

THE PEAK STATISTICS ON SIMSUN PERMUTATIONS

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ABSTRACT. In this paper, we study the relationship among left peaks, interior peaks and up-down runs of simsun permutations. Properties of the generating polynomials, including the recurrence relation, generating function and real-rootedness are studied. Moreover, we introduce and study simsun permutations of the second kind.

Keywords: Simsun permutations; Left peaks; Interior peaks; Alternating runs; Excedances

1. INTRODUCTION

Let \mathfrak{S}_n denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \dots, n\}$. Let $\pi = \pi(1)\pi(2) \cdots \pi(n) \in \mathfrak{S}_n$. A descent of π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$. We say that π has no *double descents* if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$. The permutation π is called *simsun* if for all k , the subword of π restricted to $[k]$ (in the order they appear in π) contains no double descents. For example, 35142 is simsun, but 35241 is not. Simsun permutations are useful in describing the action of the symmetric group on the maximal chains of the partition lattice (see [24, 25]). They are a variant of André permutations that was introduced by Foata and Schützenberge [12]. There has been much recent work related to simsun permutations (see [3, 5, 8, 11, 13, 15] for instance).

Let $a_i(n)$ be the number of distinct S_n -orbits such that the stabiliser of a maximal chain in the orbit is conjugate to the Young subgroup $S_2^i \times S_1^{n-2i}$. Following Sundaram [24, Theorem 3.2], the numbers $a_i(n)$ satisfy the recurrence relation

$$a_i(n+1) = ia_i(n) + (n-2i+2)a_{i-1}(n),$$

with initial conditions $a_0(1) = 1 = a_1(2)$, $a_0(n) = 0$ for $n > 1$ and $a_i(n) = 0$ if $2i > n$. Let \mathcal{RS}_n be the set of simsun permutations of length n . Simion and Sundaram [24, p. 267] discovered that $a_i(n)$ is the number of permutations in \mathcal{RS}_{n-2} with $i-1$ descents and $\#\mathcal{RS}_n = E_{n+1}$, where E_n is the n th Euler number, which also is the number alternating permutations in \mathfrak{S}_n .

The descent number of $\pi \in \mathfrak{S}_n$ is defined by $\text{des}(\pi) = \#\{i \in [n-1] : \pi(i) > \pi(i+1)\}$. Let $S(n, k) = \#\{\pi \in \mathcal{RS}_n : \text{des}(\pi) = k\}$. We define $S_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} S(n, k)x^k$. Then the numbers $S(n, k)$ satisfy the recurrence relation

$$S(n, k) = (k+1)S(n-1, k) + (n-2k+1)S(n-1, k-1), \quad (1)$$

with the initial conditions $S(0, 0) = 1$ and $S(0, k) = 0$ for $k \geq 1$, which is equivalent to

$$S_{n+1}(x) = (1+nx)S_n(x) + x(1-2x)S'_n(x), \quad (2)$$

with $S_0(x) = 1$. Let $S(x, z) = \sum_{n \geq 0} S_n(x) \frac{z^n}{n!}$. Chow and Shiu [5, Theorem 1] obtained that

$$S(x, z) = \left(\frac{\sqrt{2x-1} \sec\left(\frac{z}{2}\sqrt{2x-1}\right)}{\sqrt{2x-1} - \tan\left(\frac{z}{2}\sqrt{2x-1}\right)} \right)^2. \quad (3)$$

For convenience, here we list the first few terms of $S_n(x)$:

$$S_1(x) = 1, S_2(x) = 1 + x, S_3(x) = 1 + 4x, S_4(x) = 1 + 11x + 4x^2, S_5(x) = 1 + 26x + 34x^2.$$

The number of peaks of permutations is certainly among the most important combinatorial statistics. See, e.g., [2, 16, 18] and the references therein. A *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\text{lpk}(\pi)$ denote the number of left peaks in π . For example, $\text{lpk}(21435) = 2$. Sundaram discovered that $a_i(n)$ is also the number of André permutations in \mathfrak{S}_{n-1} with i left peaks (see [26, p. 175]). In fact, since any descent of a simsun permutation is a left peak, we have

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\text{lpk}(\pi)}.$$

Let $\widehat{W}(n, k) = \#\{\pi \in \mathfrak{S}_n : \text{lpk}(\pi) = k\}$. Let $\widehat{W}_n(x) = \sum_{k \geq 0} \widehat{W}(n, k)x^k$. The polynomials $\widehat{W}_n(x)$ satisfy the recurrence relation

$$\widehat{W}_{n+1}(x) = (1 + nx)\widehat{W}_n(x) + 2x(1-x)\widehat{W}'_n(x),$$

with initial conditions $\widehat{W}_0(x) = \widehat{W}_1(x) = 1$ (see [21, A008971]). It is well known [21, A008971] that

$$\widehat{W}(x, z) = \sum_{n \geq 0} \widehat{W}_n(x) \frac{z^n}{n!} = \frac{\sqrt{1-x}}{\sqrt{1-x} \cosh(z\sqrt{1-x}) - \sinh(z\sqrt{1-x})}. \quad (4)$$

By comparing (3) with (4), we observe that $S(x, z) = \widehat{W}(2x, z/2)^2$, which leads to the following formula:

$$S_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \widehat{W}_k(2x) \widehat{W}_{n-k}(2x). \quad (5)$$

Denote by B_n the hyperoctahedral group of rank n . Elements π of B_n are signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i , where $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$. A *snake* of type B_n is a signed permutation $\pi(1)\pi(2) \cdots \pi(n) \in B_n$ such that $0 < \pi(1) > \pi(2) < \cdots < \pi(n)$. The n th *Springer number* S_n is the number of snakes of type B_n . Springer [22] derived the following generating function:

$$\sum_{n \geq 0} S_n \frac{z^n}{n!} = \frac{1}{\cos z - \sin z},$$

which equals $\widehat{W}(2, z)$. As a special case of (5), we get

$$E_{n+1} = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} S_k S_{n-k}.$$

We refer the reader to [4] for various structures related to Springer numbers. Motivated by (5), it is natural to study peak statistics on simsun permutations.

