SUMS OF POWERS OF CATALAN TRIANGLE NUMBERS

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ABSTRACT. In this paper we consider combinatorial numbers $C_{m,k}$ for $m \geq 1$ and $k \geq 0$ which unifies the entries of the Catalan triangles $B_{n,k}$ and $A_{n,k}$ for appropriate values of parameters m and k, i.e., $B_{n,k} = C_{2n,n-k}$ and $A_{n,k} = C_{2n+1,n+1-k}$. In fact, some of these numbers are the well-known Catalan numbers C_n that is $C_{2n,n-1} = C_{2n+1,n} = C_n$.

We present new identities for recurrence relations, linear sums and alternating sum of $C_{m,k}$. After that, we check sums (and alternating sums) of squares and cubes of $C_{m,k}$ and, consequently, for $B_{n,k}$ and $A_{n,k}$. In particular, one of these equalities solves an open problem posed in [8]. We also present some linear identities involving harmonic numbers H_n and Catalan triangles numbers $C_{m,k}$. Finally, in the last section new open problems and identities involving C_n are conjectured.

1. Introduction

The well-known Catalan numbers $(C_n)_{n\geq 0}$ given by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \geq 0,$$

appear in a wide range of problems. For instance, the Catalan number C_n counts the number of ways to triangulate a regular polygon with $n+2$ sides; or, the number of ways that $2n$ people seat around a circular table are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other, see for example [17, 21].

The Catalan numbers may be defined recursively by $C_0 = 1$ and $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$ for $n \geq 1$ and first terms in this sequence are

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132, \ldots$$

Catalan numbers have been studied in depth in many papers and monographs (see for example [3]-[11], [15]-[21]) and the Catalan sequence is probably the most frequently encountered sequence. In [17] the generalized k-th Catalan numbers ${}_kC_n = \frac{1}{n}\binom{nk}{n-1}$, $k \geq 1$, are considered to count the number of ways of subdividing a convex polygon into k disjoint $(n+1)$-polygons by means of non-intersecting diagonals, $k \geq 1$, see also for example [2, 9].

In this paper, we consider combinatorial numbers $(C_{m,k})_{m\geq 1, k\geq 0}$ given by

$$C_{m,k} := \frac{m-2k}{m}\binom{m}{k}. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 05A19; 05A10; 11B65.

Key words and phrases. Catalan numbers; Combinatorial identities; Binomial coefficients, Catalan triangle, Harmonic numbers.

P. J. Miana has been partially supported by Project MTM2013-42105-P, DGI-FEDER, of the MCYTS and Project E-64-FEDER, D.G. Aragón. N. Romero has been has been partially supported by the Spanish Ministry of Economy and Competitiveness, Project MTM2014-52016-C2-1-P.

We collect the first values in the following table

(1.2)

$m \setminus k$	0	1	2	3	4	5	6	7	8	9	10	...
1	1	-1										
2	1	0	-1									
3	1	1	-1	-1								
4	1	2	0	-2	-1							
5	1	3	2	-2	-3	-1						
6	1	4	5	0	-5	-4	-1					
7	1	5	9	5	-5	-9	-5	-1				
8	1	6	14	14	0	-14	-14	-6	-1			
9	1	7	20	28	14	-14	-28	-20	-7	-1		
10	1	8	27	48	42	0	-42	-48	-27	-8	-1	
...	...	...	...	...	...	...	...	...	...	...	...	...

These combinatorial numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ are closely related to Catalan numbers C_n and generalized (or higher) Catalan numbers ${}_kC_n$. In fact, it follows that

$$\begin{aligned} C_{2n,n-1} &= C_n = C_{2n+1,n}, \\ C_{kn+1,n} &= \frac{(k-2)n+1}{kn+1}\binom{kn+1}{n} = ((k-2)n+1)\,{}_kC_n. \end{aligned}$$

These numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ appear in several Catalan triangles. For instance, $C_{2n,n-k} = B_{n,k}$, where

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k}, \quad 0 \leq k \leq n,$$

(see [8, 15]) and also $C_{2n+1,n+1-k} = A_{n,k}$, where

$$A_{n,k} = \frac{2k-1}{2n+1}\binom{2n+1}{n+1-k}, \quad 1 \leq k \leq n+1,$$

(see [11]).

This paper is organized as follows. In the second section, we present a new recurrence relation that satisfies numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ in Proposition 2.1. Moreover, we establish new identities in the sum of $C_{m,k}$ and their alternating, $(-1)^k C_{m,k}$ in Theorem 2.2. Next, as consequence in Corollary 2.3, we obtain the alternating sum of the entries of the two Catalan triangle numbers $(B_{n,k})_{n\geq k\geq 1}$ and $(A_{n,k})_{n+1\geq k\geq 1}$.

