# $q$-BERNSTEIN FUNCTIONS AND APPLICATIONS

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ABSTRACT. We characterize of the $q$-Bernstein functions in terms of $q$-Laplace transform. Moreover, we present several results of $q$-completely monotonic, $q$-log completely monotonic and $q$-Bernstein functions.

KEYWORDS: $q$-completely monotonic function; $q$-Bernstein function; $q$-infinitely divisible function

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## 1. INTRODUCTION

In last decades, the $q$-calculus has been received a lot of attentions. The field has expanded explosively due to the fact that applications of basic hypergeometric series to the diverse different branches of mathematics and applied mathematics are constantly being covered (for example, see [11] and references therein).

Recall [6, 9–11] some notation and definitions concerning $q$-calculus. The observation $\lim_{q\to 1^-}\frac{1-q^x}{1-q}=x$ plays as a basic step for the theory of $q$-analogues, where $x,q\in\mathbb{C}$. We define $[x]=\frac{1-q^x}{1-q}$, is called the $q$ *number of* $x$, which it is introduced by Heine. Clearly, $\lim_{q\to 1^-}[x]=x$. Throughout this paper we assume that $q$ satisfies $0<q<1$. The $q$-*Pochhammer symbol* is given by $[x]_k=\prod_{j=0}^{k-1}[x+j]$ with $[x]_0=1$. The $q$-*factorial* of $n$ is given by $[1]_n=[1][2]\cdots[n]=[n]!$ and the $q$-*Gauss binomial coefficients* are defined by $\left[\begin{smallmatrix} n\\ k\end{smallmatrix}\right]_q=\frac{[n]!}{[k]![n-k]!}$. For the exponential function has given two analogues (see [9, 11]) as $e_q(x)=\sum_{n\geq 0}\frac{x^n}{[n]!}$ and $E_q(x)=\sum_{n\geq 0}q^{\binom{n}{2}}\frac{x^n}{[n]!}$, where the series converges for $|x|<\frac{1}{1-q}$ and $x\in\mathbb{C}$ respectively. Clearly, $E_q(x)=e_{1/q}(x)$ and we define $E_q^x=(E_q(1))^x$. The $q$-derivative (for example, see [6, 9–11]) of an arbitrary function $f(x)$ is defined by

$$D_q(f(x))=\frac{f(qx)-f(x)}{x(q-1)},$$

where $x\neq 0$. Obviously, if the function $f$ is differentiable then $\lim_{q\to 1^-}D_q(f(x))=\frac{d}{dx}f(x)$. Clearly, $D_q(e_q(ax))=ae_q(ax)$ and $D_q(E_q(ax))=aE_q(aqx)$ for $|ax|<\frac{1}{1-q}$ and $a,x\in\mathbb{C}$ respectively. By simple induction on $n$, we obtain $D_q^n(e_q(ax))=a^ne_q(ax)$ and $D_q^n(E_q(ax))=a^nq^{\binom{n}{2}}E_q(aq^nx)$ for $n\geq 0$.

In [8] it has been introduced the definition of $q$-completely monotonic function. A positive function $f$ is said to be $q$-*completely monotonic* (respectively, $q$-*log-completely monotonic*), if it an infinitely $q$-differentiable function such that $(-1)^nD_q^nf(z)\geq 0$ for $n\geq 0$ (respectively, $(-1)^nD_q^nLog_qf(z)\geq 0$ for $n\geq 1$ ) and $z\in\mathbb{R}^+$, where we define $Log_q(f(x))=\log_{E_q}(f(x))$. When $q\to 1^-$, we obtain the classical case: a positive function $f$ is said to be *completely monotonic* (respectively, *log-completely monotonic*), if it an infinitely differentiable function such that $(-1)^n(f(z))^{(n)} \geq 0$ for $n \geq 0$ (respectively, $(-1)^n(\log f(z))^{(n)} \geq 0$ for $n \geq 1$ ) for and $z \in \mathbb{R}^+$. For instance, let $\Gamma$ be the Euler gamma function defined on $\mathbb{R}^+$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ and the digamma (or psi) function is given by $\psi = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Then It is well known that $\psi'$ is strictly completely monotonic on $\mathbb{R}^+$, see [1, Page 260]. The $q$-gamma function has the following integral representation, $\Gamma_q(x) = \int_0^\infty x^{t-1}E_q(-qx)d_qx$, and the $q$-analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the $q$-gamma function, that is, $\psi_q(x) = \frac{d}{dx}\log\Gamma_q(x)$ (see [3]). It is well known that $\psi_q'(x)$ is strictly completely monotonic on $\mathbb{R}^+$, see last section for further properties of the $q$-gamma and $q$-psi functions.