This paper is organized as follows. In Section 2, we give a constructive proof of a connection between $S(n, k)$ and the number of permutations in \mathfrak{S}_{n+1} with k interior peaks. In Section 3, we

count simsun permutations by their interior peaks. In Section 4, we count simsun permutations by their up-down runs. In Section 5, we introduce simsun permutations of the second kind.

2. RELATIONSHIP TO PERMUTATIONS WITH A GIVEN NUMBER OF INTERIOR PEAKS

We first recall some basic definitions of peak statistics. An *interior peak* in π is an index $i \in \{2, 3, \dots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$ (see [21, A008303]). Let $\text{pk}(\pi)$ denote the number of interior peaks in π . Let $W(n, k) = \#\{\pi \in \mathfrak{S}_n : \text{pk}(\pi) = k\}$. We define $W_n(x) = \sum_{k \geq 0} W(n, k)x^k$. The polynomials $W_n(x)$ satisfy the recurrence relation

$$W_{n+1}(x) = (nx - x + 2)W_n(x) + 2x(1-x)W'_n(x),$$

with initial conditions $W_1(x) = 1, W_2(x) = 2$ and $W_3(x) = 4 + 2x$. We say that π changes direction at position i if either $\pi(i-1) < \pi(i) > \pi(i+1)$, or $\pi(i-1) > \pi(i) < \pi(i+1)$, where $i \in \{2, 3, \dots, n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i where π changes direction (see [21, A059427]). Let $R(n, k)$ denote the number of permutations in \mathfrak{S}_n with k alternating runs and let $R_n(x) = \sum_{k=1}^{n-1} R(n, k)x^k$. The alternating runs of permutations was first studied by André [1] and he showed that $R(n, k)$ satisfies the following recurrence relation

$$R(n, k) = kR(n-1, k) + 2R(n-1, k-1) + (n-k)R(n-1, k-2)$$

for $n, k \geq 1$, where $R(1, 0) = 1$ and $R(1, k) = 0$ for $k \geq 1$.

Let $d(n, k)$ denote the number of increasing 1-2 trees on $[n]$ with k leaves (see [21, A094503]). Let $D_n(x) = \sum_{k \geq 1} d(n, k)x^k$. It follows from [5, Proposition 4] that $D_{n+1}(x) = xS_n(x)$ for $n \geq 0$. Using [18, Corollary 2, Theorem 11], we get

$$R_n(x) = \frac{x(1+x)^{n-2}}{2^{n-2}} W_n\left(\frac{2x}{1+x}\right) = 2x(1+x)^{n-2} S_{n-1}\left(\frac{x}{1+x}\right) \quad \text{for } n \geq 2. \quad (6)$$

Therefore, combining (6) and [17, Eq. (13)], we get the following result.

Proposition 1. *For $n \geq 1$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, we have*

$$W(n+1, k) = 2^{n-k} S(n, k). \quad (7)$$

Moreover, we have

$$S_n(x) = \frac{1}{2^{n+1}x} \sum_{k=0}^{\lfloor n/2 \rfloor + 1} p(n+1, n-2k+2)(2x-1)^k$$

for $n \geq 1$, where

$$p(n, n-2k+1) = (-1)^k \sum_{i \geq 1} i! \left\{ \begin{matrix} n \\ i \end{matrix} \right\} (-2)^{n-i} \left[\binom{i}{n-2k} - \binom{i}{n-2k+1} \right].$$

In the rest of this section, we give a constructive proof of (7). Let

$$D(\pi) = \{i \in [n-2] : \pi(i) > \pi(i+1)\}$$

be the descent set of $\pi \in \mathcal{RS}_n$. Set $\overleftarrow{D}(\pi) = \{i-1 : i \in D(\pi)\}$. It should be noted that if we get a permutation $\pi' \in \mathcal{RS}_{n+1}$ from a permutation $\pi \in \mathcal{RS}_n$ by inserting the entry $n+1$ into π , then the entry $n+1$ can not be inserted right after $\pi(j)$, where $j \in \overleftarrow{D}(\pi)$. In this

section, we always assume that permutations in \mathcal{RS}_n are prepended by 0. That is, we identify a permutation $\pi(1)\pi(2)\cdots\pi(n) \in \mathcal{RS}_n$ with the word $\pi(0)\pi(1)\pi(2)\cdots\pi(n)$, where $\pi(0) = 0$. Define $\mathcal{RS}_{n,k} = \{\pi \in \mathcal{RS}_n \mid \text{des}(\pi) = k\}$. We can now introduce a definition of *labeled simsun permutations*.

Definition 2. Let $\pi \in \mathcal{RS}_{n,k}$. Suppose $i_1 < i_2 < \cdots < i_k$ are elements of $D(\pi)$. Then we put the superscript label x_r right after $\pi(i_r)$, where $1 \leq r \leq k$. If $j_1 < j_2 < \cdots < j_{n-2k}$ are elements of the set $\{0, 1, 2, \dots, n-1\} \setminus (D(\pi) \cup \overleftarrow{D}(\pi))$, then we put the superscript label y_s right after $\pi(j_s)$, where $1 \leq s \leq n-2k$.

