 (i) that the Riordan array  $R$  is  $\text{TP}_r$  (resp.,  $\text{TP}$ ) if its  $A$ -sequence is  $\text{PF}_r$  (resp.  $\text{PF}$ ).

In particular, if  $(a_n)_{n \geq 0}$  is  $\text{PF}_2$ , then  $R$  is  $\text{TP}_2$ . It follows from Theorem 2.1 (ii) that the sequence  $(r_{n,0})_{n \geq 0}$  is log-concave. In what follows we show that each row of  $R$  is log-concave by induction. Denote  $s_i = r_{n,i}$  for  $0 \leq i \leq n$  and  $t_j = r_{n+1,j}$  for  $0 \leq j \leq n+1$ . We distinguish two cases.

First consider the case that  $R$  is a consistent Riordan array. In this case,  $t_0 = t_1$  and

$$t_k - t_{k+1} = a_0(s_{k-1} - s_k) + \cdots + a_{n-k}(s_{n-1} - s_n) + a_{n-k+1}s_n$$

for  $1 \leq k \leq n$ . It follows that each row of  $R$  is nonincreasing by induction. On the other hand,

$$\begin{bmatrix} t_{n+1} & & & & \\ t_n & t_{n+1} & & & \\ \vdots & \vdots & \ddots & & \\ t_1 & t_2 & \cdots & t_{n+1} & \end{bmatrix} = \begin{bmatrix} s_n & & & & \\ s_{n-1} & s_n & & & \\ \vdots & \vdots & \ddots & & \\ s_0 & s_1 & \cdots & s_n & \end{bmatrix} \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ \vdots & \vdots & \ddots & & \\ a_n & a_{n-1} & \cdots & a_0 & \end{bmatrix}. \quad (2.4)$$

Suppose that the sequence  $(a_n)_{n \geq 0}$  is log-concave. Then its Toeplitz matrix  $\mathcal{A} = [a_{i-j}]_{i,j \geq 0}$  is  $\text{TP}_2$ , and so are the leading principal submatrices of  $\mathcal{A}$ . Thus the second matrix on the right hand side of (2.4) is  $\text{TP}_2$ . If the  $n$ th row  $s_0, s_1, \dots, s_n$  of  $R$  is log-concave, then so is the reverse sequence  $s_n, \dots, s_1, s_0$ , which implies that the first matrix on the right hand side of (2.4) is  $\text{TP}_2$ . Thus the matrix on the left hand side of (2.4) is  $\text{TP}_2$  by Lemma 2.3, or equivalently, the sequence  $t_{n+1}, t_n, \dots, t_1$  is log-concave, and so is the reverse sequence  $t_1, \dots, t_n, t_{n+1}$ . Note that  $t_0 = t_1 \geq t_2$ . Hence the sequence  $t_0, t_1, t_2, \dots, t_{n+1}$  is also log-concave. Thus each row of  $R$  is log-concave by induction.

Next let  $R$  be a quasi-consistent Riordan array. Then

$$\begin{bmatrix} t_{n+1} & & & & \\ t_n & t_{n+1} & & & \\ \vdots & \vdots & \ddots & & \\ t_0 & t_1 & \cdots & t_{n+1} & \end{bmatrix} = \begin{bmatrix} s_n & & & & \\ s_{n-1} & s_n & & & \\ \vdots & \vdots & \ddots & & \\ s_0 & s_1 & \cdots & s_n & \\ 0 & s_0 & \cdots & s_{n-1} & s_n \end{bmatrix} \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ \vdots & \vdots & \ddots & & \\ a_{n+1} & a_n & \cdots & a_0 & \end{bmatrix}.$$

Assume that the sequence  $s_0, s_1, \dots, s_n$  is log-concave. Then the first matrix on the right hand side is  $\text{TP}_2$ . It follows that the matrix on the left hand side is  $\text{TP}_2$ . In other words, the sequence  $t_0, t_1, \dots, t_n, t_{n+1}$  is log-concave. Thus each row of  $R$  is log-concave by induction. This completes the proof.  $\square$

### 3 Further work

It is known that sequences of binomial coefficients located in a ray or a transversal of the Pascal triangle have various positivity properties (see [22, 25] for instance). Similar problems naturally arise in a Riordan array. For example, in which case each row of such a Riordan array is PF, the corresponding linear transformation can preserve the PF property (the log-concavity, the log-convexity, resp.), and each column of the array is first log-concave and then log-convex?