In [16], Kim investigated some properties on the weighted $q$-Bernstein polynomials. Moreover, he derived some new identities between the weighted $q$-Bernstein polynomials and the twisted $q$-Bernoulli numbers (also, see [7, 17]). Here, we interest on a general concept, namely, $q$-Bernstein function. A positive function $f$ on $[0, +\infty)$ is said to be *$q$-Bernstein function* if it is infinity $q$-differentiable and $(-1)^{n-1}D_q^n f(z) \geq 0$ for $n \geq 1$. Clearly, a function $f$ non-negative and infinitely $q$-differentiable on $[0, +\infty)$ is $q$-Bernstein function if and only if $D_q(f(z))$ is $q$-completely monotonic function.

In this paper, we show several results on $q$-completely monotonic functions, $q$-log-completely monotonic and $q$-Bernstein functions. Then we define $q$-analog of probability measures (for definitions, see next section) on $[0, +\infty)$ converges vaguely to a measure function $v$ (see [22]). Using this definition, we present a characterization of the $q$-Bernstein functions in terms of $q$-Laplace transform. In last section, we present applications for our results.

## 2. Main results

We start by citing [18, Proposition 2.7]:

(*) If $f$ is a positive increasing (decreasing) function for $x \in [0, +\infty)$, then $f$ is $q$-increasing ($q$-decreasing) function, namely, $D_q(f(x)) > 0$ ($D_q(f(x)) < 0$).

**Theorem 2.1.** *If $f$ is a log-completely monotonic function, then $f$ is $q$-log-completely monotonic function.*

*Proof.* Let $f(q) = \sum_{n\geq 0} \frac{q^{\binom{n}{2}}}{[n]!}$. Obviously, $\lim_{q\to 0^+} f(q) = 2$ and $\lim_{q\to 1^-} f(q) = e$. By the fact that $f(q)$ is an increasing function on $q$ we have $2 \leq E_q \leq e$ for all $q \in (0, 1)$.

We have to prove that if $(-1)^n(Log_q(f(x)))^{(n)} \geq 0$ then

$$(-1)^n D_q^n(Log_q(f(x))) \geq 0,$$

for all $n \geq 1$. We proceed the proof by induction on $n$. For $n = 1$ it holds, because if $-(Log_q f(x))' > 0$ or $(Log_q f(x))' < 0$ then by (*) it follows that $-D_q(Log_q(f(x))) > 0$ or $D_q(Log_q(f(x))) < 0$. We assume the claim holds for all $n = 1, 2, \ldots, k$, that is, if $(-1)^n(Log_q(f(x)))^{(n)} \geq 0$ then $(-1)^n D_q^n(Log_q(f(x))) \geq 0$, for all $n = 1, 2, \ldots, k$. We will prove that it holds for $n = k + 1$, that is, if $(-1)^{k+1}(\log(f(x)))^{(k+1)} \geq 0$ then $(-1)^{k+1}D_q^{k+1}(Log_q(f(x))) \geq 0$. Indeed, if $(Log_q(f(x)))^{(k+1)} > 0$ then $(Log_q(f(x)))^{(k)}$ is an increasing function, so $D_q^k(Log_q(f(x))) > 0$ and then by using

(*), we obtain that $D_q(D_q^k(Log_q(f(x)))) > 0$. Similarly, the case $(Log_q(f(x)))^{(k+1)} < 0$. □