Let $\mathfrak{S}_{n,k} = \{\pi \in \mathfrak{S}_n \mid \text{pk}(\pi) = k\}$. We introduce a definition of *labeled permutations*.

Definition 3. Let $\pi \in \mathfrak{S}_{n,k}$. Suppose $i_1 < i_2 < \cdots < i_k$ are indices of the interior peaks of π . Then we put the superscript labels p_r immediately before and right after $\pi(i_r)$, where $1 \leq r \leq k$. If $j_1 < j_2 < \cdots < j_{n-2k-1}$ are elements of the set $\{1, 2, 3, \dots, n-1\} \setminus (\{i_1, i_2, \dots, i_k\} \cup \{i_1-1, i_2-1, \dots, i_k-1\})$, then we put the superscript label q_s right after $\pi(j_s)$, where $1 \leq s \leq n-2k-1$.

In the following discussion, we always add labels to permutations in $\mathcal{RS}_{n,k}$ and $\mathfrak{S}_{n,k}$. As an example, for $\pi = 34125$, if we say that $\pi \in \mathcal{RS}_{5,1}$, then the labels of π is given by $y^1 34^{x^1} 1^{y^2} 2^{y^3} 5$; if we say that $\pi \in \mathfrak{S}_{5,1}$, then the labels of π is given by $3^{p^1} 4^{p^1} 1^{q^1} 2^{q^2} 5$.

Now we construct a correspondence, denoted by Φ , between $\mathcal{RS}_{n,k}$ and $\mathfrak{S}_{n+1,k}$. When $n = 1$, the correspondence between $\mathcal{RS}_{1,0}$ and $\mathfrak{S}_{2,0}$ is given by

$$y^1 1 \xleftarrow{\Phi} \{1^{q^1} 2, 2^{q^1} 1\}.$$

When $n = 2$, the correspondence between $\mathcal{RS}_{2,k}$ and $\mathfrak{S}_{3,k}$ is given by

$$\begin{aligned} y^1 1^{y^2} 2 &\xleftarrow{\Phi} \{1^{q^1} 2^{q^2} 3, 3^{q^1} 1^{q^2} 2, 2^{q^1} 1^{q^2} 3, 3^{q^1} 2^{q^2} 1\}; \\ 2^{x^1} 1 &\xleftarrow{\Phi} \{1^{p^1} 3^{p^1} 2, 2^{p^1} 3^{p^1} 1\}. \end{aligned}$$

Let $n = m$. Suppose Φ is a correspondence between $\mathcal{RS}_{m,k}$ and $\mathfrak{S}_{m+1,k}$ for all k . More precisely, given an element $\pi \in \mathcal{RS}_{m,k}$. Suppose we have the correspondence

$$\pi \xleftarrow{\Phi} \{\sigma_1, \sigma_2, \dots, \sigma_{2^{m-k}}\},$$

where $\sigma_i \in \mathfrak{S}_{m+1,k}$ for $1 \leq i \leq 2^{m-k}$. Consider the case $n = m+1$. Suppose $\widehat{\pi} \in \mathcal{RS}_{m+1}$ is obtained from π by inserting the entry $m+1$ into π . We distinguish three cases:

- (i) If $\widehat{\pi}(m+1) = m+1$, then we insert the entry $m+2$ at the front or at the end of each σ_i . In this case, the obtained elements in $\Phi(\widehat{\pi})$ all have k interior peaks. Therefore, we get $2 \cdot 2^{m-k} = 2^{m+1-k}$ elements in $\mathfrak{S}_{m+2,k}$.
- (ii) If the entry $m+1$ is inserted to the position of π with label x_r , then we insert the entry $m+2$ to one of the positions of each σ_i with label p_r . In this case, $\text{des}(\widehat{\pi}) = k$ and we get $2 \cdot 2^{m-k} = 2^{m+1-k}$ elements in $\mathfrak{S}_{m+2,k}$.
- (iii) If the entry $m+1$ is inserted to the position of π with label y_s , then we insert the entry $m+2$ to the position of each σ_i with label q_s . In this case, $\text{des}(\widehat{\pi}) = k+1$ and we get $2^{m-k} = 2^{(m+1)-(k+1)}$ elements in $\mathfrak{S}_{m+2,k+1}$.

It is straightforward to show that each labeled permutation in $\Phi(\widehat{\pi})$ will be obtained exactly once in this way. Conversely, given an element τ of $\mathfrak{S}_{m+2,k}$. Removing the entry $m+2$ of τ , we can find the position of the largest entry of the corresponding simsun permutation in \mathcal{RS}_{m+1} . As illustrated in example 4, we can get an unique element of \mathcal{RS}_{m+1} by repeatedly removing the largest entry. By induction, we see that Φ is the desired correspondence between $\mathcal{RS}_{m,k}$ and $\mathfrak{S}_{m+1,k}$, which also gives a constructive proof of (7).

