On Enumeration of Paths in Catalan–Schröder Lattices*

Max A. Alekseyev

George Washington University, Washington, D.C., U.S.A.

Abstract

We address the problem of enumerating paths in square lattices, where allowed steps include $(1,0)$ and $(0,1)$ everywhere, and $(1,1)$ above the diagonal $y = x$. We consider two such lattices differing in whether the $(1,1)$ steps are allowed along the diagonal itself. Our analysis leads to explicit generating functions and an efficient way to compute terms of many sequences in the Online Encyclopedia of Integer Sequences, proposed by Clark Kimberling over a decade and a half ago.

1 Introduction

The *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ (sequence A000108 in the OEIS [3]) enumerate among other combinatorial objects [2] the paths from $(0,0)$ to (n,n) in the integer lattice bounded by lines $y = 0$ and $y = x$ with unit steps $(0,1)$ and $(1,0)$, called *Dyck paths*. We represent these restrictions as a directed lattice, which we will refer to the *Catalan lattice* $\mathcal{L}_C$ (Fig. 1, left panel). If we allow diagonal steps $(1,1)$ in this lattice, the paths in it become known as *Schröder paths* [1], and the number of such paths from $(0,0)$ to (n,n) is given by the *large Schröder numbers* S_n (sequence A006318 in the OEIS). So we refer to the resulting lattice as the *Schröder lattice* $\mathcal{L}_S$ (Fig. 1, right panel).

Over one and a half decades ago, Clark Kimberling contributed the sequences A026769–A026790 to the OEIS [3], concerning a composition of the Catalan and Schröder lattices that below the diagonal $y = x$ represents the Catalan lattice and above the diagonal represents the transposed Schröder lattice. There are two such lattices $\mathcal{L}_{CS}$ and $\mathcal{L}^{\star}_{CS}$, where the $(1,1)$ steps along the diagonal are allowed and disallowed, respectively (Fig. 2). In this note, we address the problem of enumerating paths in the lattices $\mathcal{L}_{CS}$ and $\mathcal{L}^{\star}_{CS}$. We begin our analysis by recalling some useful facts about the Catalan and Schröder lattices.

The generating functions for Catalan and Schröder numbers are given by

$$C(x) = \sum_{n=0}^{\infty} C_n \cdot x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

and

$$\mathcal{S}(x) = \sum_{n=0}^{\infty} S_n \cdot x^n = \frac{1 - x - \sqrt{1-6x+x^2}}{2x} = \frac{1}{1-x} \cdot C\left(\frac{x}{(1-x)^2}\right),$$

respectively [1]. So the number of paths from $(0,0)$ to (n,n) in $\mathcal{L}_C$ and $\mathcal{L}_S$ is given by $[x^n]\, C(x)$ and $[x^n]\, \mathcal{S}(x)$, respectively, where $[x^n]$ denotes the operator of taking the coefficient of x^n.

*The work is supported by the National Science Foundation under grant No. IIS-1462107.