In the third section, we obtain the value of $\sum_{k=0}^{n} C_{m,k}^2$ and $\sum_{k=0}^{n}(-1)^k C_{m,k}^2$ for $m, n \geq 1$ in Theorem 3.1. We also show two identities which allows to decompose squares of combinatorial numbers as sum of squares of other combinatorial numbers. In particular, the nice identity

$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \frac{3n-2k}{n}\binom{2n-1-k}{n-1}^2, \qquad n \geq 1,$$

is presented in Theorem 3.3.

The forth section is dedicated to the sum of cubes of numbers $(C_{m,k})_{m\geq 1,k\geq 0}$. For $m \geq 1$ and $n \geq 1$, we present the identity

$$\sum_{k=0}^{n} C_{m,k}^3 = 4\binom{m-1}{n}^3 - 3\binom{m-1}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1},$$

in Theorem 4.1(i). Thus, from this identity we obtain

$$\begin{aligned}\sum_{k=1}^{n} B_{n,k}^3 &= \frac{n+1}{2} C_n b(n),\\ \sum_{k=1}^{n+1} A_{n,k}^3 &= (n+1)C_n \left((2(n+1)C_n)^2 - 3a(n)\right), \qquad n \in \mathbb{N},\end{aligned}$$

in Theorem 4.3 and Corollary 4.2 respectively, where integer sequences $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 1}$ are defined by

$$a(n) := \sum_{k=0}^{n} \binom{n+k}{n}^2 \qquad \text{and} \qquad b(n) := \sum_{k=0}^{n} \frac{n-k}{n} \binom{n-1+k}{n-1}^2.$$

This first sum solves the third open problem posed in [8, Section 3]. These sequences $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 1}$ appear in the On-Line Encyclopedia of Integer Sequences ([18]). We also present the value of the alternating sum $\sum_{k=0}^{n}(-1)^k C_{m,k}^3$ in Theorem 4.1(ii).

Identities which involved harmonic numbers $(H_n)_{n\geq 1}$ where

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad n \in \mathbb{N}, \tag{1.3}$$

have received a notable attention in last decades. We only mention shortly papers [4, 13, 19], the monograph [1, Chapter 7] and the reference therein.

In the fifth section we present a new identity which involves harmonic numbers $(H_n)_{n\geq 1}$ and Catalan triangle numbers $(C_{m,k})_{m\geq 1, k\geq 0}$ in Theorem 5.1 (and then for $B_{n,k}$ and $A_{n,k}$ in Corollary 5.2). This identity includes, as particular case, a known equality proved in [13].

In the last section we conjecture some identities which involves numbers $B_{n,k}$ and $A_{n,k}$. Although the WZ-theory (see for example [10, 14, 13, 22]) allows to give computer proofs, authors can not find an analytic proof of these equalities. Note that analytic proofs give additional information about the nature of these sequences which remains hidden in computer proofs.

Notation. We follow the usual convention that $\binom{u}{v}$ is zero if $u < v$ (in particular $\binom{u}{-1}$ for $u \geq 0$) and a sum is zero if its range of summation is empty.

2. Recurrence relation and sums of Catalan triangle numbers

One of the main aim of this section is to prove a recurrence relation (Proposition 2.1) that satisfies combinatorial numbers $(C_{m,k})_{m\geq 1, k\geq 0}$ given in (1.1). Moreover, for $m \geq 2$ and $n \geq 1$, we obtain the sum of combinatorial numbers $(C_{m,k})_{m\geq 1, k\geq 0}$ and the alternating sum, that is, $\sum_{k=0}^{n}(-1)^k C_{m,k}$ in Theorem 2.2 which includes some known identities for Catalan triangle numbers $(B_{n,k})_{n\geq k\geq 1}$ and $(A_{n,k})_{n+1\geq k\geq 1}$.

These numbers also are related to the entries $B_{n,k}$ and $A_{n,k}$ of the two particular Catalan triangles. In fact, the combinatorial numbers $B_{n,k}$ are the entries of the following Catalan triangle introduced in [15]:

(2.4)

$n \setminus k$	1	2	3	4	5	6	...
1	1						
2	2	1					
3	5	4	1				
4	14	14	6	1			
5	42	48	27	8	1		
6	132	165	110	44	10	1	
...	...	...	...	...	...	...	...

which are given by

$$B_{n,k} := \frac{k}{n}\binom{2n}{n-k}, \ n,k \in \mathbb{N}, \ k \le n. \tag{2.5}$$

Notice that $B_{n,1} = C_n$ and $C_{2n,n-k} = B_{n,k}$ for $k \le n$.

