# Some New Results On Even Almost Perfect Numbers Which Are Not Powers Of Two

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**Abstract:** In this note, we present some new results on even almost perfect numbers which are not powers of two. In particular, we show that $2^{r+1} < b$, if $2^r b^2$ is an even almost perfect number.
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## 1 Introduction

Let $\sigma(x)$ denote the sum of divisors of $x$. If $\sigma(y) = 2y - 1$, we say that $y$ is *almost perfect*.

In [3], Dris gives the following criterion for almost perfect numbers in terms of the abundancy index $I(x) = \sigma(x)/x$:

**Theorem 1.1.** *Let $m$ be a positive integer. Then $m$ is almost perfect if and only if*

$$\frac{2m}{m+1} \leq I(m) < \frac{2m+1}{m+1}.$$

Dris also obtains the following result [3]:

**Theorem 1.2.** *Let $M$ be a positive integer. Then $M$ is deficient if and only if*

$$\frac{2M}{M+D(M)} \leq I(M) < \frac{2M+D(M)}{M+D(M)},$$

*where $D(M) = 2M - \sigma(M)$ is the deficiency of $M$.*

It is currently an open problem to determine if the only even almost perfect numbers are those of the form $2^k$, where $k \geq 1$. (Note that $1$ is the single currently known odd almost perfect number, as $\sigma(1) = 2 \cdot 1 - 1 = 1$.)

Antalan and Tagle showed in [2] that, if $M \neq 2^k$ is an even almost perfect number, then $M$ takes the form $M = 2^r b^2$, where $b$ is an odd composite integer. Antalan also proved in [1] that $3 \nmid M$.

## 2 Main Results

Our penultimate goal is, of course, to show that if $n$ is an even almost perfect number, then $n = 2^k$ for some positive integer $k$.

Assume to the contrary that there exists an even almost perfect number $M \neq 2^k$. By [2], $M$ then takes the form $M = 2^r b^2$, where $r \geq 1$ and $b$ is an odd composite integer. Note that $b^2$ is deficient, as it is a factor of the deficient number $M = 2^r b^2$.

(The following proof for the assertion that $b^2$ is not almost perfect, is from [6].)

Since $M$ is almost perfect, we have

$$(2^{r+1} - 1)\sigma(b^2) = \sigma(2^r)\sigma(b^2) = \sigma(2^r b^2) = \sigma(M) = 2M - 1 = 2^{r+1}b^2 - 1.$$

So we have

$$\sigma(b^2) = \frac{2^{r+1}b^2 - 1}{2^{r+1} - 1} = b^2 + \frac{b^2 - 1}{2^{r+1} - 1}.$$

Now,

$$2b^2 - \sigma(b^2) = b^2 - \frac{b^2 - 1}{2^{r+1} - 1}.$$

If $b^2$ is also almost perfect, then we have

$$1 = 2b^2 - \sigma(b^2) = b^2 - \frac{b^2 - 1}{2^{r+1} - 1},$$

which, since $b > 1$, gives

$$2^{r+1} - 1 = 1 \Longleftrightarrow r = 0.$$

This contradicts $r \geq 1$. Consequently, since $b^2$ is deficient, we can write $\sigma(b^2) = 2b^2 - c$, where $c > 1$.

Note that we have proved the following propositions:

**Lemma 2.1.** *Let $M = 2^r b^2$ be an even almost perfect number, with $\sigma(b^2) = 2b^2 - c$. Then*

$$c = b^2 - \frac{b^2 - 1}{2^{r+1} - 1}.$$

**Lemma 2.2.** *Let $M = 2^r b^2$ be an even almost perfect number, with $\sigma(b^2) = 2b^2 - c$. Then*

$$c \geq \frac{2b^2 + 1}{3}.$$

Notice that, since $b$ is an odd composite, and since $3 \nmid M$ (see [1]), then $b \geq 5 \cdot 7 = 35$, so that we have the estimate $c \geq \frac{2 \cdot 35^2 + 1}{3} = 817.\overline{333}$, which implies that $c \geq 819$ since $c$ is an odd integer.

Recall that the abundancy index of $x$ is defined to be the ratio $I(x) = \frac{\sigma(x)}{x}$. We call a number $S$ solitary if the equation $I(S) = I(d)$ has exactly one solution $d = S$. A sufficient (but not necessary) condition for $T$ to be solitary is $\gcd(T, \sigma(T)) = 1$, where $\gcd$ is the greatest common divisor function.

The following result was communicated to the second author by Dagal last October 4, 2015.