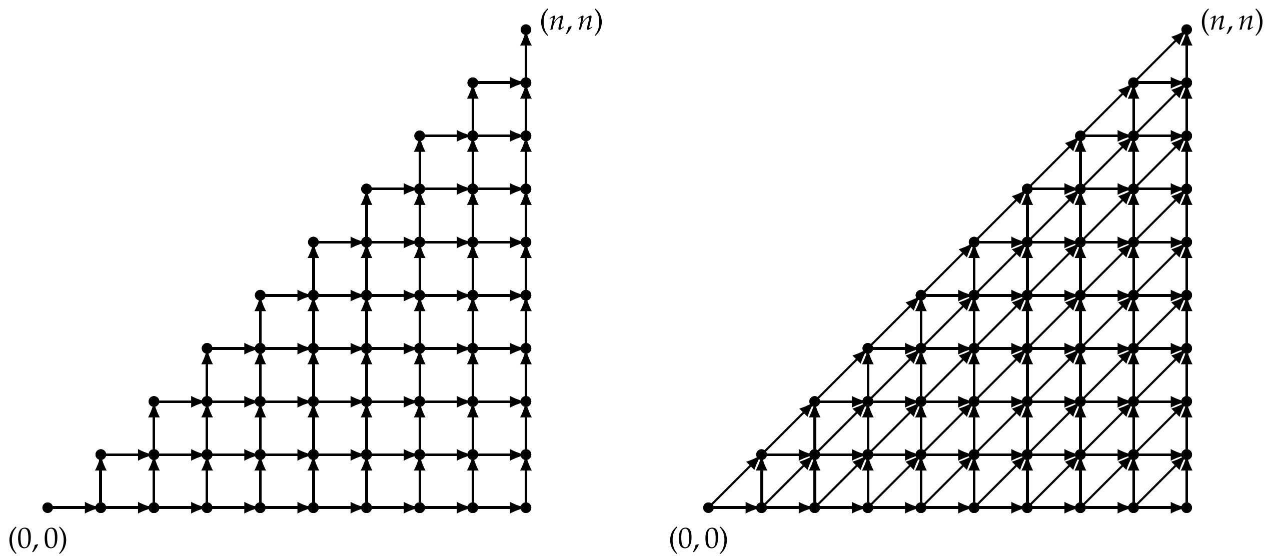

Figure 1: Catalan lattice $\mathcal{L}_C$ and Schröder lattice $\mathcal{L}_S$.

We will need the following lemma, which states well-known facts about the number of paths and the number of *subdiagonal paths* (i.e., paths that lay below the line $y = x$, except possibly for their endpoints) from $(0,0)$ to (n,k) in $\mathcal{L}_C$ and $\mathcal{L}_S$.

Lemma 1. *For any integers $n \geq k \geq 0$,*

(i) the number of paths from $(0,0)$ to (n,k) in $\mathcal{L}_C$ and $\mathcal{L}_S$ equals $[x^k]\, C(x)^{n-k+1}$ and $[x^k]\, \mathcal{S}(x)^{n-k+1}$, respectively.

(ii) the number of subdiagonal paths from $(0,0)$ to (n,n) in $\mathcal{L}_C$ and $\mathcal{L}_S$ equals $[x^{n-1}]\, C(x)$ and $[x^{n-1}]\, \mathcal{S}(x)$, respectively.

(iii) for $n > k$, the number of subdiagonal paths from $(0,0)$ to (n,k) in $\mathcal{L}_C$ and $\mathcal{L}_S$ equals $[x^k]\, C(x)^{n-k}$ and $[x^k]\, \mathcal{S}(x)^{n-k}$, respectively.

2 Enumeration of paths in Catalan–Schröder lattices

Theorem 2. *Let f_n and $f_n^\star$ be the number of paths from $(0,0)$ to (n,n) in the lattices $\mathcal{L}_{CS}$ and $\mathcal{L}_{CS}^\star$, respectively. Then*

$$\mathcal{F}^\star(x) = \sum_{n=0}^{\infty} f_n^\star \cdot x^n = \frac{1}{1 - x \cdot (C(x) + \mathcal{S}(x))}$$

and

$$\mathcal{F}(x) = \sum_{n=0}^{\infty} f_n \cdot x^n = \frac{1}{1 - x \cdot (C(x) + \mathcal{S}(x) + 1)} = \frac{\mathcal{S}(x)}{1 - x \cdot C(x) \cdot \mathcal{S}(x)}.$$

Proof. Any path from $(0,0)$ to (n,n) in $\mathcal{L}_{CS}^\star$ consists of subdiagonal and/or supdiagonal[1] subpaths from (p_0,p_0) to (p_1,p_1), from (p_1,p_1) to (p_2,p_2), …, from (p_{m-1},p_{m-1}) to (p_m,p_m), where $m \geq 0$ and

[1]Similarly to subdiagonal paths, we define supdiagonal paths as those that lay above the diagonal $y = x$, except possibly for their endpoints.