Although numbers $B_{n,k}$ are not as well known as Catalan numbers, they have also several applications, for example, $B_{n,k}$ is the number of walks of n steps, each in direction N, S, W or E, starting at the origin, remaining in the upper half-plane and ending at height k; see more details in [5, 15, 18] for more information.

In the last years, Catalan triangle (2.4) has been studied in detail. For instance, the formula

$$\sum_{k=1}^{i} B_{n,k}B_{n,n+k-i}(n+2k-i) = (n+1)C_n\binom{2(n-1)}{i-1}, \quad i \le n, \tag{2.6}$$

which appears in a problem related with the dynamical behavior of a family of iterative processes has been proved in [8, Theorem 5]. These numbers $(B_{n,k})_{n\ge k\ge 1}$ have been analyzed in many ways. For instance, symmetric functions have been used in [3], recurrence relations in [16], or in [7] the Newton interpolation formula, which is applied to conclude divisibility properties of sums of products of binomial coefficients.

Other combinatorial numbers $A_{n,k}$ defined as follows

$$A_{n,k} := \frac{2k-1}{2n+1}\binom{2n+1}{n+1-k}, \ n,k \in \mathbb{N}, \ k \le n+1, \tag{2.7}$$

appear as the entries of this second Catalan triangle,

(2.8)

$n \setminus k$	1	2	3	4	5	6	...
1	1	1					
2	2	3	1				
3	5	9	5	1			
4	14	28	20	7	1		
5	42	90	75	35	9	1	
6	132	297	275	154	54	11	1
...	...	...	...	...	...	...	...

which is considered in [11]. Notice that $A_{n,1} = C_n$ and $C_{2n+1,n-k+1} = A_{n,k}$ for $k \le n+1$.

The entries $B_{n,k}$ and $A_{n,k}$ of the above two particular Catalan triangles satisfy the recurrence relations

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \qquad k \ge 2, \tag{2.9}$$

and

$$A_{n,k} = A_{n-1,k-1} + 2A_{n-1,k} + A_{n-1,k+1}, \qquad k \geq 2. \tag{2.10}$$

Now, we show that numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ also satisfy a recurrence relation which extends recurrence relations (2.9) and (2.10).

Proposition 2.1. *For $m \geq 1$ and $k \geq 2$, the following identity holds:*

$$C_{m+2,k} = C_{m,k} + 2C_{m,k-1} + C_{m,k-2}.$$

Proof. Note that

$$C_{m,k} + 2C_{m,k-1} + C_{m,k-2} \quad = \quad \frac{(m-1)!}{k!(m-k+2)!} P(m,k),$$

where

$$\begin{aligned} P(m,k) \quad &= \quad (m-2k)(m-k+2)(m-k+1) + 2(m-2k+2)k(m-k+2) \\ &\qquad +(m-2k+4)k(k-1) \\ &= \quad m(m+1)(m+2-2k). \end{aligned}$$

Finally we conclude that

$$C_{m,k} + 2C_{m,k-1} + C_{m,k-2} = \frac{m+2-2k}{m+2}\binom{m+2}{k} = C_{m+2,k},$$

and the proof is finished. □

As it was shown in [15], the values of the sums of $B_{n,k}$ and $A_{n,k}$ in terms of Catalan numbers is given by:

$$\sum_{k=1}^{n} B_{n,k} = \frac{n+1}{2} C_n \qquad \text{and} \qquad \sum_{k=1}^{n+1} A_{n,k} = (n+1)C_n, \tag{2.11}$$

and the sums of its squares by

$$\sum_{k=1}^{n} B_{n,k}^2 = C_{2n-1} \qquad \text{and} \qquad \sum_{k=1}^{n+1} A_{n,k}^2 = C_{2n}, \quad n \in \mathbb{N}. \tag{2.12}$$

However the sums of its cubes $\sum_{k=1}^{n} B_{n,k}^3$ (posed in [8, Section 3]) and $\sum_{k=1}^{n+1} A_{n,k}^3$ in terms of Catalan numbers were unknown until now. This and other questions are studied in the in the next two sections.

To conclude this section we give the sum of numbers $(C_{m,k})_{m\geq 1,k\geq 0}$ and for their alternating sum in the following theorem.

Theorem 2.2. *For $m \geq 2$ and $n \geq 1$, we obtain the following identities:*

(i) $\displaystyle\sum_{k=0}^{n} C_{m,k} = \binom{m-1}{n}$,

(ii) $\displaystyle\sum_{k=0}^{n} (-1)^k C_{m,k} = (-1)^n C_{m-1,n}$.

