Total positivity of recursive matrices

Xi Chen, Huyile Liang and Yi Wang *

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

Abstract

Let $A = [a_{n,k}]_{n,k\geq 0}$ be an infinite lower triangular matrix defined by the recurrence

$$a_{0,0} = 1, \quad a_{n+1,k} = r_k a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1},$$

where $a_{n,k} = 0$ unless $n \geq k \geq 0$ and r_k, s_k, t_k are all nonnegative. Many well-known combinatorial triangles are such matrices, including the Pascal triangle, the Stirling triangle (of the second kind), the Bell triangle, the Catalan triangles of Aigner and Shapiro. We present some sufficient conditions such that the recursive matrix A is totally positive. As applications we give the total positivity of the above mentioned combinatorial triangles in a unified approach.

MSC: 05A20; 15B36; 15A45

Keywords: Totally positive matrix; Recursive matrix; Tridiagonal matrix

1 Introduction

Let $A = [a_{n,k}]_{n,k\geq 0}$ be an infinite matrix. It is called *totally positive of order* r (or shortly, TP_r), if its minors of all orders $\leq r$ are nonnegative. It is called *TP* if its minors of all orders are nonnegative. Let $(a_n)_{n\geq 0}$ be an infinite sequence of nonnegative numbers. It is called a *Pólya frequency sequence of order* r (or shortly, a PF_r sequence), if its Toeplitz matrix

$$[a_{i-j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & \cdots & & \ddots \end{bmatrix}$$

is TP_r. It is called *PF* if its Toeplitz matrix is TP. We say that a finite sequence $a_0, a_1, \ldots, a_n$ is PF_r (PF, resp.) if the corresponding infinite sequence $a_0, a_1, \ldots, a_n, 0, \ldots$

Email address: wangyi@dlut.edu.cn (Y. Wang)

is PF_r (PF, resp.). We say that a nonnegative sequence (a_n) is *log-convex* (*log-concave*, resp.) if $a_i a_{j+1} \geq a_{i+1} a_j$ ($a_i a_{j+1} \leq a_{i+1} a_j$, resp.) for $0 \leq i < j$. Clearly, the sequence (a_n) is log-concave if and only if it is PF_2, i.e., its Toeplitz matrix $[a_{i-j}]_{i,j\geq 0}$ is TP_2, and the sequence is log-convex if and only if its Hankel matrix

$$[a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots \\ a_2 & a_3 & a_4 & a_5 & \cdots \\ a_3 & a_4 & a_5 & a_6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is TP_2 [5].

Let $\pi = (r_k)_{k\geq 1}, \sigma = (s_k)_{k\geq 0}, \tau = (t_k)_{k\geq 1}$ be three sequences of nonnegative numbers and define an infinite lower triangular matrix

$$A := A^{\pi,\sigma,\tau} = [a_{n,k}]_{n,k\geq 0} = \begin{bmatrix} a_{0,0} & & & & \\ a_{1,0} & a_{1,1} & & & \\ a_{2,0} & a_{2,1} & a_{2,0} & & \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \\ \vdots & & & & \ddots \end{bmatrix}$$

by the recurrence

$$a_{0,0} = 1, \quad a_{n+1,0} = s_0 a_{n,0} + t_1 a_{n,1}, \quad a_{n+1,k} = r_k a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1}, \tag{1.1}$$

where $a_{n,k} = 0$ unless $n \geq k \geq 0$. Following Aigner [3], we say that $A^{\pi,\sigma,\tau}$ is the *recursive matrix* and $a_{n,0}$ are the *Catalan-like numbers* corresponding to (π, σ, τ). Such triangles arise often in combinatorics and many well-known counting coefficients are the Catalan-like numbers. The following are several basic examples of recursive matrices.

Example 1.1. (i) The Pascal triangle $P = \left[\binom{n}{k}\right]_{n,k\geq 0}$ satisfies $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

(ii) The Stirling triangle (of the second kind) $S = [S(n,k)]_{n,k\geq 0}$ satisfies $S(n+1,k) = S(n,k-1) + (k+1)S(n,k)$.