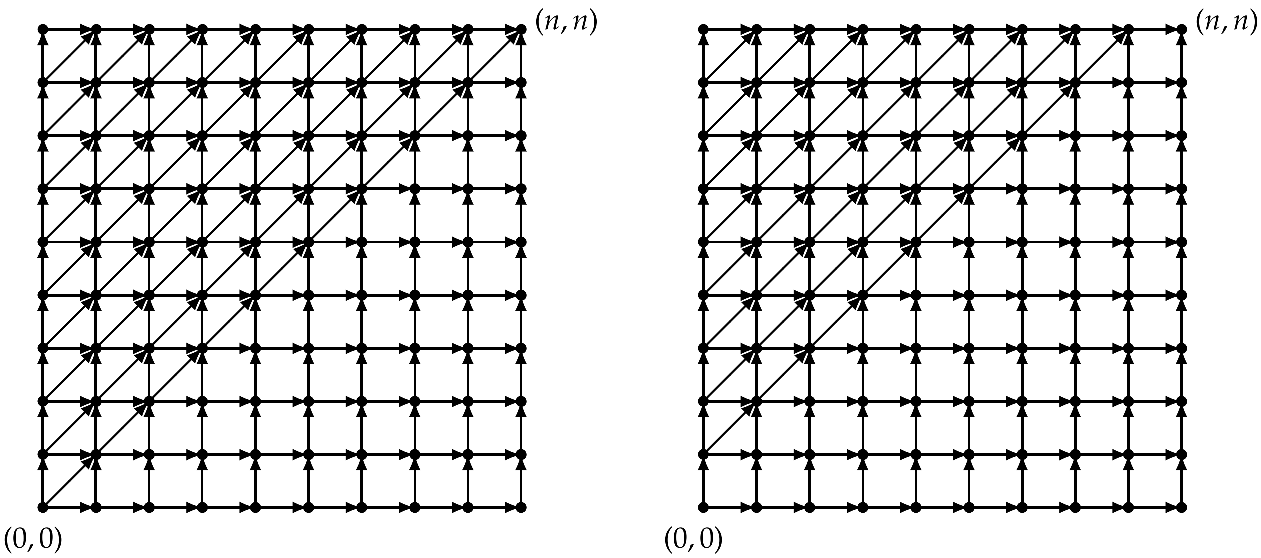

Figure 2: Catalan–Schröder lattices $\mathcal{L}_{CS}$ and $\mathcal{L}^{\star}_{CS}$.

$0 = p_0 < p_1 < \cdots < p_m = n$ are integers (in other words, p_i represent the coordinates of vertices where the path visits the diagonal). By Lemma 1, for any $k = 0, 1, \ldots, m-1$, the number of subdiagonal and/or supdiagonal paths from (p_k, p_k) to (p_{k+1}, p_{k+1}) equals

$$[x^{p_{k+1}-p_k-1}]\ (C(x) + \mathcal{S}(x)).$$

Indeed, every such path is either subdiagonal (enumerated by $C(x)$) or supdiagonal (enumerated by $\mathcal{S}(x)$).

Hence, the total number of paths in from $(0, 0)$ to (n, n) in $\mathcal{L}^{\star}_{CS}$ equals

$$\sum_{m=0}^{\infty} \sum_{0<p_1<\cdots<p_{m-1}<n} \prod_{k=0}^{m-1} [x^{p_{k+1}-p_k-1}]\ (C(x) + \mathcal{S}(x)) = \sum_{m=0}^{\infty} [x^{n-m}]\ (C(x) + \mathcal{S}(x))^m$$
$$= [x^n]\ \sum_{m=0}^{\infty} (x \cdot (C(x) + \mathcal{S}(x)))^m = [x^n]\ \frac{1}{1 - x \cdot (C(x) + \mathcal{S}(x))}.$$

In the lattice $\mathcal{L}_{CS}$, in addition to subdiagonal and/or supdiagonal subpaths, we need to account for single diagonal steps (when $p_{k+1} = p_k + 1$), which brings the additional summand 1 to $C(x) + \mathcal{S}(x)$. That is, the total number of paths in from $(0, 0)$ to (n, n) in $\mathcal{L}_{CS}$ equals the coefficient of x^n in

$$\frac{1}{1 - x \cdot (C(x) + \mathcal{S}(x) + 1)} = \frac{\mathcal{S}(x)}{1 - x \cdot C(x) \cdot \mathcal{S}(x)}.$$

The last equality follows from the algebraic identity:

$$x \cdot \mathcal{S}(x)^2 - (1 - x) \cdot \mathcal{S}(x) + 1 = 0.$$

□

Theorem 3. *For integers n, k, the number of paths from $(0, 0)$ to (n, k) in $\mathcal{L}_{CS}$ equals*

$$\begin{cases} [x^k]\ \mathcal{F}(x) \cdot C(x)^{n-k}, & \text{if } n \geq k; \\ [x^n]\ \mathcal{F}(x) \cdot \mathcal{S}(x)^{k-n}, & \text{if } n \leq k. \end{cases}$$