To present our next result, we recall that the $q$-analogue of the partial Bell polynomials are given by (see [4, 21])

$$B_{n,k,q}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{b_1+b_2+\cdots+b_k=n,\, b_i\geq 1} \frac{[n]!\prod_{j=1}^{k} x_{b_j}}{\prod_{j=1}^{k}[b_1+\cdots+b_j]\prod_{j=1}^{k}[b_j-1]!}.$$

The $q$-analogue of Faá di Bruno's formula is given by (see [24])

$$D_q^n g(h(x)) = \sum_{k=1}^{n} D_q^k(g(x)) \circ h(x) B_{n,k,q}(h_{b_1,0}, h_{b_2,b_1}, h_{b_3,b_1+b_2}, \dots),$$

for all $n \geq 1$, where $h_{i,j} = h_{i,j}(x) = D_q^i(h(q^j x))$. Thus, for $n \geq 1$,

(1)

$$D_q^n g(h(x)) = \sum_{k=1}^{n} D_q^k(g(x)) \circ h(x) \sum_{b_1+\cdots+b_k=n,\, b_i\geq 1} \frac{[n]!\prod_{j=1}^{k} D_q^{b_j}(h(q^{b_1+\cdots+b_{j-1}}x))}{\prod_{j=1}^{k}[b_1+\cdots+b_j]\prod_{j=1}^{k}[b_j-1]!}.$$

**Theorem 2.2.** *Let $f(x)$ be any $q$-log-completely monotonic function. Then $f$ is a $q$-completely monotonic function.*

*Proof.* By applying (1) with $g(x) = E_q^x$ and $h(x) = Loq_q(f(x))$, we obtain

$$\begin{aligned} &D_q^n g(h(x)) \\ &= \sum_{k=1}^{n} D_q^k(g(x)) \circ h(x) \sum_{b_1+b_2+\cdots+b_k=n,\, b_i\geq 1} \frac{[n]!\prod_{j=1}^{k} D_q^{b_j}(Loq_q f(q^{b_1+\cdots+b_{j-1}}x))}{\prod_{j=1}^{k}[b_1+\cdots+b_j]\prod_{j=1}^{k}[b_j-1]!}. \end{aligned}$$

which implies

$$\begin{aligned} &(-1)^n D_q^n g(h(x)) \\ &= \sum_{k=1}^{n} D_q^k(E_q^x) \circ Loq_q(f(x)) \sum_{b_1+\cdots+b_k=n,\, b_i\geq 1} \frac{[n]!\prod_{j=1}^{k} D_q^{b_j}(Loq_q f(q^{b_1+\cdots+b_{j-1}}x))}{\prod_{j=1}^{k}[b_1+\cdots+b_j]\prod_{j=1}^{k}[b_j-1]!}. \end{aligned}$$

Since $f$ is a $q$-log-completely monotonic function, we have

$$(-1)^{b_j} D_q^{b_j}(Loq_q f(q^{b_1+\cdots+b_{j-1}}x)) \geq 0,$$

for any $b_j \geq 0$, and $\frac{d^k(E_q^x)}{dx^k} = (\log E_q)^k \cdot E_q^x > 0$, by Theorem 2.8 see [18] we have $\frac{d_q^k(E_q^x)}{d_q x^k} > 0$, then $D_q^k(E_q^x) \circ Loq_q(f(x)) > 0$. Hence, $(-1)^n D_q^n f(x) = (-1)^n D_q^n g(h(x)) \geq 0$, for all $n \geq 0$, which completes the proof. □

**Theorem 2.3.** *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$. Then, $f$ is a $q$-Bernstein function if and only if $E_q^{-tf}$, $t > 0$ is a $q$-completely monotonic function.*

*Proof.* Let $f$ be any $q$-Bernstein function on $\mathbb{R}^+$, and we define $g(x) = E_q^{-xt}$, $t > 0$, to be a $q$-completely monotonic function. By [18, Proposition 2.12 ] we have that $g \circ f = E_q^{-tf}$ is a $q$-completely monotonic function. Conversely, suppose that $E_q^{-tf}$ is a $q$-completely monotonic, then $1 - E_q^{-tf}$ is a $q$-Bernstein function. Now, we find that $f = \lim_{t\to 0^+} \frac{1-E_q^{-tf}}{t\cdot\log E_q}$ is a $q$-Bernstein function. □