Example 4. Given $\pi = 3412 \in \mathcal{RS}_{4,1}$. The correspondence between π and $\Phi(\pi)$ is built up as follows:

$$\begin{aligned} y_1 1 &\xleftarrow{\Phi} \{1^{q_1} 2, 2^{q_1} 1\}; \\ y_1 1 y_2 2 &\xleftarrow{\Phi} \{1^{q_1} 2^{q_2} 3, 3^{q_1} 1^{q_2} 2, 2^{q_1} 1^{q_2} 3, 3^{q_1} 2^{q_2} 1\}; \\ 3^{x_1} 1 y_1 2 &\xleftarrow{\Phi} \{1^{p_1} 4^{p_1} 2^{q_1} 3, 3^{p_1} 4^{p_1} 1^{q_1} 2, 2^{p_1} 4^{p_1} 1^{q_1} 3, 3^{p_1} 4^{p_1} 2^{q_1} 1\}; \\ y_1 3 4^{x_1} 1 y_2 2 &\xleftarrow{\Phi} \{1^{p_1} 5^{p_1} 4^{q_1} 2^{q_2} 3, 3^{p_1} 5^{p_1} 4^{q_1} 1^{q_2} 2, 2^{p_1} 5^{p_1} 4^{q_1} 1^{q_2} 3, 3^{p_1} 5^{p_1} 4^{q_1} 2^{q_2} 1, \\ &\quad 1^{q_1} 4^{p_1} 5^{p_1} 2^{q_2} 3, 3^{q_1} 4^{p_1} 5^{p_1} 1^{q_2} 2, 2^{q_1} 4^{p_1} 5^{p_1} 1^{q_2} 3, 3^{q_1} 4^{p_1} 5^{p_1} 2^{q_2} 1\}. \end{aligned}$$

3. INTERIOR PEAKS OF SIMSUN PERMUTATIONS

Let $\mathcal{RS}_n^+ = \{\pi \in \mathcal{RS}_n : \pi(1) > \pi(2)\}$ and $\mathcal{RS}_n^- = \{\pi \in \mathcal{RS}_n : \pi(1) < \pi(2)\}$. For $\pi \in \mathcal{RS}_n^+$, we have $\text{lpk}(\pi) = \text{pk}(\pi) + 1$. While for $\pi \in \mathcal{RS}_n^-$, we have $\text{lpk}(\pi) = \text{pk}(\pi)$.

We define

$$\begin{aligned} P_n(x) &= \sum_{\pi \in \mathcal{RS}_n} x^{\text{pk}(\pi)} = \sum_{k \geq 0} P(n, k) x^k, \\ P_n^+(x) &= \sum_{\pi \in \mathcal{RS}_n^+} x^{\text{pk}(\pi)} = \sum_{k \geq 0} P^+(n, k) x^k, \\ P_n^-(x) &= \sum_{\pi \in \mathcal{RS}_n^-} x^{\text{pk}(\pi)} = \sum_{k \geq 0} P^-(n, k) x^k. \end{aligned}$$

The following lemma is a fundamental result.

Lemma 5. For $n \geq 2$, we have

$$P^+(n+1, k) = (k+1)P^+(n, k) + (n-2k)P^+(n, k-1) + P^-(n, k), \quad (8)$$

$$P^-(n+1, k) = (k+1)P^-(n, k) + (n-2k+1)P^-(n, k-1) + P^+(n, k-1). \quad (9)$$

Equivalently, the polynomials $P_n^+(x)$ and $P_n^-(x)$ satisfy the following recurrence relations

$$\begin{aligned} P_{n+1}^+(x) &= ((n-2)x+1)P_n^+(x) + x(1-2x)\frac{d}{dx}P_n^+(x) + P_n^-(x), \\ P_{n+1}^-(x) &= ((n-1)x+1)P_n^-(x) + x(1-2x)\frac{d}{dx}P_n^-(x) + xP_n^+(x). \end{aligned}$$

Proof. We now prove (8). There are three ways we can get a permutation $\pi' \in \mathcal{RS}_{n+1}^+$ with k interior peaks from a permutation $\pi \in \mathcal{RS}_n$ by inserting the entry $n+1$ into π :

- (a) If $\pi \in \mathcal{RS}_n^+$ and $\text{pk}(\pi) = k$, then we can insert the entry $n + 1$ right after an interior peak of π or put the entry $n + 1$ at the end of π . As we have $P^+(n, k)$ choices for π , this accounts for $(k + 1)P^+(n, k)$ possibilities.
- (b) If $\pi \in \mathcal{RS}_n^+$ and $\text{pk}(\pi) = k - 1$, then there are $n - 2k$ positions could be inserted the entry $n + 1$, since we can't insert the entry $n + 1$ immediately before or right after each left peak of π . As we have $P^+(n, k - 1)$ choices for π , this accounts for $(n - 2k)P^+(n, k - 1)$ possibilities.
- (c) If $\pi \in \mathcal{RS}_n^-$ and $\text{pk}(\pi) = k$, then we have to put the entry $n + 1$ at the front of π .

This completes the proof of (8). In the same way, one can get (9). \square

The first few terms of the $P_n(x)$, $P_n^+(x)$ and $P_n^-(x)$ are respectively given as follows:

$$\begin{aligned} P_1(x) &= 1, P_2(x) = 2, P_3(x) = 3 + 2x, P_4(x) = 4 + 12x, P_5(x) = 5 + 44x + 12x^2; \\ P_1^+(x) &= 1, P_2^+(x) = 1, P_3^+(x) = 2, P_4^+(x) = 3 + 4x, P_5^+(x) = 4 + 22x; \\ P_1^-(x) &= 1, P_2^-(x) = 1, P_3^-(x) = 1 + 2x, P_4^-(x) = 1 + 8x, P_5^-(x) = 1 + 22x + 12x^2. \end{aligned}$$

By Lemma 5, it is easy to deduce that

$$\deg P_n^+(x) = \lfloor (n - 2)/2 \rfloor, \quad \deg P_n(x) = \deg P_n^-(x) = \lfloor (n - 1)/2 \rfloor.$$