Similarly, the number of paths from $(0,0)$ *to* (n,k) *in* $\mathcal{L}^\star_{CS}$ *equals*

$$\begin{cases}[x^k]\,\mathcal{F}^\star(x)\cdot C(x)^{n-k}, & \text{if } n\ge k;\\ [x^n]\,\mathcal{F}^\star(x)\cdot \mathcal{S}(x)^{k-n}, & \text{if } n\le k.\end{cases}$$

Proof. Any path from $(0,0)$ to (n,k) in $\mathcal{L}_{CS}$ is formed by a path from $(0,0)$ to (m,m) for some $m\ge 0$ and a path from (m,m) to (n,k) that never visits the diagonal again. Clearly, this decomposition is unique and so is m. The number of paths from $(0,0)$ to (m,m) equals $[x^m]\,\mathcal{F}(x)$. If $n>k$, then the number of paths from (m,m) to (n,k) avoiding the diagonal equals the number of subdiagonal paths from $(0,0)$ to $(n-m,k-m)$ in $\mathcal{L}_C$, which is $[x^{k-m}]\,C(x)^{n-k}$ (by Lemma 1). In this case, the number of paths $(0,0)$ to (n,k) in $\mathcal{L}_{CS}$ equals

$$\sum_{m=0}^{\infty}[x^m]\,\mathcal{F}(x)\cdot[x^{n-m}]\,C(x)^{n-k}=[x^n]\,\mathcal{F}(x)\cdot C(x)^{n-k}.$$

Similarly, if $n<k$, then the number of paths from (m,m) to (n,k) avoiding the diagonal equals the number of subdiagonal paths from $(0,0)$ to $(k-m,n-m)$ in $\mathcal{L}_S$, which is $[x^{n-m}]\,\mathcal{S}(x)^{k-n}$ (by Lemma 1). In this case, the number of paths $(0,0)$ to (n,k) in $\mathcal{L}_{CS}$ equals $[x^n]\,\mathcal{F}(x)\cdot\mathcal{S}(x)^{k-n}$. It is easy to see that the formulae in both cases are also consistent with the case $n=k$, where the number of paths equals $[x^n]\,\mathcal{F}(x)$ by the definition of $\mathcal{F}(x)$.

The lattice $\mathcal{L}^\star_{CS}$ is considered similarly. □

3 Sequences in the OEIS

Below we derive formulae for sequences A026769–A026779 (concerning the lattice $\mathcal{L}^\star_{CS}$) and sequences A026780–A026790 (concerning the lattice $\mathcal{L}_{CS}$) in the Online Encyclopedia of Integer Sequences [3].

3.1 Sequences A026769 and A026780

A026769(n,k) and A026780(n,k) give the number of paths from $(0,0)$ to $(k,n-k)$ in the lattices $\mathcal{L}^\star_{CS}$ and $\mathcal{L}_{CS}$, respectively. Formulae for these numbers are given in Theorem 3.

3.2 Sequences A026770–A026774 and A026781–A026785

The n-th term of A026770–A026774 enumerate paths in $\mathcal{L}^\star_{CS}$ from $(0,0)$ to (n,n), $(n-1,n+1)$, $(n-2,n+2)$, $(n-1,n)$, and $(n-2,n+1)$, respectively. By Theorem 3, the ordinary generating function for the number of such paths is $\mathcal{F}^\star(x)$, $x\cdot\mathcal{F}^\star(x)\cdot\mathcal{S}(x)^2$, $x^2\cdot\mathcal{F}^\star(x)\cdot\mathcal{S}(x)^4$, $x\cdot\mathcal{F}^\star(x)\cdot\mathcal{S}(x)$, and $x^2\cdot\mathcal{F}^\star(x)\cdot\mathcal{S}(x)^3$, respectively.

The sequences A026781–A026785 enumerate similar paths in $\mathcal{L}_{CS}$ and have ordinary generating functions $\mathcal{F}(x)$, $x\cdot\mathcal{F}(x)\cdot\mathcal{S}(x)^2$, $x^2\cdot\mathcal{F}(x)\cdot\mathcal{S}(x)^4$, $x\cdot\mathcal{F}(x)\cdot\mathcal{S}(x)$, and $x^2\cdot\mathcal{F}(x)\cdot\mathcal{S}(x)^3$, respectively.