**Theorem 2.4.** *Let $g$ be any $q$-completely monotonic function. Then, $g \circ f$ is a $q$-completely monotonic if and only if $E_q^{-f}$ is a $q$-log-completely monotonic.*

*Proof.* Let $g$ be any $q$-completely monotonic function and suppose that $g \circ f$ is a $q$-completely monotonic function. Now, we define $g(x) = E_q^{-x}$, since the function $g(x) = E_q^{-x}$ is a $q$-completely monotonic, then $g \circ f = E_q^{-f}$ is a $q$-completely monotonic. Conversely, suppose that $E_q^{-f}$ is a $q$-log-completely monotonic function, which implies that $Log_q(E_q^{-f}) = -f$, in other words $(-1)^n D_q^n(Log_q(E_q^{-f})) = (-1)^{n-1} D_q^n f > 0$, then we have $f$ is a $q$-Bernstein function, since $g \circ f \geq 0$, by using $q$-analogue of Faá di Bruno's formula (or [18, Proposition 2.12 ]), we have that $g \circ f$ is a $q$-completely monotonic function . $\square$

**Theorem 2.5.** *Let $f, g$ be any $q$-Bernstein functions. Then $g \circ f$ is a $q$-Bernstein function.*

*Proof.* Suppose that $f, g$ are $q$-Bernstein functions. For any $h$, $q$-completely monotonic function, we use [18, Proposition 2.12] to get that $h \circ f$ is a $q$-completely monotonic function, and then $h \circ (g \circ f) = (h \circ g) \circ f$ is a $q$-completely monotonic function. Now, since $h \circ (g \circ f)$ is a $q$-completely monotonic function, by [18, Proposition 2.12 ] we have that $g \circ f$ is a $q$-Bernstein function. $\square$

**Theorem 2.6.** *(i) Let $f(x) \geq 0$ be any $q$-completely monotonic function on $\mathbb{R}^+$ and let $a > 0$. Then $f(x) - f(x+a)$ is a $q$-completely monotonic function on $\mathbb{R}^+$. (ii) Let $f(x) \geq 0$ and let $f(x) - f(x+a)$ be a $q$-completely monotonic function on $\mathbb{R}^+$ for each $a$ in some right-hand neighborhood of $0$. Then $f(x)$ is a $q$-completely monotonic function on $\mathbb{R}^+$.*

*Proof.* (i) Let $f(x)$ be a $q$-completely monotonic on $\mathbb{R}^+$ and let $a > 0$, by [18, Theorem 1.2] we have $f(x) = \int_0^\infty E_q(-xt) d_q(\mu(t))$, where $\mu(t)$ is a positive measure on $\mathbb{R}^+$. Hence,

$$\begin{aligned} &(-1)^n D_q^n(f(x) - f(x+a)) \\ &= q^{\binom{n}{2}} \int_0^\infty \Big( E_q(-q^n xt) - E_q(-q^n(x+a)t) \Big) t^n d_q(\mu(t)) \geq 0. \end{aligned}$$

(ii) Let $f(x) \geq 0$ and let $f(x) - f(x+a)$ be a $q$-completely monotonic on $\mathbb{R}^+$ for each a in some right-hand neighborhood of 0. Since $D_q(h(0)) = h'(0)$, we have that $-(D_q f(x)) = \lim_{a \to 0^+} \frac{f(x) - f(x+a)}{a}$ is a $q$-completely monotonic on $\mathbb{R}^+$. Then, we have $f(x)$ is a $q$-completely monotonic on $\mathbb{R}^+$. $\square$