Lemma 6. *For $n \geq 1$, we have*

$$P^+(n + 1, k) = (n - 2k)S(n, k), \quad P^-(n + 1, k) = (1 + k)S(n, k). \quad (10)$$

Equivalently,

$$P_{n+1}^+(x) = nS_n(x) - 2xS_n'(x), \quad P_{n+1}^-(x) = S_n(x) + xS_n'(x). \quad (11)$$

Proof. We prove (10) by induction. If $n = 1$, the result is obvious, so we proceed to the inductive step. Suppose the result holds for $n = m$. For $n = m + 1$, combining (1) and (8), we have

$$\begin{aligned} P^+(m + 2, k) &= (k + 1)P^+(m + 1, k) + (m + 1 - 2k)P^+(m + 1, k - 1) + P^-(m + 1, k) \\ &= (k + 1)(m - 2k)S(m, k) + (m + 1 - 2k)(m + 2 - 2k)S(m, k - 1) + (k + 1)S(m, k) \\ &= (k + 1)(m - 2k)S(m, k) + (m + 1 - 2k)(m + 2 - 2k)S(m, k - 1) + \\ &\quad S(m + 1, k) - (m + 2 - 2k)S(m, k - 1) \\ &= (m - 2k)[(k + 1)S(m, k) + (m + 2 - 2k)S(m, k - 1)] + S(m + 1, k) \\ &= (m - 2k)S(m + 1, k) + S(m + 1, k) \\ &= (m + 1 - 2k)S(m + 1, k). \end{aligned}$$

Along the same lines, one can prove $P^-(n + 1, k) = (1 + k)S(n, k)$. \square

It is clear that $P(n, k) = P^+(n, k) + P^-(n, k)$ for $n \geq 2$. We can now conclude the following result from the discussion above.

Theorem 7. *For $n \geq 1$, we have*

$$P(n + 1, k) = (n + 1 - k)S(n, k), \quad (12)$$

or equivalently,

$$P_{n+1}(x) = (n+1)S_n(x) - xS'_n(x). \quad (13)$$

Furthermore, we have

$$P(n+1, k) = \frac{(k+1)(n-k+1)}{n-k} P(n, k) + (n-2k+1)P(n, k-1) \quad (14)$$

for $0 \leq k \leq \lfloor n/2 \rfloor$. In particular, $P(n, 0) = n$ and $P(n, 1) = (n-1)(2^{n-1} - n)$ for $n \geq 1$.

It should be noted that (14) follows immediately from (1) and (12).

We now recall some notations from [14] concerning the zeros of polynomials. Let RZ denote the set of real polynomials with only real zeros. Furthermore, denote by $\text{RZ}(I)$ the set of such polynomials all whose zeros are in the interval I . Suppose that $p, q \in \text{RZ}$, that those of p are $\xi_1 \leq \dots \leq \xi_n$, and that those of q are $\theta_1 \leq \dots \leq \theta_m$. We say that p *interlaces* q if $\deg p = 1 + \deg q$ and the zeros of p and q satisfy

$$\theta_1 \leq \xi_1 \leq \theta_2 \leq \dots \leq \xi_n \leq \theta_{n+1}.$$

We also say that p *alternates left of* q if $\deg p = \deg q$ and the zeros of p and q satisfy

$$\xi_1 \leq \theta_1 \leq \xi_2 \leq \dots \leq \xi_n \leq \theta_n.$$

We use the notation $p \dagger q$ for “ p interlaces q ,” $p \ll q$ for “ p alternates left of q ,” and $p \prec q$ for either $p \dagger q$ or $p \ll q$. For notational convenience, let $a \prec bx + c$ for any real constants a, b, c .

We now recall a result on the real-rootedness of $S_n(x)$.

Lemma 8 ([5, Theorem 1]). *For $n \geq 2$, we have $S_n(x) \in \text{RZ}(-\infty, 0)$ and $S_n(x) \prec S_{n+1}(x)$.*

Let sgn denote the sign function defined on \mathbb{R} .

Theorem 9. *For $n \geq 2$, we have $P_n(x), P_n^+(x), P_n^-(x) \in \text{RZ}(-\infty, 0)$ and*

$$P_{n+1}(x) \ll S_n(x), P_{n+1}^+(x) \prec S_n(x), S_n(x) \ll P_{n+1}^-(x).$$

Proof. For $n \geq 2$, let $r_{\lfloor n/2 \rfloor} < r_{\lfloor n/2 \rfloor - 1} < \dots < r_2 < r_1$ be the distinct zeros of $S_n(x)$. Then by (13), we get $\text{sgn } P_{n+1}(r_i) = (-1)^{i-1}$ for $i = 1, 2, \dots, \lfloor n/2 \rfloor$. Hence $P_{n+1}(x)$ has precisely one zero in each of $\lfloor n/2 \rfloor - 1$ intervals $(r_{\lfloor n/2 \rfloor}, r_{\lfloor n/2 \rfloor - 1}), \dots, (r_2, r_1)$. Recall that $\deg P_{n+1}(x) = \lfloor n/2 \rfloor$. If $n = 2k + 1$ is odd, then $\text{sgn } P_{n+1}(r_k) = (-1)^{k-1}$ and $\text{sgn } P_{n+1}(-\infty) = (-1)^k$. If $n = 2k + 2$ is even, then $\text{sgn } P_{n+1}(r_k) = (-1)^k$ and $\text{sgn } P_{n+1}(-\infty) = (-1)^{k+1}$. Thus $P_{n+1}(x)$ has an additional zero in the interval $(-\infty, r_{\lfloor n/2 \rfloor})$. Therefore, we have $P_{n+1}(x) \ll S_n(x)$. Similarly, by using (11), one can derive $P_{n+1}^+(x) \prec S_n(x)$ and $S_n(x) \ll P_{n+1}^-(x)$. \square