Aigner [2] gave combinatorial interpretations for recursive matrices in terms of weighted Motzkin paths. Cheon *et al.* [6] provided combinatorial interpretations for consistent Riordan arrays in terms of weighted Łukasiewicz paths. It is not difficult to give a similar combinatorial interpretation for a quasi-consistent Riordan array. Brenti [5] gave combinatorial proofs of total positivity of many well-known matrices by means of lattice path techniques. It is natural to ask for combinatorial proofs of Theorems 2.4 and 2.8.

### Acknowledgement

This work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11071030, 11371078) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20110041110039).

### References

- [1] M. Aigner, Catalan-like numbers and determinants, *J. Combin. Theory Ser. A* 87 (1999) 33–51.
- [2] M. Aigner, Catalan and other numbers — A recurrent theme, in: H. Crapo, D. Senato (Eds.), *Algebraic Combinatorics and Computer Science*, Springer Italian, Milan, 2001, 347–390.
- [3] M. Aigner, Enumeration via ballot numbers, *Discrete Math.* 308 (2008) 2544–2563.
- [4] F. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.* 413 (1989).
- [5] F. Brenti, Combinatorics and total positivity, *J. Combin. Theory Ser. A* 71 (1995) 175–218.
- [6] G.-S. Cheon, H. Kim and L.W. Shapiro, Combinatorics of Riordan arrays with identical  $A$  and  $Z$  sequences, *Discrete Math.* 312 (2012) 2040–2049.
- [7] S.M. Fallat and C.R. Johnson, *Totally Nonnegative Matrices*, Princeton University Press, Princeton, 2011.
- [8] S. Fomin and A. Zelevinsky, Total positivity: tests and parameterizations, *Math. Intelligencer* 22 (2000), 23–33.

- [9] T.-X. He, Parametric Catalan numbers and Catalan triangles, *Linear Algebra Appl.* 438 (2013) 1467–1484.
- [10] T.-X. He and R. Sprugnoli, Sequence characterization of Riordan arrays, *Discrete Math.* 309 (2009) 3962–3974.
- [11] S. Karlin, *Total Positivity, Volume 1*, Stanford University Press, 1968.
- [12] L.L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, *Adv. in Appl. Math.* 39 (2007) 453–476.
- [13] L.L. Liu and Y. Wang, A unified approach to polynomial sequences with only real zeros, *Adv. in Appl. Math.* 38 (2007) 542–560.
- [14] D. Merlini, D.G. Rogers, R. Sprugnoli and M.C. Verri, On some alternative characterizations of Riordan arrays, *Canad. J. Math.* 49 (1997) 301–320.
- [15] A. Pinkus, *Totally positive matrices*, Cambridge University Press, Cambridge, 2010.
- [16] D.G. Rogers, Pascal triangles, Catalan numbers and renewal arrays, *Discrete Math.* 22 (1978) 301–310.
- [17] L.W. Shapiro, A Catalan triangle, *Discrete Math.* 14 (1976) 83–90.
- [18] L.W. Shapiro, S. Getu, W.-J. Woan and L.C. Woodson, The Riordan group, *Discrete Appl. Math.* 34 (1991) 229–239.
- [19] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>.
- [20] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* 132 (1994) 267–290.
- [21] R. Sprugnoli, Combinatorial sums through Riordan arrays, *J. Geom.* 101 (2011) 195–210.
- [22] X.-T. Su and Y. Wang, On unimodality problems in Pascal’s triangle, *Electron. J. Combin.*, 15 (2008), Research Paper 113, 12 pp.
- [23] Y. Wang and Y.-N. Yeh, Polynomials with real zeros and Pólya frequency sequences, *J. Combin. Theory Ser. A* 109 (2005) 63–74.
- [24] Y. Wang and Y.-N. Yeh, Log-concavity and LC-positivity, *J. Combin. Theory Ser. A* 114 (2007) 195–210.
- [25] Y. Yu, Conforming two conjectures of Su and Wang on binomial coefficients, *Adv. in Appl. Math.* 43 (2009) 317–322.