For our next step, we define $\mathbb{R}_{q,+} = \{q^n \mid n \in \mathbb{Z}\}$ and

$$L_{q,\lambda}(\mathbb{R}_{q,+}, \mu(t)) = \left\{ f \mid \int_0^\infty E_q(-\lambda f(t)) d_q(\mu(t)) < \infty \right\},$$

where $\mu(t)$ is any positive measure. We consider the $q$-Wiener algebra

$$\mathcal{A}_{q,\lambda} = \{ f \in L_{q,\lambda}(\mathbb{R}_q, +) \mid \mathcal{L}_{q,\lambda}(f) \in L_{q,\lambda}(\mathbb{R}_q, +) \}.$$

Let $\mathcal{M}_q^+$ be the set of positives and bounded measures on $\mathbb{R}_{q,+}$. The $q$-convolution semigroup (see [8]) of probability measures is a family $(\pi_t)_{t>0}$ of probability measures in $\mathcal{M}_q^+$ such that

$$\mathcal{L}_{q,\lambda}(\pi_t) \in \mathcal{A}_{q,\lambda}, \quad \pi_t *_q \pi_s = \pi_{t+s}, \quad \text{for } x, t > 0. \tag{2}$$

We define the $q$-convolution product of two measures $\nu, \mu \in \mathcal{M}_q^+$ is given by

$$\int_0^\infty f(t)(\mu *_q \nu) d_q t = \int_0^\infty \int_0^\infty f(t+s)\mu(d_q t)\nu(d_q s).$$

**Theorem 2.7.** *Let $(\mu_t)_{t>0}$ be a $q$-convolution semigroup of probability measures on $\mathbb{R}_{q,+}$. Then there exists a $q$-Bernstein function $f$ such that the $q$-Laplace transform of $\mu_t$ is given by $\mathcal{L}_q \mu_t = E_q^{-tf}$, for all $t \geq 0$. Conversely, if $f$ is a $q$-Bernstein function, there exists a $q$-convolution semigroup of probability measures $(\mu_t)_{t>0}$ on $\mathbb{R}_{q,+}$ such that $\mathcal{L}_q \mu_t = E_q^{-tf}$, for all $t \geq 0$.*

*Proof.* Suppose that $(\mu_t)_{t>0}$ be a $q$-convolution semigroup of probability measures on $\mathbb{R}_{q,+}$. Set $t \geq 0$. Since $\mathcal{L}_q \mu_t > 0$, we define a function $f_t : [0, +\infty) \to \mathbb{R}$ by $f_t(\lambda) = -Loq_q \mathcal{L}_{q,\lambda}(\mu_t)$. By (2) we have $f_{t+s}(\lambda) = f_t(\lambda) + f_s(\lambda)$, for all $t, s \geq 0$, that is, $t \to f_t(\lambda)$ satisfies the Cauchy's functional equation. By the continuous of $f_t$, we obtain that there is a unique solution $f_t(\lambda) = t \cdot f(\lambda)$, where $f(\lambda) = f_1(\lambda)$. Therefore, $Log_q(\mathcal{L}_{q,\lambda}(\mu_t)) = -f_t(\lambda)$, which implies that $\mathcal{L}_{q,\lambda}(\mu_t) = E_q^{-t \cdot f(\lambda)}$, in particular, $E_q^{-tf}$ is a $q$-completely monotonic function for all $t > 0$. By Theorem 2.3, $f$ is $q$-Bernstein function.

Conversely, suppose that $f$ is a $q$-Bernstein function, by Theorem 2.3, we have $E_q^{-tf}$ is a $q$-completely monotonic function. Therefore, for every $t \geq 0$ there exists a measure $\mu_t$ on $\mathbb{R}_{q,+}$ such that $\mathcal{L}_q \mu_t = E_q^{-tf}$. (Note that by definition this family is a $q$-convolution semigroup of probability measures $(\mu_t)_{t>0}$ on $\mathbb{R}_{q,+}$.) □

In order to state our next results, we need the following definition. A $q$-completely monotonic function $f$ is said to be *$q$-infinitely divisible* if for every $t > 0$ the function $f^t$ is again a $q$-completely monotonic function.