4. UP-DOWN RUNS OF SIMSUN PERMUTATIONS

An *alternating subsequence* of $\pi \in \mathfrak{S}_n$ is a subsequence $\pi(i_1), \pi(i_2), \dots, \pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \dots > \pi(i_k),$$

where $i_1 < i_2 < \dots < i_k$. Motivated by the study of longest increasing subsequences, Stanley [23] studied the longest alternating subsequences. Let $\ell_n(\pi)$ be the length of the longest alternating subsequence of a permutation $\pi \in \mathfrak{S}_n$. The *up-down runs* of a permutation π are the alternating

runs of π endowed with a 0 in the front (see [21, A186370]). For example, the permutation $\pi = 514623$ has 4 alternating runs and 5 up-down runs. Let $\text{uprun}(\pi)$ be the number of up-down runs of π . It is clear that $\text{uprun}(\pi) = \ell_n(\pi)$ for $\pi \in \mathfrak{S}_n$. We define

$$T_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\text{uprun}(\pi)} = \sum_{k=1}^n T(n, k)x^k.$$

The first few terms of $T_n(x)$ are

$$T_1(x) = x, \quad T_2(x) = x + x^2, \quad T_3(x) = x + 2x^2 + 2x^3, \quad T_4(x) = x + 3x^2 + 8x^3 + 4x^4.$$

Theorem 10. *For $n \geq 1$, the numbers $T(n, k)$ satisfy the recurrence relation*

$$T(n, k) = \lceil k/2 \rceil T(n-1, k) + T(n-1, k-1) + (n-k+1)T(n-1, k-2), \quad (15)$$

with initial conditions $T(0, 0) = 1$ and $T(0, k) = 0$ for $k > 0$.

Proof. There are three ways in which a permutation $\pi' \in \mathcal{RS}_n$ with $\text{uprun}(\pi') = k$ can be obtained from a permutation $\pi \in \mathcal{RS}_{n-1}$ by inserting the entry n into π .

- (a) If $\text{uprun}(\pi) = k$, then we can insert the entry n right after the end of each ascending run. This accounts for $\lceil k/2 \rceil T(n-1, k)$ possibilities.
- (b) If $\text{uprun}(\pi) = k-1$, then we distinguish two cases: when π ends with an ascending run, we insert the entry n to the front of the last entry of π ; when π ends with descending run, we insert the entry n at the end of π . This gives $T(n-1, k-1)$ possibilities.
- (c) If $\text{uprun}(\pi) = k-2$, then we can insert the entry n into the remaining $n-k+1$ positions. This gives $(n-k+1)T(n-1, k-2)$ possibilities.

This completes the proof of (15). □

Note that $S(n, 0) = T(n, 1) = 1$, corresponding to the permutation $12 \cdots n$. Recall that an element π of \mathfrak{S}_n is *alternating* if $\pi(1) > \pi(2) < \pi(3) > \cdots > \pi(n)$. In other words, $\pi(i) < \pi(i+1)$ if i is even and $\pi(i) > \pi(i+1)$ if i is odd. If $n = 2m$ is even, then $S(2m, m) = T(2m, 2m)$, corresponding to the number of alternating permutations in \mathcal{RS}_{2m} . If $n = 2m+1$ is odd, applying the complement operation ϕ to $\pi \in \mathcal{RS}_n$, i.e., $\phi(\pi(i)) = \pi(n+1-i)$, it is clear that $P(2m+1, m) = T(2m+1, 2m+1)$ counts the number of alternating permutations in \mathcal{RS}_{2m+1} . In general, by analyzing permutations in \mathcal{RS}_n^+ and \mathcal{RS}_n^- , it is easy to verify that

$$S(n, k) = T(n, 2k) + T(n, 2k+1), \quad P(n, k) = T(n, 2k+1) + T(n, 2k+2).$$

Equivalently, we have

$$(1+x)T_n(x) = xS_n(x^2) + x^2P_n(x^2) \quad \text{for } n \geq 1. \quad (16)$$

Theorem 11. *For $n \geq 0$, we have*

$$T_{n+1}(x) = x(1+nx)S_n(x^2) + \frac{1}{2}x^2(1-2x)S'_n(x^2).$$

Proof. It follows from (2), (13) and (16) that

$$\begin{aligned}
 (1+x)T_{n+1}(x) &= xS_{n+1}(x^2) + x^2P_{n+1}(x^2) \\
 &= x((1+nx^2)S_n(x^2) + \frac{1}{2}x(1-2x^2)S'_n(x^2)) + x^2((n+1)S_n(x^2) - \frac{x}{2}S'_n(x^2)) \\
 &= x(1+(n+1)x+nx^2)S_n(x^2) + \frac{1}{2}x^2(1-x-2x^2)S'_n(x^2) \\
 &= x(1+x)(1+nx)S_n(x^2) + \frac{1}{2}x^2(1+x)(1-2x)S'_n(x^2),
 \end{aligned}$$

the statement immediately follows. \square

We call the simsun permutations discussed above to be the simsun permutations of the first kind. In the next section, we shall introduce the simsun permutations of the second kind.