**Lemma 2.3.** *If $2^r b^2$ is an almost perfect number with $\gcd(2, b) = 1$ and $b > 1$, then $b^2$ is solitary.*

(Note: The proof that follows is different from that of Dagal's [4].)

*Proof.* Since $2^r b^2$ is almost perfect, we have

$$(2^{r+1} - 1)\sigma(b^2) = \sigma(2^r)\sigma(b^2) = \sigma(2^r b^2) = 2^{r+1}b^2 - 1.$$

We want to show that

$$\gcd(b^2, \sigma(b^2)) = 1.$$

It suffices to find a linear combination of $b^2$ and $\sigma(b^2)$ that is equal to 1. Such a linear combination is given by the equation

$$1 = (1 - 2^{r+1})\sigma(b^2) + 2^{r+1}b^2.$$

□

From the equation

$$1 = (1 - 2^{r+1})\sigma(b^2) + 2^{r+1}b^2$$

we obtain

$$2^{r+1}\left(\sigma(b^2) - b^2\right) = \sigma(b^2) - 1$$

so that

$$2^{r+1} = \frac{\sigma(b^2) - 1}{\sigma(b^2) - b^2} = 1 + \frac{b^2 - 1}{\sigma(b^2) - b^2}.$$

This last equation gives the divisibility constraint in the following result:

**Lemma 2.4.** *If $2^r b^2$ is an almost perfect number with $\gcd(2, b) = 1$ and $b > 1$, then*

$$\left(\sigma(b^2) - b^2\right) \mid \left(b^2 - 1\right).$$

Numbers $n$ such that $\sigma(n) - n$ divides $n - 1$ are listed in OEIS sequence A059046 [7], the first $62$ terms of which are given below:

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59, 61, 64, 67, 71, 73, 77, 79,$$

$$81, 83, 89, 97, 101, 103, 107, 109, 113, 121, 125, 127, 128, 131, 137, 139, 149, 151, 157, 163, 167, 169, 173,$$

$$179, 181, 191, 193, 197, 199, 211.$$

**Remark 2.1.** *Does OEIS sequence A059046 contain any odd squares $u^2$, with $\omega(u) \geq 2$? MSE user Charles (*`http://math.stackexchange.com/users/1778`*) checked and found that "'there are no such squares with $u^2 < 10^{22}$."' [5]*

Suppose that $M = 2^r b^2$ is an almost perfect number with $\gcd(2, b) = 1$ and $b > 1$. Let us call $b^2$ the *odd part* of $M$.

The following result shows that distinct even almost perfect numbers (other than the powers of $2$) cannot share the same odd part.

**Lemma 2.5.** *Suppose that there exist at least two distinct even almost perfect numbers*

$$M_1 = 2^{r_1}{b_1}^2$$

*and*

$$M_2 = 2^{r_2}{b_2}^2,$$

*with $\gcd(2, b_1) = \gcd(2, b_2) = 1$, $b_1 > 1$, $b_2 > 1$, and $r_1 \neq r_2$. Then $b_1 \neq b_2$.*

*Proof.* Assume to the contrary that $1 < b_1 = b_2 = b$. This implies that $1 < {b_1}^2 = {b_2}^2 = b^2$, so that

$$\frac{2^{r_1+1}b^2 - 1}{2^{r_2+1}b^2 - 1} = \frac{2M_1 - 1}{2M_2 - 1} = \frac{\sigma(M_1)}{\sigma(M_2)} = \frac{(2^{r_1+1} - 1)\sigma(b^2)}{(2^{r_2+1} - 1)\sigma(b^2)} = \frac{2^{r_1+1} - 1}{2^{r_2+1} - 1}.$$

Solving for $b^2$ gives

$$(2^{r_2+1} - 1)(2^{r_1+1}b^2 - 1) = (2^{r_1+1} - 1)(2^{r_2+1}b^2 - 1)$$

$$2^{r_1+r_2+2}b^2 - 2^{r_1+1}b^2 - 2^{r_2+1} + 1 = 2^{r_1+r_2+2}b^2 - 2^{r_2+1}b^2 - 2^{r_1+1} + 1$$

$$(2^{r_1+1} - 2^{r_2+1})b^2 = 2^{r_1+1}b^2 - 2^{r_2+1}b^2 = 2^{r_1+1} - 2^{r_2+1}.$$