**Theorem 2.8.** *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be any function. Then, $g$ is a $q$-infinitely divisible if and only if $g = E_q^{-f}$ where $f$ is a $q$-Bernstein function.*

*Proof.* Suppose that $g$ is a $q$-infinitely divisible. Since $g^t$ is a $q$-completely monotonic function, we obtain, by [18, Theorem 1.4], that there exists a measure $\mu_t$ on $[0, +\infty)$ such that $g^t(\lambda) = \mathcal{L}_q(\mu_t, \lambda)$. By Theorem 2.7, there exists $q$-Bernstein function $f$ such that $\mathcal{L}_q(\mu_t, \lambda) = E_q^{-tf(\lambda)}$. Hence, $g^t = E_q^{-tf}$, which implies that $g = E_q^{-f}$.

Now, suppose that $g = E_q^{-f}$ where $f$ is a $q$-Bernstein function. Then, by Theorem 2.3, we have, $g^t = E_q^{-tf}$ is a $q$-completely monotonic function, which completes the proof. □

**Theorem 2.9.** *Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be any function. Then, $g$ is a $q$-infinitely divisible if and only if $g$ is a $q$-log completely monotonic.*

*Proof.* Suppose that $g$ is a $q$-infinitely divisible. So $g^t$ is a $q$-completely monotonic. By Theorem 2.8, there exits a $q$-Bernstein function $f$ such that $g = E_q^{-f}$, which implies that $Log_q(g) = -f$, in other words

$$(-1)^n D_q^n (Log_q g) = (-1)^{n-1} D_q^n f > 0.$$

Conversely, if $g$ is a $q$-log completely monotonic function, then the function $g$ is a $q$-completely monotonic. By Theorem 2.8, there exits a $q$-Bernstein function $f$ such that

$$g^t = E_q^{-t\cdot f}$$

is a $q$-completely monotonic function, for all $t > 0$, as required. □

## 3. Applications

In this section we present applications for our results. In order to do that, we recall $q$-analogue of several known functions. Jackson (for example, see [3, 9, 13, 14, 17]) defined the $q$-analogue of the gamma function as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\ 0 < q < 1, \tag{3}$$

and

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}},\ q > 1, \tag{4}$$

where $(a;q)_\infty = \prod_{j\geq 0}(1-aq^j)$.

The $q$-analogue of the psi function is defined for $0 < q < 1$ as the logarithmic derivative of the $q$-gamma function, namely, $\psi_q(x) = \frac{d}{dx}\log\Gamma_q(x)$. Askey [3] considered several properties of the $q$-gamma function. It is well known that $\Gamma_q(x) \to \Gamma(x)$ and $\psi_q(x) \to \psi(x)$ as $q \to 1^-$. For $0 < q < 1$ and $x > 0$, by (3) we obtain that

$$\psi_q(x) = -\log(1-q) + \log(q)\sum_{n\geq 0}\frac{q^{n+x}}{1-q^{n+x}} = -\log(1-q) + \log(q)\sum_{n\geq 1}\frac{q^{nx}}{1-q^n} \tag{5}$$

and for $q > 1$ and $x > 0$, by (4) we have that

$$\begin{aligned}\psi_q(x) &= -\log(q-1) + \log(q)\left(x - \tfrac{1}{2} - \sum_{n\geq 0}\tfrac{q^{-n-x}}{1-q^{-n-x}}\right)\\ &= -\log(q-1) + \log(q)\left(x - \tfrac{1}{2} - \sum_{n\geq 1}\tfrac{q^{-nx}}{1-q^{-n}}\right).\end{aligned} \tag{6}$$

We set $\psi_1 = \psi$. A Stieltjes integral representation for $\psi_q(x)$ with $0 < q < 1$ is given in [12]. It is well-known that $\psi'$ is strictly completely monotonic function on $(0,\infty)$ (see [1, Page 260]). From (5) and (6) we conclude that $\psi'_q$ has the same property for any $q > 0$, namely, $(-1)^n(\psi'_q(x))^{(n)} > 0$, for $x > 0$ and $n \geq 0$. If $q \in (0,1)$, then by (5), we have that

$$\psi_q^{(k)}(x) = \log^{k+1} q\sum_{n\geq 1}\frac{n^k\cdot q^{nx}}{1-q^n}. \tag{7}$$