Proof. Note that it is enough to check by induction process the identities. We only prove item (ii). For $n = 1$, we directly check the identity. Now suppose that the identity holds for n. Then

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^k C_{m,k} &= (-1)^n C_{m-1,n} + (-1)^{n+1} C_{m,n+1} \\ &= (-1)^n \frac{m-2n-1}{m-1}\binom{m-1}{n} + (-1)^{n+1}\frac{m-2n-2}{n+1}\binom{m-1}{n} \\ &= (-1)^{n+1}\frac{(m-2n-3)(m-n-1)}{(m-1)(n+1)}\binom{m-1}{n} \\ &= (-1)^{n+1}\frac{m-2n-3}{m-1}\binom{m-1}{n+1} = (-1)^{n+1} C_{m-1,n+1}.\end{aligned}$$

□

Notice that item (i) in Theorem 2.2 includes the identities given in (2.11). On the other hand, item (i) in the next corollary was proved in [6] and we present an alternative proof.

Corollary 2.3. *For $n \geq 1$, we have*

(i) $\displaystyle\sum_{k=1}^{n}(-1)^k B_{n,k} = -C_{n-1}$,

(ii) $\displaystyle\sum_{k=1}^{n+1}(-1)^k A_{n,k} = 0$.

Proof. By Theorem 2.2, we have

$$\sum_{k=1}^{n}(-1)^k B_{n,k} = \sum_{k=0}^{n}(-1)^k C_{2n,n-k} = \sum_{k=0}^{n}(-1)^{n-k} C_{2n,k} = C_{2n-1,n} = -C_{n-1},$$

and

$$\sum_{k=1}^{n+1}(-1)^k A_{n,k} = \sum_{k=1}^{n+1}(-1)^k C_{2n+1,n-k+1} = \sum_{k=0}^{n}(-1)^{n-k+1} C_{2n+1,k} = -C_{2n,n} = 0.$$

□

3. Sums of squares of Catalan triangle numbers

In the section, our main objective is twofold. Firstly, we check $\sum_{k=0}^{n} C_{m,k}^2$ and $\sum_{k=0}^{n}(-1)^k C_{m,k}^2$ in Theorem 3.1. As a consequence of this result, the identities presented in (2.12) are proved in Corollary 3.2.

Secondly, a key result of this paper is to decompose the binomial number $\binom{2n}{n}^2$ in sum of squares of other combinatorial numbers, i.e.

$$\binom{2n}{n}^2 = \sum_{k=0}^{n}\frac{3n-2k}{n}\binom{2n-1-k}{n-1}^2, \qquad n \geq 1.$$

To do that we present a straightforward proof as a consequence of a more general identity in combinatorial numbers in Theorem 3.3 (i). This equality is essential to check $\sum_{k=1}^{n} B_{n,k}^3$ in Theorem 4.3.

Theorem 3.1. *For $n \geq 1$ and $m \geq 1$, we have*

(i) $\displaystyle\sum_{k=0}^{n} C_{m,k}^2 = \frac{m-2n}{m}\binom{m-1}{n}^2 + \frac{2}{m}\sum_{k=0}^{n-1}\binom{m-1}{k}^2,$

(ii) $\displaystyle\sum_{k=0}^{n} (-1)^k C_{m,k}^2 = 2(-1)^n\binom{m-1}{n}^2 - \sum_{k=0}^{n}(-1)^k\binom{m}{k}^2.$

Proof. We prove the identities by invoking an inductive process for n.

(i) For $n = 1$, we directly check it. Now we assume that the desired identity holds for n. For $n+1$, we have

$$\sum_{k=0}^{n+1} C_{m,k}^2 = \frac{m-2n}{m}\binom{m-1}{n}^2 + \frac{2}{m}\sum_{k=0}^{n-1}\binom{m-1}{k}^2 + \left(\frac{m-2n-2}{m}\binom{m}{n+1}\right)^2.$$

On the other hand, observe that

$$\left(\frac{m-2n-2}{m}\binom{m}{n+1}\right)^2 = \frac{m-2n-2}{m}\binom{m-1}{n+1}^2 - \frac{m-2n}{m}\binom{m-1}{n}^2 + \frac{2}{m}\binom{m-1}{n}^2.$$

Therefore, we obtain the identity

$$\sum_{k=0}^{n+1} C_{m,k}^2 = \frac{m-2(n+1)}{m}\binom{m-1}{n+1}^2 + \frac{2}{m}\sum_{k=0}^{n}\binom{m-1}{k}^2.$$

(ii) For $n = 1$, we directly check it. Now we assume that the desired identity holds for n. For $n+1$, we have

$$\sum_{k=0}^{n+1}(-1)^k C_{m,k}^2 = 2(-1)^n\binom{m-1}{n}^2 - \sum_{k=0}^{n}(-1)^k\binom{m}{k}^2 + (-1)^{n+1}\left(\frac{m-2n-2}{m}\binom{m}{n+1}\right)^2.$$