(iii) The Catalan triangle of Aigner is

$$C = [C_{n,k}] = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 9 & 5 & 1 & & \\ 14 & 28 & 20 & 7 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix},$$

where $C_{n+1,0} = C_{n,0} + C_{n,1}, C_{n+1,k} = C_{n,k-1} + 2C_{n,k} + C_{n,k+1}$ [1]. The corresponding Catalan-like numbers $C_{n,0}$ are precisely the Catalan numbers C_n.

The Catalan triangle of Shaprio is

$$B = [B_{n,k}] = \begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix},$$

where $B_{n+1,k} = B_{n,k-1}+2B_{n,k}+B_{n,k+1}$ [15]. The corresponding Catalan-like numbers $B_{n,0}$ are precisely the Catalan numbers C_{n+1}. There are a lot of papers to consider combinatorics of the Catalan triangle [1, 4, 9, 14–16]. See also Sloane's OEIS [17, A039598].

(iv) The Bell triangle, introduced by Aigner [2], is

$$X = [X_{n,k}] = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 3 & 1 & & & \\ 5 & 10 & 6 & 1 & & \\ 15 & 37 & 31 & 10 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix},$$

where $X_{n+1,k} = X_{n,k-1}+(k+1)X_{n,k}+(k+1)X_{n,k+1}$. The corresponding Catalan-like numbers $X_{n,0}$ are the Bell numbers B_n.

Aigner [1–4] studied various combinatorial properties of recursive matrices and Hankel matrices of the Catalan-like numbers. It is well known that the Pascal triangle is TP [11, p. 137]. Brenti [6] showed, among other things, that the Stirling triangle is TP. Very recently, Zhu [19, Theorem 3.1] showed that if $s_{k-1}s_k \geq r_k t_k$ for $k \geq 1$, then the sequence $(a_{n,0})_{n\geq 0}$ of Catalan-like numbers defined by (1.1) is log-convex. Zhu [20, Theorem 2.1] also showed that if r_k, s_k are nonnegative quadratic polynomials in k and $t_k = 0$ for all k, then the corresponding matrix A is TP. The object of this paper is to give some sufficient conditions for total positivity of recursive matrices. In the next section, we present our main results. As applications, we show that many well-known combinatorial triangles, including the Pascal triangle, the Stirling triangle, the Bell triangle, the Catalan triangles of Aigner and Shapiro are TP in a certain unified approach. As consequences, the corresponding Catalan-like numbers, including the Catalan numbers and the Bell numbers, form a log-convex sequence respectively. In Section 3, we point out that our results can be carried over verbatim to their q-analogue. We also propose a couple of problems for further work.

2 Main results and applications

We first review some basic facts about TP matrices. The first is direct by definition and the second follows immediately from the classic Cauchy-Binet formula.

Lemma 2.1. *A matrix is* TP_r *(TP, resp.) if and only if its leading principal submatrices are all* TP_r *(TP, resp.).*

Lemma 2.2. *The product of two* TP_r *(TP, resp.) matrices is still* TP_r *(TP, resp.).*

Rewrite the recursive relation (1.1) as

$$\begin{bmatrix} a_{1,0} & a_{1,1} & & & \\ a_{2,0} & a_{2,1} & a_{2,2} & & \\ a_{3,0} & a_{3,1} & a_{3,2} & a_{3,3} & \\ & & \cdots & & \ddots \end{bmatrix} = \begin{bmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ & \cdots & & \ddots \end{bmatrix} \begin{bmatrix} s_0 & r_1 & & \\ t_1 & s_1 & r_2 & \\ & t_2 & s_2 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

or briefly,

$$\overline{A} = AJ \tag{2.1}$$

where $\overline{A}$ is obtained from A by deleting the 0th row and J is the Jacobi matrix

$$J := J^{\pi,\sigma,\tau} = \begin{bmatrix} s_0 & r_1 & & & \\ t_1 & s_1 & r_2 & & \\ & t_2 & s_2 & r_3 & \\ & & t_3 & s_3 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}. \tag{2.2}$$

Clearly, the recursive relation (1.1) is decided completely by the tridiagonal matrix J. Call J the *coefficient matrix* of the recursive relation (1.1). For convenience, we also call J the *coefficient matrix* of the recursive matrix A.