5. SIMSUN PERMUTATIONS OF THE SECOND KIND

In this section, we always write $\pi \in \mathfrak{S}_n$ in *standard cycle decomposition*, where each cycle is written with its smallest entry first and the cycles are written in increasing order of their smallest entry. For each $\pi \in \mathfrak{S}_n$, we say that π has an *excedance* at i if $\pi(i) > i$. The excedance number of π is defined by $\text{exc}(\pi) = \#\{i \in [n-1] : \pi(i) > i\}$. Following [20], for $\pi \in \mathfrak{S}_n$, a value $x = \pi(i)$ is called a *double excedance* if $i = \pi^{-1}(x) < x < \pi(x)$, and we say that $x = \pi(i)$ is a *cyclic peak* if $i = \pi^{-1}(x) < x > \pi(x)$. Let $\text{cpk}(\pi)$ denote the number of cyclic peaks of π .

Definition 12. *We say that $\pi \in \mathfrak{S}_n$ is a simsun permutation of the second kind if for all $k \in [n]$, after removing the k largest letters of π , the resulting permutation has no double excedances.*

For example, $(1, 5, 3, 4)(2)$ is not a simsun permutation of the second kind since when we remove the letter 5, the resulting permutation $(1, 3, 4)(2)$ contains a double excedance. Let \mathcal{SS}_n be the set of the simsun permutations of the second kind of length n . It is clear that $\text{exc}(\pi) = \text{cpk}(\pi)$ for $\pi \in \mathcal{SS}_n$.

In the following, we first present a constructive proof of the following identity:

$$\sum_{\pi \in \mathcal{RS}_n} x^{\text{des}(\pi)} = \sum_{\pi \in \mathcal{SS}_n} x^{\text{exc}(\pi)}. \tag{17}$$

Let $\mathcal{SS}_{n,k} = \{\pi \in \mathcal{SS}_n \mid \text{exc}(\pi) = k\}$. As a variant of Definition 2, we introduce a definition of *labeled simsun permutations of the the second kind*.

Definition 13. *Let $\sigma \in \mathcal{SS}_{n,k}$. Suppose $i_1 < i_2 < \dots < i_k$ are the excedances of σ . Then we put the superscript labels u_r right after i_r , where $1 \leq r \leq k$. In the remaining positions except the first position of each cycle and the positions right after $\sigma(i_r)$, we put the superscript labels $v_1, v_2, \dots, v_{n-2k}$ from left to right.*

As an example, for $\sigma = (1, 3)(2, 4)(5) \in \mathcal{SS}_{5,2}$, the labeled σ is given by $(1^{u_1}3)(2^{u_2}4)(5^{v_1})$.

Now we start to construct a bijection, denoted by Ψ , between $\mathcal{RS}_{n,k}$ and $\mathcal{SS}_{n,k}$. When $n = 1$, we have $\mathcal{RS}_{1,0} = \{y^1 1\}$. Set $\Psi(y^1 1) = (1^{v_1})$. This gives a bijection between $\mathcal{RS}_{1,0}$ and $\mathcal{SS}_{1,0}$. Let $n = m$. Suppose Ψ is a bijection between $\mathcal{RS}_{m,k}$ and $\mathcal{SS}_{m,k}$ for all k . Given $\pi \in \mathcal{RS}_{m,k}$. Suppose $\Psi(\pi) = \sigma$. Consider the following three cases:

- (i) If $\hat{\pi}$ is obtained from π by inserting the entry $m + 1$ to the position of π with label x_r , then we insert $m + 1$ to σ with label u_r . In this case, $\text{des}(\hat{\pi}) = \text{exc}(\Psi(\hat{\pi})) = k$. Hence $\hat{\pi} \in \mathcal{RS}_{m+1,k}$ and $\Psi(\hat{\pi}) \in \mathcal{SS}_{m+1,k}$.
- (ii) If $\hat{\pi}$ is obtained from π by inserting the entry $m + 1$ to the position of π with label y_r , then we insert $m + 1$ to σ with label v_r . In this case, $\text{des}(\hat{\pi}) = \text{exc}(\Psi(\hat{\pi})) = k + 1$. Hence $\hat{\pi} \in \mathcal{RS}_{m+1,k+1}$ and $\Psi(\hat{\pi}) \in \mathcal{SS}_{m+1,k+1}$.
- (iii) If $\hat{\pi}$ is obtained from π by inserting the entry $m + 1$ at the end of π , then we append $(m + 1)$ to σ as a new cycle. Hence $\hat{\pi} \in \mathcal{RS}_{m+1,k}$ and $\Psi(\hat{\pi}) \in \mathcal{SS}_{m+1,k}$.

By induction, we see that Ψ is the desired bijection between $\mathcal{RS}_{m,k}$ and $\mathcal{SS}_{m,k}$ for all k , which also gives a constructive proof of (17).

Example 14. Given $\pi = 3412 \in \mathcal{RS}_{4,1}$. The correspondence between π and $\Psi(\pi)$ is built up as follows:

$$\begin{aligned} y_1 1 &\Leftrightarrow (1^{v_1}); \\ y_1 1 y_2 2 &\Leftrightarrow (1^{v_1})(2^{v_2}); \\ 3^{x_1} 1 y_1 2 &\Leftrightarrow (1^{u_1} 3)(2^{v_1}); \\ y_1 3 4^{x_1} 1 y_2 2 &\Leftrightarrow (1^{u_1} 4 3^{v_1})(2^{v_2}). \end{aligned}$$