On the other hand, observe that

$$\left(\frac{m-2n-2}{m}\binom{m}{n+1}\right)^2 = 2\binom{m-1}{n+1}^2 + 2\binom{m-1}{n}^2 - \binom{m}{n+1}^2.$$

Therefore, we obtain the identity

$$\sum_{k=0}^{n+1}(-1)^k C_{m,k}^2 = 2(-1)^{n+1}\binom{m-1}{n+1}^2 - \sum_{k=0}^{n+1}(-1)^k\binom{m}{k}^2.$$

□

Now, taking into account the well-known Vandermonde identity $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ and identity $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2 = (-1)^n\binom{2n}{n}$ for $n \geq 0$, the following corollary is obtained. Note that the Corollary 3.2 (iv) was proved in [23, Theorem 2.2].

Corollary 3.2. *For $n \geq 1$, we have*

(i) $\displaystyle\sum_{k=0}^{n} C_{n,k}^2 = 2C_{n-1},$

(ii) $\displaystyle\sum_{k=1}^{n} B_{n,k}^2 = C_{2n-1},$

(iii) $\displaystyle\sum_{k=1}^{n+1} A_{n,k}^2 = C_{2n}$,

(iv) $\displaystyle\sum_{k=1}^{n} (-1)^k B_{n,k}^2 = -\frac{n+1}{2} C_n$.

Proof. From Theorem 3.1 (i), we have

$$\sum_{k=1}^{n} C_{n,k}^2 = \frac{2}{n} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 = 2C_{n-1},$$

$$\sum_{k=0}^{n} B_{n,k}^2 = \sum_{k=0}^{n} C_{2n,k}^2 = \frac{2}{2n} \sum_{k=0}^{n-1} \binom{2n-1}{k}^2 = \frac{1}{2n} \sum_{k=0}^{2n-1} \binom{2n-1}{k}^2 = C_{2n-1},$$

and

$$\sum_{k=1}^{n+1} A_{n,k}^2 = \sum_{k=0}^{n} C_{2n+1,k}^2 = \frac{1}{2n+1} \left(\binom{2n}{n}^2 + 2 \sum_{k=0}^{n-1} \binom{2n}{k}^2 \right) = \frac{1}{2n+1} \sum_{k=0}^{2n} \binom{2n}{k}^2 = C_{2n}.$$

As a consequence of Theorem 3.1 (ii), item (iv) is obtained

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k B_{n,k}^2 &= \sum_{k=0}^{n} (-1)^{n+k} C_{2n,k}^2 = 2\binom{2n-1}{n}^2 - \sum_{k=0}^{n} (-1)^{n+k} \binom{2n}{k}^2 \\ &= \frac{-1}{2} \sum_{k=0}^{2n} (-1)^{n+k} \binom{2n}{k}^2 = \frac{-1}{2} \binom{2n}{n} = -\frac{n+1}{2} C_n. \end{aligned}$$

□

Theorem 3.3. *For $m \geq n \geq 1$, we have*

(i) $\displaystyle\binom{m}{n}^2 = \sum_{j=n}^{m} \frac{2j-n}{n} \binom{j-1}{n-1}^2$,

(ii) $\displaystyle\binom{2n}{n}^2 = \sum_{k=0}^{n} \frac{3n-2k}{n} \binom{2n-1-k}{n-1}^2$.

Proof. To prove item (i) we invoke an inductive process for m. For $m = n$, we check directly the identity. Now we assume that the identity holds for m and we prove it for $m+1$. Thus, it follows that

$$\begin{aligned} \sum_{j=n}^{m+1} \frac{2j-n}{n} \binom{j-1}{n-1}^2 &= \binom{m}{n}^2 + \frac{2m+2-n}{n} \binom{m}{n-1}^2 \\ &= \left(\frac{m-n+1}{m+1} \binom{m+1}{n} \right)^2 + \frac{2m+2-n}{n} \left(\frac{n}{m+1} \binom{m+1}{n} \right)^2 = \binom{m+1}{n}^2. \end{aligned}$$

Then we conclude the identity holds for $m \geq n \geq 1$.