For example, the coefficient matrix of the Bell triangle is

$$\begin{bmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 2 & 3 & 1 & \\ & & 3 & 4 & \ddots \\ & & & \ddots & \ddots \end{bmatrix},$$

the coefficient matrices of Catalan triangles of Aigner and Shapiro are

$$\begin{bmatrix} 1 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & 1 & 2 & \ddots \\ & & & \ddots & \ddots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & 1 & 2 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}.$$

Theorem 2.3. *Let A be a recursive matrix with the coefficient matrix J.*

(i) *If J is TP_r (TP, resp.), then so is A.*

(ii) *If A is TP_2, then the sequence $(a_{n,0})_{n\geq 0}$ of the Catalan-like numbers is log-convex.*

Proof. (i) Clearly, it suffices to consider the TP_r case. Let

$$A_n = \begin{bmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ \vdots & \vdots & \ddots & \\ a_{n,0} & a_{n,1} & \cdots & a_{n,n} \end{bmatrix}, \qquad J_n = \begin{bmatrix} s_0 & r_1 & & \\ t_1 & s_1 & \ddots & \\ & \ddots & \ddots & r_n \\ & & t_n & s_n \end{bmatrix}$$

and

$$\overline{A}_{n+1} = \begin{bmatrix} a_{1,0} & a_{1,1} & & & \\ a_{2,0} & a_{2,1} & a_{2,2} & & \\ \cdots & \cdots & \cdots & \ddots & \\ a_{n,0} & a_{n,1} & a_{n,2} & \cdots & a_{n,n} \\ a_{n+1,0} & a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} \end{bmatrix}$$

be the nth leading principal submatrices of A, J and $\overline{A}$ respectively. Then $\overline{A}_{n+1} = A_n J_n$ by (2.1). Now J is TP_r, so is J_n. Assume that A_n is TP_r. Then the product $\overline{A}_{n+1} = A_n J_n$ is also TP_r. It follows that A_{n+1} is TP_r. Thus A is TP_r by induction.

(ii) By (1.1), we have

$$\begin{bmatrix} a_{0,0} & a_{1,0} \\ a_{1,0} & a_{2,0} \\ a_{2,0} & a_{3,0} \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} a_{0,0} & & & \\ a_{1,0} & a_{1,1} & & \\ a_{2,0} & a_{2,1} & a_{2,2} & \\ & & \cdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & s_0 \\ 0 & t_1 \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}. \tag{2.3}$$

Clearly, the second matrix in the right hand side of (2.3) is TP_2 since s_0 and t_1 are nonnegative. If A is TP_2, then so is the matrix in the left hand side of (2.3), which is equivalent to the log-convexity of the sequence $(a_{n,0})_{n\geq 0}$. This completes the proof. □

So we may focus our attention on the total positivity of tridiagonal matrices. We first give two simple applications of Theorem 2.3 from this point of view.

Corollary 2.4 ([19, Theorem 3.1])**.** *If $s_{k-1}s_k \geq r_k t_k$ for $k \geq 1$, then the sequence $(a_{n,0})_{n\geq 0}$ of Catalan-like numbers defined by (1.1) is log-convex.*

Proof. If $s_{k-1}s_k \geq r_k t_k$ for $k \geq 1$, then J is TP_2, and so is A by Theorem 2.3 (i). Thus $(a_{n,0})_{n\geq 0}$ is log-convex by Theorem 2.3 (ii). □

Corollary 2.5. *Let $A = [a_{n,k}]_{n,k\geq 0}$ be a recursive matrix defined by*

$$a_{0,0} = 1, \quad a_{n+1,k} = r_k a_{n,k-1} + s_k a_{n,k}. \tag{2.4}$$

If r_k and s_k are nonnegative, then A is TP.