3.3 Sequences A026775 and A026786

The terms A026775(n) and A026786(n) give the number of paths from $(0,0)$ to $(\lfloor n/2\rfloor,\lceil n/2\rceil)$ in the lattices $\mathcal{L}^\star_{CS}$ and $\mathcal{L}_{CS}$, respectively. By Theorem 3, these are the coefficients of $x^{\lfloor n/2\rfloor}$ in $\mathcal{F}^\star(x)\cdot\mathcal{S}(x)^{n \bmod 2}$ and $\mathcal{F}(x)\cdot\mathcal{S}(x)^{n \bmod 2}$, which are the same as the coefficients of x^n in $\mathcal{F}^\star(x^2)\cdot(1+x\cdot\mathcal{S}(x^2))$ and $\mathcal{F}(x)\cdot(1+x\cdot\mathcal{S}(x^2))$.

3.4 Sequences A026776 and A026787

The terms A026776(n) and A026787(n) give the total number of paths from $(0,0)$ to $(i, n-i)$, where $i = 0, 1, \dots, n$, in the lattices in $\mathcal{L}^{\star}_{CS}$ and $\mathcal{L}_{CS}$, respectively.

Theorem 4. *The ordinary generating function for A026776 is*

$$\mathcal{F}^{\star}(x^2) \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^2)} + \frac{1}{1 - x \cdot C(x^2)} - 1 \right).$$

The ordinary generating function for A026787 is

$$\mathcal{F}(x^2) \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^2)} + \frac{1}{1 - x \cdot C(x^2)} - 1 \right).$$

Proof. By Theorem 3, A026776(n) equals

$$\begin{aligned}
&\sum_{i=0}^{\lfloor n/2 \rfloor} [x^i]\, \mathcal{F}^{\star}(x) \cdot \mathcal{S}(x)^{n-2i} + \sum_{i=\lfloor n/2 \rfloor + 1}^{n} [x^{n-i}]\, \mathcal{F}^{\star}(x) \cdot C(x)^{2i-n} \\
=&\sum_{i=0}^{\lfloor n/2 \rfloor} [x^{2i}]\, \mathcal{F}^{\star}(x^2) \cdot \mathcal{S}(x^2)^{n-2i} + \sum_{i=\lfloor n/2 \rfloor + 1}^{n} [x^{2n-2i}]\, \mathcal{F}^{\star}(x^2) \cdot C(x^2)^{2i-n} \\
=&[x^n]\, \mathcal{F}^{\star}(x^2) \cdot \left(\sum_{i=0}^{\lfloor n/2 \rfloor} \left(x \cdot \mathcal{S}(x^2) \right)^{n-2i} + \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \left(x \cdot C(x^2) \right)^{2i-n} \right).
\end{aligned}$$

We notice that in the first sum, the powers of $x \cdot \mathcal{S}(x^2)$ go over the nonnegative integers up to n of the same oddness as n. Since we are interested only in the coefficient of x^n, we can drop both these restrictions. Namely, the powers above n have the coefficient of x^n equal zero, while the power m of the opposite oddness than n may have nonzero coefficients only for x in powers of the same oddness as m. The same arguments apply for the powers of $x \cdot C(x^2)$, except that in this case they go over the positive integers. That is, the above expression simplifies to

$$\begin{aligned}
&[x^n]\, \mathcal{F}^{\star}(x^2) \cdot \left(\sum_{m=0}^{\infty} \left(x \cdot \mathcal{S}(x^2) \right)^m + \sum_{m=1}^{\infty} \left(x \cdot C(x^2) \right)^m \right) \\
=&[x^n]\, \mathcal{F}^{\star}(x^2) \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^2)} + \frac{1}{1 - x \cdot C(x^2)} - 1 \right),
\end{aligned}$$

which gives the ordinary generating function for A026776.