We now consider the following enumerative polynomials

$$S_n(x, q) = \sum_{\pi \in \mathcal{SS}_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)},$$

where $\text{cyc}(\pi)$ is the number of cycles of π . Let

$$S = S(x, q; z) = \sum_{n \geq 0} S_n(x, q) \frac{z^n}{n!}.$$

Theorem 15. The polynomials $S_n(x, q)$ satisfy the recurrence relation

$$S_{n+1}(x, q) = (q + nx)S_n(x, q) + x(1 - 2x) \frac{\partial}{\partial x} (S_n(x, q)), \quad (18)$$

with the initial condition $S_0(x, q) = 1$. Furthermore,

$$S(x, q; z) = S(x, z)^q. \quad (19)$$

Proof. Let n be a fixed positive integer and given $\sigma \in \mathcal{SS}_n$. Let σ_i be an element of \mathcal{SS}_{n+1} obtained from σ by inserting the entry $n + 1$, in the standard cycle decomposition of σ , right after i if i is not a cyclic peak of σ and $i \in [n]$ or as a new cycle $(n + 1)$ if $i = n + 1$. It is clear that

$$\text{cyc}(\sigma_i) = \begin{cases} \text{cyc}(\sigma), & \text{if } i \in [n]; \\ \text{cyc}(\sigma) + 1, & \text{if } i = n + 1. \end{cases}$$

Therefore, we have

$$\begin{aligned}
 S_{n+1}(x, q) &= \sum_{\pi \in \mathcal{SS}_{n+1}} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)} \\
 &= \sum_{i=1}^{n+1} \sum_{\sigma \in \mathcal{SS}_n} x^{\text{exc}(\sigma_i)} q^{\text{cyc}(\sigma_i)} \\
 &= \sum_{\sigma \in \mathcal{SS}_n} x^{\text{exc}(\sigma)} q^{\text{cyc}(\sigma)+1} + \sum_{i=1}^n \sum_{\sigma \in \mathcal{SS}_n} x^{\text{exc}(\sigma_i)} q^{\text{cyc}(\sigma)} \\
 &= qS_n(x, q) + \sum_{\sigma \in \mathcal{SS}_n} (\text{exc}(\sigma)x^{\text{exc}(\sigma)} + (n - 2\text{exc}(\sigma))x^{\text{exc}(\sigma)+1})q^{\text{cyc}(\sigma)} \\
 &= qS_n(x, q) + nxS_n(x, q) + \sum_{\pi \in \mathcal{RS}_n} (1 - 2x)\text{exc}(\pi)x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)},
 \end{aligned}$$

and (18) follows. By rewriting (18) in terms of generating function S , we have

$$(1 - xz)S_z = qS + x(1 - 2x)S_x. \quad (20)$$

It is routine to check that the generating function $\tilde{S} = \tilde{S}(x, q; z) = S(x, z)^q$ satisfies (20). Also, this generating function gives $\tilde{S}(x, q; 0) = 1$, $\tilde{S}(x, 0; z) = 1$ and $\tilde{S}(0, q; z) = e^{qz}$. Hence $S = \tilde{S}$. \square

Combining (18) and [19, Theorem 2], we get the following corollary.

Corollary 16. *If $q > 0$, then $S_n(x, q)$ has nonpositive and simple zeros for $n \geq 2$.*

Using (19), it is easy to verify that

$$S(1, q; z) = \frac{1}{(1 - \sin z)^q}.$$

and for $n \geq 1$,

$$S_n(x, -1) = \begin{cases} (1-x)(1-2x)^{m-1}, & \text{if } n = 2m; \\ -(1-2x)^m, & \text{if } n = 2m+1. \end{cases}$$

A cycle $(b(1), b(2), \dots)$ is said to be up-down if it satisfies $b(1) < b(2) > b(3) < \dots$. We say that a permutation π is *cycle-up-down* if it is a product of up-down cycles. Let Δ_n be the set of cycle-up-down permutations in \mathfrak{S}_n . Deutsch and Elizalde [7, p. 193] discovered that

$$\sum_{n \geq 0} \sum_{\pi \in \Delta_n} q^{\text{cyc}(\pi)} \frac{z^n}{n!} = S(1, q; z).$$

Therefore, we have

$$\sum_{\pi \in \mathcal{SS}_n} q^{\text{cyc}(\pi)} = \sum_{\pi \in \Delta_n} q^{\text{cyc}(\pi)}.$$

We define

$$S_n(x, y, q) = \sum_{\pi \in \mathcal{SS}_n} x^{\text{exc}(\pi)} y^{\text{fix}(\pi)} q^{\text{cyc}(\pi)},$$

where $\text{fix}(\pi)$ is the number of fixed points of π , i.e., $\text{fix}(\pi) = \#\{i \in [n] : \pi(i) = i\}$. Note that

$$\begin{aligned}
 S_n(x, y, q) &= \sum_{i=0}^n \binom{n}{i} (yq - q)^i \sum_{\pi \in \mathcal{SS}_{n-i}} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)} \\
 &= \sum_{i=0}^n \binom{n}{i} (yq - q)^i S_{n-i}(x, q).
 \end{aligned}$$

Using (19), we obtain

$$\sum_{n \geq 0} S_n(x, y, q) \frac{z^n}{n!} = e^{qz(y-1)} S(x, z)^q.$$

6. CONCLUDING REMARKS

In this paper we study the peak statistics on simsum permutations. It is well known that the descent statistic is equidistributed over n -simsum permutations and n -André permutations (see [5]), and there are bijections between simsum permutations and increasing 1-2 trees (see [6] for instance). Therefore, one can find corresponding results on André permutations and increasing 1-2 trees. For example, $S(1, q; z)$ also is the (shifted) exponential generating function that counts André permutations with respect to the size and the number of right-to-left minima (see [9, Proposition 1]) Furthermore, it would be interesting to derive similar results on signed simsum permutations introduced by Ehrenborg and Readdy [10].

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