If $q > 1$, then by (6) we obtain that

$$\psi'_q = \log q\Big(1 + n\log q\sum_{n\geq 1}\frac{q^{-nx}}{1-q^{-nx}}\Big), \tag{8}$$

and for $k \geq 2$

$$\psi_q^{(k)} = (-1)^k n^k \log^{k+1} q\sum_{n\geq 1}\frac{q^{-nx}}{1-q^{-nx}}. \tag{9}$$

The *polylogarithm* is a special function that is defined by the infinite sum, or power series:

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots .$$

Let $\alpha \in R$ and $\beta \geq 0$ be real numbers, we define

$$f_{\alpha,\beta,q}(x) = (1-q)^x \frac{e^{h(x)}\Gamma_q(x+\beta)}{[x]^{x+\beta-\alpha}}, \tag{10}$$

where $h(x) = -\frac{Li_2(q^x)+x\log(q)\log(1-q^x)}{\log(q)}$.

Now we ready to present applications of the previous results.

**Theorem 3.1.** *Let $2\alpha \leq 1 \leq \beta$ and $0 < q < 1$. Then the function $f_{\alpha,\beta,q}(x)$ is a $q$-log-completely monotonic function on $(0,\infty)$.*

*Proof.* It is clear that

$$\ln f_{\alpha,\beta,q}(x) = x\log(1-q) + h(x) + \ln\Gamma_q(x+\beta) - (x+\beta-\alpha)\ln[x],$$

which implies

$$\begin{aligned}&[\ln f_{\alpha,\beta,q}(x)]' \\ &= \log(1-q) + h'(x) + \psi_q(x+\beta) - \ln[x] - \frac{q^x(\beta-\alpha)\log(q)}{1-q^x} - \frac{xq^x\log(q)}{1-q^x}.\end{aligned}$$

Since $h'(x) = \frac{xq^x\log(q)}{1-q^x}$, we have that

$$[\ln f_{\alpha,\beta,q}(x)]' = \log(1-q) + \psi_q(x+\beta) - \ln[x] - \frac{q^x(\beta-\alpha)\log(q)}{1-q^x}.$$

On the other hand, Lemma 2.3 (see [19]) gives

$$\Big(\frac{-q^x\log q}{1-q^x}\Big)^{(n)} = (-1)^n \int_0^\infty t^n e^{-xt} d\gamma_q(t), \qquad n \geq 0, \tag{11}$$

Hence,

$$\begin{aligned}&(-1)^n[\ln f_{\alpha,\beta,p}(x)]^{(n)} \\ &= (-1)^n\Big[\psi_q^{(n-1)}(x+\beta) - \Big(\frac{-q^x\log q}{1-q^x}\Big)^{(n-1)} + (\beta-\alpha)\Big(\frac{-q^x\log q}{1-q^x}\Big)^{(n)}\Big] \\ &= \int_0^\infty \frac{t^{n-1}e^{-(x+\beta)t}}{1-e^{-t}}d\gamma_q(t) - \int_0^\infty t^{n-2}e^{-xt}d\gamma_q(t) + (\beta-\alpha)\int_0^\infty t^{n-1}e^{-xt}d\gamma_q(t) \\ &= \int_0^\infty g_{\alpha,\beta}(t)\frac{t^{n-2}e^{-xt}}{1-e^{-t}}d\gamma_q(t), \qquad (12)\end{aligned}$$

where $g_{\alpha,\beta}(t) = t + [(\beta-\alpha)t-1][e^{\beta t} - e^{(\beta-1)t}]$. Note that $g_{\alpha,\beta}(t) > 0$ (see [5]). So, by (12), we see that when $n \geq 2$

$$(-1)^n[\ln f_{\alpha,\beta,p}(x)]^{(n)} > 0$$

on $(0,\infty)$ for $2\alpha \leq 1 \leq \beta$. Thus, by Theorem 2.1 we have

$$(-1)^n D_q^n(Log_q f_{\alpha,\beta,p}(x)) > 0$$

on $(0,\infty)$ for $2\alpha \le 1 \le \beta$ and $n \ge 2$.