The generating function for A026787 is be obtained by replacing $\mathcal{F}^{\star}(x^2)$ with $\mathcal{F}(x^2)$. □

3.5 Sequences A026777 and A026788

The term A026777(n) gives the total number of paths in $\mathcal{L}^{\star}_{CS}$ from $(0,0)$ to $(i, n-i)$, where $i = 0, 1, \dots, \lfloor n/2 \rfloor$. The ordinary generating function for A026777 is $\frac{\mathcal{F}^{\star}(x^2)}{1 - x \cdot \mathcal{S}(x^2)}$, which can be easily obtained from our analysis of the sequence A026776 above.

The ordinary generating function for A026788, which enumerates similar paths in $\mathcal{L}_{CS}$, equals $\frac{\mathcal{F}(x^2)}{1 - x \cdot \mathcal{S}(x^2)}$.

3.6 Sequences A026778 and A026789

The term A026778(n) gives the total number of paths in $\mathcal{L}^{\star}_{CS}$ from $(0,0)$ to $(i, i-j)$, where $0 \le j \le i \le n$. It is easy to see that

$$\texttt{A026778}(n) = \sum_{m=0}^{n} \texttt{A026776}(m)$$

and thus the ordinary generating function for A026778 can be obtained from the one for A026776 by multiplying it by $\frac{1}{1-x}$. That is, the ordinary generating function for A026778 equals

$$\frac{\mathcal{F}^{\star}(x^2)}{1-x} \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^2)} + \frac{1}{1 - x \cdot C(x^2)} - 1 \right).$$

The ordinary generating function for A026789, which enumerates similar paths in $\mathcal{L}_{CS}$, equals

$$\frac{\mathcal{F}(x^2)}{1-x} \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^2)} + \frac{1}{1 - x \cdot C(x^2)} - 1 \right).$$

3.7 Sequences A026779 and A026790

The terms A026779(n) and A026790(n) give the total number of paths from $(0,0)$ to $(i, n-2i)$, where $i = 0, 1, \ldots, \lfloor n/2 \rfloor$, in the lattices in $\mathcal{L}^{\star}_{CS}$ and $\mathcal{L}_{CS}$, respectively.

Theorem 5. *The ordinary generating function for* A026779 *is*

$$\mathcal{F}^{\star}(x^3) \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^3)} + \frac{1}{1 - x^2 \cdot C(x^3)} - 1 \right).$$

The ordinary generating function for A026790 *is*

$$\mathcal{F}(x^3) \cdot \left(\frac{1}{1 - x \cdot \mathcal{S}(x^3)} + \frac{1}{1 - x^2 \cdot C(x^3)} - 1 \right).$$

Proof. By Theorem 3, A026779 equals

$$\begin{aligned}
&\sum_{i=0}^{\lfloor n/3 \rfloor} [x^i]\, \mathcal{F}^{\star}(x) \cdot \mathcal{S}(x)^{n-3i} + \sum_{i=\lfloor n/3 \rfloor + 1}^{\lfloor n/2 \rfloor} [x^{n-2i}]\, \mathcal{F}^{\star}(x) \cdot C(x)^{3i-n} \\
=&\sum_{i=0}^{\lfloor n/3 \rfloor} [x^{3i}]\, \mathcal{F}^{\star}(x^3) \cdot \mathcal{S}(x^3)^{n-3i} + \sum_{i=\lfloor n/3 \rfloor + 1}^{\lfloor n/2 \rfloor} [x^{3n-6i}]\, \mathcal{F}^{\star}(x^3) \cdot C(x^3)^{3i-n} \\
=&[x^n]\, \mathcal{F}^{\star}(x^3) \cdot \left(\sum_{i=0}^{\lfloor n/3 \rfloor} \left(x \cdot \mathcal{S}(x^3) \right)^{n-3i} + \sum_{i=\lfloor n/3 \rfloor + 1}^{\lfloor n/2 \rfloor} \left(x^2 \cdot C(x^3) \right)^{3i-n} \right),
\end{aligned}$$

from where we conclude the proof with arguments similar to those in the proof of Theorem 4. □

References

[1] Petersen, T. K. *Eulerian Numbers*. Birkhäuser, Basel, 2015.

[2] Stanley, R. P. *Catalan Numbers*. Cambridge University Press, New York, NY, 2015.

[3] The OEIS Foundation. *The On-Line Encyclopedia of Integer Sequences*. Published electronically at http://oeis.org, 2016.