To show item (ii) observe that

$$\sum_{k=0}^{n} \frac{2(2n-k)-n}{n} \binom{2n-k-1}{n-1}^2 = \sum_{j=n}^{2n} \frac{2j-n}{n} \binom{j-1}{n-1}^2 = \binom{2n}{n}^2,$$

where we have applied item (i). □

Remark. Notice that item (ii) in Theorem 3.3 gives a decomposition of sum of squares of $\binom{2n}{n}^2$ for $n \geq 1$, that can be written in this form

$$\binom{2n}{n}^2 = \sum_{j=0}^{n} \frac{n+2j}{n} \binom{n-1+j}{n-1}^2.$$

4. Sums of cubes of Catalan triangle numbers

In this section we check the sum of cubes and alternating cubes of numbers $(C_{m,k})_{m\geq 1, k\geq 0}$ in Theorem 4.1. For $m \geq 1$ and $n \geq 1$, we use the identity

$$\sum_{k=0}^{n} (m-2k)\binom{m}{k}^3 = (m-n)\binom{m}{n} \sum_{j=0}^{m-1} \binom{j}{n}\binom{j}{m-n-1}, \tag{4.1}$$

which is proven in [12]. We also present some expressions of $\sum_{k=1}^{n} B_{n,k}^3$, $\sum_{k=1}^{n+1} A_{n,k}^3$ and $\sum_{k=1}^{n+1}(-1)^k A_{n,k}^3$ in Corollary 4.2. The equality presented in Theorem 4.3 solves the third open problem posed in [8, Section 3].

Theorem 4.1. *For $m \geq 1$ and $n \geq 1$,we have*

(i) $\displaystyle\sum_{k=0}^{n} C_{m,k}^3 = 4\binom{m-1}{n}^3 - 3\binom{m-1}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1},$

(ii) $\displaystyle\sum_{k=0}^{n} (-1)^k C_{m,k}^3 = \frac{m-3n}{m}(-1)^n\binom{m-1}{n}^3 - \frac{m-3}{m}\sum_{k=0}^{n-1}(-1)^k\binom{m-1}{k}^3.$

Proof. (i) We apply (4.1) to get that

$$\begin{aligned}
&m^3\sum_{k=0}^{n} C_{m,k}^3 + 3m^3\binom{m-1}{n}\sum_{j=0}^{m-1}\binom{j}{n}\binom{j}{m-n-1}\\
&= m^3\sum_{k=0}^{n} C_{m,k}^3 + 3m^2\sum_{k=0}^{n}(m-2k)\binom{m}{k}^3\\
&= \sum_{k=0}^{n}\left((m-2k)^3 + 3m^2(m-2k)\right)\binom{m}{k}^3\\
&= \sum_{k=0}^{n} 4\left((m-k)^3 - k^3\right)\binom{m}{k}^3 = 4\sum_{k=0}^{n}\left((m-k)^3\binom{m}{m-k}^3 - k^3\binom{m}{k}^3\right)\\
&= 4\sum_{k=0}^{n}\left(m^3\binom{m-1}{m-k-1}^3 - m^3\binom{m-1}{k-1}^3\right)\\
&= 4m^3\sum_{k=0}^{n}\left(\binom{m-1}{k}^3 - \binom{m-1}{k-1}^3\right) = 4m^3\left(\binom{m-1}{n}^3 - \binom{m-1}{-1}^3\right)\\
&= 4m^3\binom{m-1}{n}^3.
\end{aligned}$$

Therefore, we obtain the desired identity.

(ii) We prove the identity by invoking an inductive process for n. For $n=1$, we directly check it. Now we assume that the desired identity holds for n. For $n+1$, we have

$$\begin{aligned}\sum_{k=0}^{n+1}(-1)^k C_{m,k}^3 &= \frac{m-3n}{m}(-1)^n\binom{m-1}{n}^3-\frac{m-3}{m}\sum_{k=0}^{n-1}(-1)^k\binom{m-1}{k}^3\\ &\quad +(-1)^{n+1}\left(\frac{m-2n-2}{m}\binom{m}{n+1}\right)^3.\end{aligned}$$

On the other hand, observe that

$$\left(\frac{m-2n-2}{m}\binom{m}{n+1}\right)^3=\frac{m-3n-3}{m}\binom{m-1}{n+1}^3+\frac{2m-3n-3}{m}\binom{m-1}{n}^3.$$

Therefore, we obtain the identity

$$\sum_{k=0}^{n+1}(-1)^k C_{m,k}^3=\frac{m-3(n+1)}{m}(-1)^{n+1}\binom{m-1}{n+1}^3-\frac{m-3}{m}\sum_{k=0}^{n}(-1)^k\binom{m-1}{k}^3.$$

□

As a nice consequence of Theorem 4.1, we obtain expressions of $\sum_{k=1}^{n} B_{n,k}^3$, $\sum_{k=1}^{n+1} A_{n,k}^3$ and $\sum_{k=1}^{n+1}(-1)^k A_{n,k}^3$ in the next corollary. To check this last sum, we use Dixon's identity,

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\binom{2n}{n}\binom{3n}{n},\qquad n\geq 1.$$

Corollary 4.2. *For $n\geq 1$, we have*

(i) $\displaystyle\sum_{k=0}^{n} B_{n,k}^3=\frac{1}{2}\binom{2n}{n}^3-\frac{3}{2}\binom{2n}{n}\sum_{j=n}^{2n-1}\binom{j}{n}\binom{j}{n-1}$,

(ii) $\displaystyle\sum_{k=1}^{n+1} A_{n,k}^3=\binom{2n}{n}^3-3\binom{2n}{n}\sum_{j=n}^{2n-1}\binom{j}{n}^2$,

(iii) $\displaystyle\sum_{k=1}^{n+1}(-1)^k A_{n,k}^3=\frac{n-1}{2n+1}\binom{2n}{n}\binom{3n}{n}$.