Proof. In this case, the coefficient matrix is a bidiagonal matrix, which is obviously TP, and so is the recursive matrix by Theorem 2.3 (i). $\square$

Remark 2.1. An immediate consequence of Corollary 2.5 is Zhu's result [20, Theorem 2.1], which states that if r_k, s_k are nonnegative quadratic polynomials in k, then the matrix $[a_{n,k}]_{n,k\geq 0}$ defined by (2.4) is TP. In particular, the Pascal triangle and the Stirling triangle are TP.

There are many well-known results about the total positivity of tridiagonal matrices. The following is one of them.

Lemma 2.6 ([13, Theorem 4.3])**.** *A finite nonnegative tridiagonal matrix is TP if and only if all its principal minors containing consecutive rows and columns are nonnegative.*

Actually, it is also known that an irreducible nonnegative tridiagonal matrix is TP if and only if all its leading principal minors are positive [12, Example 2.2].

We next consider the problem in which case a tridiagonal matrix has nonnegative determinant. Let $M = [m_{ij}]_{1\leq i,j\leq n}$ be a real $n \times n$ matrix. We say that M is *row diagonally dominant* if

$$m_{ii} \geq |m_{i,1}| + \cdots + |m_{i,i-1}| + |m_{i,i+1}| + \cdots + |m_{i,n}|, \quad i = 1, 2, \ldots, n. \tag{2.5}$$

If all inequalities in (2.5) are strict, then we say that M is *strictly row diagonally dominant*. It is well known [18] that if M is strictly row diagonally dominant, then $|M| > 0$. Moreover, if M is irreducible row diagonally dominant and there is at least one strict inequality in (2.5), then $|M| > 0$. The case for nonnegative tridiagonal matrices is simpler.

Lemma 2.7. *Let*

$$J_n = \begin{bmatrix} y_0 & x_1 & & & & \\ z_1 & y_1 & x_2 & & & \\ & z_2 & y_2 & x_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & z_{n-1} & y_{n-1} & x_n \\ & & & & z_n & y_n \end{bmatrix},$$

where x_k, y_k, z_k are all nonnegative.

(i) *If J_n is row diagonally dominant, then $|J_n| \geq 0$.*

(ii) *If J_n is column diagonally dominant, then $|J_n| \geq 0$.*

Proof. (i) We proceed by induction on n. Assume that $y_n = z_n$. Then

$$|J_n| = \begin{vmatrix} y_0 & x_1 & & & \\ z_1 & y_1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & z_{n-1} & y_{n-1} - x_n & x_n \\ & & & 0 & y_n \end{vmatrix} = y_n \begin{vmatrix} y_0 & x_1 & & & \\ z_1 & y_1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & z_{n-2} & y_{n-2} & x_{n-1} \\ & & & z_{n-1} & y_{n-1} - x_n \end{vmatrix}.$$

Thus $|J_n|$ is nonnegative by the inductive hypothesis. Assume that $y_n > z_n$. Then

$$|J_n| = \begin{vmatrix} y_0 & x_1 & & & \\ z_1 & y_1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & z_{n-1} & y_{n-1} & x_n \\ & & & z_n & z_n \end{vmatrix} + \begin{vmatrix} y_0 & x_1 & & & \\ z_1 & y_1 & x_2 & & \\ & \ddots & \ddots & \ddots & \\ & & z_{n-1} & y_{n-1} & x_n \\ & & & 0 & y_n - z_n \end{vmatrix}.$$

Clearly, two determinants on the right hand side are nonnegative, so is $|J_n|$.

(ii) Apply (i) to the transpose J_n^T of J_n. □

Combining Theorem 2.3 (i) and Lemma 2.7 we obtain the following criterion.