By assumption, we have $r_1 \neq r_2$, so that $2^{r_1+1} - 2^{r_2+1} \neq 0$. Finally, we get

$$b^2 = 1,$$

which is a contradiction. □

Since $b^2$ is composite, $\sigma(b^2) > b^2 + b + 1$. In particular, we obtain

$$b^2 - b - 1 > 2b^2 - \sigma(b^2).$$

From the equation

$$2^{r+1} = 1 + \frac{b^2 - 1}{\sigma(b^2) - b^2}$$

and the inequality

$$b^2 + b + 1 < \sigma(b^2),$$

we obtain the following result:

**Theorem 2.1.** *If $2^r b^2$ is an almost perfect number with $\gcd(2, b) = 1$ and $b > 1$, then*

$$r < \log_2 b - 1.$$

This last inequality implies that

$$2^r < 2^{r+1} < b < \sigma(b)$$

and

$$\sigma(2^r) = 2^{r+1} - 1 < b - 1 < b$$

so that we have

$$\frac{\sigma(2^r)}{b} < 1 < 2 < \frac{\sigma(b)}{2^r}.$$

Additionally, since $b^2$ is deficient, we can write $\sigma(b^2) = 2b^2 - c$, where we compute $c$ to be

$$c = b^2 - \frac{b^2 - 1}{\sigma(2^r)}$$

from which we obtain the upper bound

$$\frac{\sigma(b)}{b} < \frac{\sigma(b^2)}{b^2} < \frac{4}{3}.$$

(Note that $I(b^2) < 4/3$ implies $3 \nmid b$. For suppose to the contrary that $I(b^2) < 4/3$ and $3 \mid b$. Then $3^2 \mid b^2$, so that $13/9 = I(3^2) \leq I(b^2) < 4/3$, which is a contradiction. This approach provides an alternative to Antalan's proof [1].)

Lastly, since $r \geq 1$ and $2 \mid 2^r$, then

$$\frac{3}{2} = \frac{\sigma(2)}{2} \leq \frac{\sigma(2^r)}{2^r},$$

so that we have the following series of inequalities:

**Theorem 2.2.** *If $2^r b^2$ is an almost perfect number with $\gcd(2, b) = 1$ and $b > 1$, then*

$$\frac{\sigma(2^r)}{b} < 1 < \frac{\sigma(b)}{b} < \frac{4}{3} < \frac{3}{2} \leq \frac{\sigma(2^r)}{2^r} < 2 < \frac{\sigma(b)}{2^r}.$$

We can obtain a tighter lower bound for $\sigma(b^2)/b^2$ via the following method (using the result from Dris [3] cited earlier):

**Theorem 2.3.** *If $2^r b^2$ is an almost perfect number with $\gcd(2,b)=1$ and $b>1$, then*

$$\frac{2b-1}{2b-2} < I(b^2).$$

*In particular,*

$$\sqrt{\frac{2b-1}{2b-2}} < I(b).$$

*Proof.* We start with

$$\frac{2b^2+1}{3} \le D(b^2) < b^2-b-1.$$

Since $D(b^2) \ge 819$, we can use the following bounds from [3]:

$$\frac{2b^2}{b^2+D(b^2)} < I(b^2) < \frac{2b^2+D(b^2)}{b^2+D(b^2)}.$$

This simplifies to

$$\frac{2b^2}{2b^2-b-1} < I(b^2) < \frac{9b^2-3b-3}{5b^2+1},$$

from which it follows that

$$\frac{2b^2}{2b^2-b-1} = \frac{2b^2-b-1}{2b^2-b-1} + \frac{b+1}{2b^2-b-1} = 1 + \frac{b+1}{2b^2-b-1} = 1 + \frac{b+1}{(2b+1)(b-1)},$$

of which the last quantity is bounded below by

$$1 + \frac{b+1}{(2b+1)(b-1)} > 1 + \frac{b+1}{2(b+1)(b-1)} = \frac{2b-1}{2b-2}.$$

The last assertion in the theorem follows from

$$\left(I(b)\right)^2 > I(b^2).$$

□

Proceeding similarly as before, we can prove the following result.

**Theorem 2.4.** *If $2^r b^2$ is an almost perfect number with $\gcd(2,b)=1$ and $b>1$, then*

- $r=1 \Longrightarrow 8/7 < I(b^2) < 4/3 \Longrightarrow 3 \nmid b$; *and*
- $r>1 \Longrightarrow I(b^2) < 8/7 \Longrightarrow 7 \nmid b$.

*Proof.* The details of the proof (as well as other relevant hyperlinks) are in the following MathOverflow post: `http://mathoverflow.net/q/238824`. □

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