Proof. From Theorem 4.1(i), we have

$$\begin{aligned}\sum_{k=0}^{n} B_{n,k}^3 &= \sum_{k=0}^{n} C_{2n,k}^3=4\binom{2n-1}{n}^3-3\binom{2n-1}{n}\sum_{j=0}^{2n-1}\binom{j}{n}\binom{j}{n-1}\\ &= \frac{1}{2}\binom{2n}{n}^3-\frac{3}{2}\binom{2n}{n}\sum_{j=n}^{2n-1}\binom{j}{n}\binom{j}{n-1},\end{aligned}$$

and

$$\sum_{k=1}^{n+1} A_{n,k}^3=\sum_{k=0}^{n} C_{2n+1,k}^3=4\binom{2n}{n}^3-3\binom{2n}{n}\sum_{j=0}^{2n}\binom{j}{n}\binom{j}{n}=\binom{2n}{n}^3-3\binom{2n}{n}\sum_{j=n}^{2n-1}\binom{j}{n}^2.$$

Now, from Theorem 4.1(ii), we have

$$\begin{aligned}\sum_{k=1}^{n+1}(-1)^k A_{n,k}^3 &= \sum_{k=0}^{n}(-1)^{n+1-k}C_{2n+1,k}^3 = \frac{n-1}{2n+1}\left(\binom{2n}{n}^3 + 2\sum_{k=0}^{n-1}(-1)^{n+k}\binom{2n}{k}^3\right)\\ &= \frac{n-1}{2n+1}\sum_{k=0}^{2n}(-1)^{n+k}\binom{2n}{k}^3 = \frac{n-1}{2n+1}\binom{2n}{n}\binom{3n}{n}.\end{aligned}$$

□

Remark. Note that the part (ii) of Corollary 4.2 may be written as

$$\sum_{k=1}^{n+1} A_{n,k}^3 = (n+1)C_n\left((2(n+1)C_n)^2 - 3a(n)\right), \qquad n \geq 1,$$

where the integer sequence of numbers $(a(n))_{n\geq 0}$ is defined by

$$a(n) := \sum_{k=0}^{n}\binom{n+k}{n}^2, \qquad n \in \mathbb{N}\cup\{0\}.$$

Note that $a(0) = 1$, $a(1) = 5$, $a(2) = 46$, $a(3) = 517$, $a(4) = 6376...$. This sequence appears indexed in the On-Line Encyclopedia of Integer Sequences by N.J.A. Sloane ([18]) with the reference $A112029$.

The sequence $\left(\binom{2n}{n}\binom{3n}{n}\right)_{n\geq 0}$ is known as De Bruijn's $S(3,n)$ and appears in the Sloane's On-Line Encyclopedia with the reference $A006480$.

Theorem 4.3. *For $n \geq 1$, the following identity holds:*

$$\sum_{k=1}^{n} B_{n,k}^3 = \frac{1}{2n}\binom{2n}{n}\sum_{k=1}^{n} k\binom{2n-k-1}{n-1}^2.$$

Proof. By Corollary 4.2 (i), we have

$$\sum_{k=1}^{n} B_{n,k}^3 = \frac{1}{2}\binom{2n}{n}\left(\binom{2n}{n}^2 - 3\sum_{j=n}^{2n-1}\binom{j}{n}\binom{j}{n-1}\right)$$

and we claim that

$$\binom{2n}{n}^2 - 3\sum_{j=n}^{2n-1}\binom{j}{n}\binom{j}{n-1} = \sum_{k=1}^{n}\frac{k}{n}\binom{2n-k-1}{n-1}^2.$$