For $n=1$, since $[\ln f_{\alpha,\beta,q}(x)]'$ is an increasing function, we have that

$$\begin{aligned}
&[\ln f_{\alpha,\beta,q}(x)]' \\
&< \lim_{x\to\infty}\left[\log(1-q)+\psi_q(x+\beta)-\ln[x]-\frac{q^x(\beta-\alpha)\log(q)}{1-q^x}\right]=\log(1-q)<0,
\end{aligned}$$

which implies that $D_q(Log_q f_{\alpha,\beta,p}(x))<0$.

Hence, for $2\alpha \le 1 \le \beta$ and $n \in \mathbb{N}$, $(-1)^n D_q^n(Log_q(f_{\alpha,\beta,p}(x)))>0$ in $(0,\infty)$, as required. □

**Theorem 3.2.** *Let $a_i, b_i \in \mathbb{R}$ such that $0<a_1 \leqq \cdots \leqq a_n$, $0<b_1 \leqq b_2 \leqq \cdots \leqq b_n$ and $\sum_{i=1}^k a_i \leqq \sum_{i=1}^k b_i$ for all $k=1,2,\ldots,n$. Then the function $G_{p,q}$*

$$G_{p,q}(x)=G_q(x;a_1,b_1,\cdots,a_n,b_n)=\prod_{i=1}^n \frac{\Gamma_q(x+a_i)}{\Gamma_q(x+b_i)} \qquad (0<q<1) \tag{13}$$

*is a $q$-completely monotonic on $(0,\infty)$.*

*Proof.* First, we define

$$h(x)=\sum_{i=1}^n\left[\log\Gamma_q(x+b_i)-\log\Gamma_q(x+a_i)\right].$$

Then, for $k\in\mathbb{N}_0$, we have

$$\begin{aligned}
(-1)^k\big(h'(x)\big)^{(k)} &= (-1)^k\sum_{i=1}^n\left[\psi_q^{(k)}(x+b_i)-\psi_q^{(k)}(x+a_i)\right]\\
&=(-1)^k\sum_{i=1}^n(-1)^{k+1}\int_0^\infty \frac{t^k\,\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}}\cdot\left(\mathrm{e}^{-bi}-\mathrm{e}^{-ai}\right)d\gamma_q(t)\\
&=(-1)^{2k+1}\int_0^\infty \frac{t^k\,\mathrm{e}^{-xt}}{1-\mathrm{e}^{-t}}\cdot\sum_{i=1}^n\left(\mathrm{e}^{-bi}-\mathrm{e}^{-ai}\right)d\gamma_q(t).
\end{aligned}$$

Alzer [2] showed that, if $f$ is a decreasing and convex function on $\mathbb{R}$, then

$$\sum_{i=1}^n f(b_i)\leqq\sum_{i=1}^n f(a_i). \tag{14}$$

Thus, since the function $z\mapsto \mathrm{e}^{-z}$ $(z>0)$ is a decreasing and convex on $\mathbb{R}$, we have

$$\sum_{i=1}^n\left(\mathrm{e}^{-ai}-\mathrm{e}^{-bi}\right)\geqq 0,$$

so that

$$(-1)^k\left[G_q'(x)\right]^{(k)}\geqq 0 \qquad (k\in\mathbb{N}_0).$$

Hence $h'$ is a completely monotonic function on $(0,\infty)$, Using the fact that if $h'$ is a completely monotonic function on $(0,\infty)$, then $\exp(-h)$ is also a completely monotonic function on $(0,\infty)$ (see [2]), by [18, Theorem 2.8 ] we have $\exp(-h)$ is also a $q$-completely monotonic function on $(0,\infty)$, which completes the proof. □

**Remark 3.3.** *This is a corrected version of paper (see [20]).*

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