Theorem 2.8. *Let A be the recursive matrix defined by (1.1).*

(i) *If $s_0 \geq r_1$ and $s_k \geq r_{k+1} + t_k$ for $k \geq 1$, then A is TP.*

(ii) *If $s_0 \geq t_1$ and $s_k \geq r_k + t_{k+1}$ for $k \geq 1$, then A is TP.*

Theorem 2.9. *Let A be the recursive matrix defined by (1.1). If $s_0 \geq 1$ and $s_k \geq r_k t_k + 1$ for $k \geq 1$, then A is TP.*

Proof. By Theorem 2.3, we need to show that the corresponding coefficient matrix J is TP. By Lemma 2.6, it suffices to show that the tridiagonal matrix of form

$$J_n = \begin{bmatrix} y_0 & x_1 & & & \\ z_1 & y_1 & x_2 & & \\ & z_2 & y_2 & \ddots & \\ & & \ddots & \ddots & x_n \\ & & & z_n & y_n \end{bmatrix} \tag{2.6}$$

has nonnegative determinant if $y_0 \geq 1$ and $y_k \geq x_k z_k + 1$ for $1 \leq k \leq n$. Denote $D_{-1} := 1, D_0 = y_0$ and $D_n = |J_n|$ for $n \geq 1$. We show that $D_n \geq D_{n-1} \geq 1$ by induction on n. Assume that $D_{n-1} \geq D_{n-2} \geq 1$. Note that

$$D_n = y_n D_{n-1} - x_n z_n D_{n-2}$$

by expanding the determinant (2.6) along the last row or column. Hence

$$D_n \geq y_n D_{n-1} - x_n z_n D_{n-1} = (y_n - x_n z_n) D_{n-1} \geq D_{n-1} \geq 1,$$

as desired. Thus J is TP, and so is A. □

Finally, we apply Theorem 2.9 to two particularly interesting classes of recursive matrices, which are introduced by Aigner in [1] and [3] respectively. Many well-known combinatorial triangles are of such recursive matrices (we refer the reader to Aigner [1, 3] for more information). The motivation of this paper is to study the total positivity of these combinatorial triangles.

Corollary 2.10. *Let $A = [a_{n,k}]_{n,k\geq 0}$ be an admissible matrix defined by*

$$a_{0,0} = 1, \quad a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + a_{n,k+1}.$$

If $s_0 \geq 1$ and $s_k \geq 2$ for $k \geq 1$, then A is TP.

Corollary 2.11. *Let $A = [a_{n,k}]_{n,k\geq 0}$ be a recursive matrix defined by*

$$a_{0,0} = 1, \quad a_{n+1,k} = a_{n,k-1} + s_k a_{n,k} + t_{k+1} a_{n,k+1}.$$

If $s_0 \geq 1$ and $s_k \geq t_k + 1$ for $k \geq 1$, then A is TP.

Corollary 2.12. *The Bell triangle, the Catalan triangles of Aigner and Shapiro are TP respectively.*

3 Concluding remarks and further work

For two real polynomials $f(q)$ and $g(q)$ in q, denote $f(q) \geq_q g(q)$ if coefficients of the difference $f(q)-q(q)$ are all nonnegative. Let $A(q)$ be an infinite matrix all whose elements are real polynomials in q. It is called *q-TP* if its minors of all orders have nonnegative coefficients as polynomials in q. Theorems 2.3, 2.8 and 2.9 can be carried over verbatim to their q-analogue.

Theorem 3.1. *Let $\pi = (r_k(q))_{k\geq 1}, \sigma = (s_k(q))_{k\geq 0}, \tau = (t_k(q))_{k\geq 1}$ be three sequences of polynomials in q with nonnegative coefficients and $A(q) = [a_{n,k}(q)]_{n,k\geq 0}$ be an infinite lower triangular matrix defined by*

$$a_{0,0}(q) = 1, \quad a_{n+1,k}(q) = r_k(q)a_{n,k-1}(q) + s_k(q)a_{n,k}(q) + t_{k+1}(q)a_{n,k+1}(q),$$

where $a_{n,k}(q) = 0$ unless $n \geq k \geq 0$. Then the q-recursive matrix $A(q)$ is q-TP if one of the following conditions holds:

(i) *$s_0(q) \geq_q r_1(q)$ and $s_k(q) \geq_q r_{k+1}(q) + t_k(q)$ for $k \geq 1$.*

(ii) *$s_0(q) \geq_q t_1(q)$ and $s_k(q) \geq_q r_k(q) + t_{k+1}(q)$ for $k \geq 1$.*

(iii) *$s_0(q) \geq_q 1$ and $s_k(q) \geq_q r_k(q)t_k(q) + 1$ for $k \geq 1$.*

There are other forms of recursive matrices. For example, the Eulerian triangle

$$A = [A(n,k)]_{n,k\geq 1} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 4 & 1 & & & \\ 1 & 11 & 11 & 1 & & \\ 1 & 26 & 66 & 26 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix},$$

where $A(n,k)$ is the Eulerian number and satisfies the recursive relation

$$A(n+1,k) = (n-k+2)A(n,k-1) + kA(n,k).$$

Brenti suggested the following.

Conjecture 3.2 ([7, Conjecture 6.10]). *The Eulerian triangle $A = [A(n,k)]_{n,k\geq 1}$ is TP.*

The Narayana triangle

$$N = [N(n,k)]_{n,k\geq 1} = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 6 & 1 & & \\ 1 & 10 & 20 & 10 & 1 & \\ \vdots & & & & & \ddots \end{bmatrix},$$

where $N(n,k) = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}$ is the Narayana number and satisfies the recursive relation

$$N(n+1,k) = \frac{n(n+1)}{2k(k-1)}N(n,k-1) + \frac{n(n+1)}{2(n-k+1)(n-k+2)}N(n,k)$$

for $k \geq 2$. Sometimes N is called the Catalan triangle since its row sum is precisely the Catalan number:

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

We refer the reader to Sloane's OEIS [17, A001263] for more information about the Narayana triangle. Here we propose the following conjecture.

Conjecture 3.3. *The Narayana triangle $N = [N(n,k)]_{n,k\geq 1}$ is TP.*

Acknowledgement

This work was supported in part by the National Natural Science Foundation of China (Grant No. 11371078) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20110041110039).

References

[1] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1999) 33–51.

[2] M. Aigner, A characterization of the Bell numbers, Discrete Math. 205 (1999), 207–210.

[3] M. Aigner, Catalan and other numbers — A recurrent theme, in: H. Crapo, D. Senato (Eds.), Algebraic Combinatorics and Computer Science, Springer, Berlin, 2001, 347–390.

[4] M. Aigner, Enumeration via ballot numbers, Discrete Math. 308 (2008) 2544–2563.

[5] F. Brenti, Unimodal, log-concave and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc. 413 (1989).

[6] F. Brenti, Combinatorics and total positivity, J. Comb. Theory Ser. A 71 (1995) 175–218.

[7] F. Brenti, The applications of total positivity to combinatorics, and conversely, in: Total Positivity and Its Applications, Jaca, 1994, in: Math. Appl., vol. 359, Kluwer, Dordrecht, 1996, pp. 451–473.

[8] S.M. Fallat and C.R. Johnson, Totally Nonnegative Matrices, Princeton University Press, Princeton, 2011.

[9] T.-X. He, Parametric Catalan numbers and Catalan triangles, Linear Algebra Appl. 438 (2013) 1467–1484.

[10] R.A. Horn and C.R. Johnson, Matrix Analysis, 2nd, Cambridge University Press, Cambridge, 2012.

[11] S. Karlin, Total Positivity, Vol. 1, Stanford University Press, Stanford, 1968.

[12] H. Minc, Nonnegative Matrices, Wiley-Interscienee, New York, 1988.

[13] A. Pinkus, Totally positive matrices, Cambridge University Press, Cambridge, 2010.

[14] D.G. Rogers, Pascal triangles, Catalan numbers and renewal arrays, Discrete Math. 22 (1978) 301–310.

[15] L.W. Shapiro, A Catalan triangle, Discrete Math. 14 (1976) 83–90.

[16] L.W. Shapiro, S. Getu, W.-J. Woan and L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.

[17] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[18] O. Taussky, A recurring theorem on determinants, Amer. Math. Monthly 56 (1949) 672–676.

[19] B.-X. Zhu, Log-convexity and strong q-log-convexity for some triangular arrays, Adv. in. Appl. Math. 50 (2013) 595–606.

[20] B.-X. Zhu, Some positivities in certain triangular arrays, Proc. Amer. Math. Soc. 142 (2014) 2943–2952.