Note that

$$\sum_{j=n}^{2n-1}\binom{j}{n}\binom{j}{n-1} = \sum_{j=n-1}^{2n-1}\frac{j-n+1}{n}\binom{j}{n-1}^2 = \sum_{k=0}^{n}\frac{n-k}{n}\binom{2n-k-1}{n-1}^2,$$

and then we obtain

$$3\sum_{j=n}^{2n-1}\binom{j}{n}\binom{j}{n-1} + \sum_{k=0}^{n}\frac{k}{n}\binom{2n-k-1}{n-1}^2 = \sum_{k=0}^{n}\frac{3n-2k}{n}\binom{2n-k-1}{n-1}^2 = \binom{2n}{n}^2,$$

where we have applied Theorem 3.3 (ii). □

Remark. Note that the identity of Theorem 4.3 may be written as

$$\sum_{k=1}^{n} B_{n,k}^3 = \frac{n+1}{2} C_n b(n),$$

where the integer sequence of numbers $(b(n))_{n\geq 1}$ is defined by

$$b(n) := \sum_{k=0}^{n} \frac{k}{n}\binom{2n-k-1}{n-1}^2 = \sum_{k=0}^{n} \frac{n-k}{n}\binom{n-1+k}{n-1}^2, \qquad n \in \mathbb{N}.$$

Note that $b(1) = 1$, $b(2) = 3$, $b(3) = 19$, $b(4) = 163$, $b(5) = 1625, \ldots$. This sequence also appears indexed in the On-Line Encyclopedia of Integer Sequences by N.J.A. Sloane ([18]) with the reference $A183069$.

5. Identities involving harmonic numbers and Catalan triangle numbers

A large number of identities which included harmonic numbers $(H_n)_{n\geq 1}$, defined by (1.3), have appeared in several papers: a systematic study of explicit formulas for sums of the form $\sum_{k=1}^{n} a_k H_k$ are given in [19]; some other finite summation identities involving harmonic numbers are considered in [13] and proved by the WZ-theory; infinite series involving harmonic numbers are presented in [4]. See other approaches in [1, Chapter 7] and reference therein.

However, the next nice relation between Catalan triangle numbers $(C_{m,k})_{m\geq 1, k\geq 0}$ and harmonic numbers $(H_n)_{n\geq 1}$ seems to be new. We also present the particular case of $B_{n,k}$ and $A_{n,k}$ in Corollary 5.2.

Theorem 5.1. *For $m \geq 1$ and $n \geq 1$, we have*

$$\sum_{k=1}^{n} C_{m,k} H_k = \binom{m-1}{n} H_n - \frac{1}{m}\sum_{k=1}^{n}\binom{m}{k}. \tag{5.1}$$

Proof. We prove the identities by invoking an induction process for n. For $n = 1$, we directly check it. Now we assume that the identity (5.1) holds for n. For $n+1$, we have

$$\sum_{k=1}^{n+1} C_{m,k} H_k = \binom{m-1}{n} H_n - \frac{1}{m}\sum_{k=1}^{n}\binom{m}{k} + \frac{m-2n-2}{m}\binom{m}{n+1} H_{n+1}.$$

On the other hand, observe that

$$\binom{m-1}{n} H_n - \frac{n+1}{m}\binom{m}{n+1} H_{n+1} = -\frac{1}{m}\binom{m}{n+1}.$$

Therefore, we obtain the identity

$$\sum_{k=1}^{n+1} C_{m,k} H_k = \frac{m-n-1}{m}\binom{m}{n+1} H_{n+1} - \frac{1}{m}\sum_{k=1}^{n+1}\binom{m}{k} = \binom{m-1}{n+1} H_{n+1} - \frac{1}{m}\sum_{k=1}^{n+1}\binom{m}{k}.$$

□

Using Theorem 5.1, we will show the relationship of the harmonic numbers and the Catalan triangle numbers.

Corollary 5.2. *For $n \geq 1$, we have*

(i) $\displaystyle\sum_{k=1}^{n} C_{n,k} H_k = \frac{1-2^n}{n}$,

(ii) $\displaystyle\sum_{k=0}^{n-1} B_{n,k} H_{n-k} = \frac{2nH_n - 1}{4n}\binom{2n}{n} - \frac{2^{2n-1}-1}{2n}$,

(iii) $\displaystyle\sum_{k=1}^{n} A_{n,k} H_{n-k+1} = H_n\binom{2n}{n} - \frac{2^{2n}-1}{2n+1}$.

Proof. On the one hand, we have

$$\sum_{k=1}^{n} C_{n,k} H_k = \frac{-1}{n}\sum_{k=1}^{n}\binom{n}{k} = \frac{1-2^n}{n}.$$

On the other hand, taking into account identities

$$2\sum_{k=0}^{n}\binom{2n}{k} - \binom{2n}{n} = \sum_{k=0}^{2n}\binom{2n}{k} = 2^{2n} \quad \text{and} \quad 2\sum_{k=0}^{n}\binom{2n+1}{k} = \sum_{k=0}^{2n+1}\binom{2n+1}{k} = 2^{2n+1},$$

we have

$$\sum_{k=0}^{n-1} B_{n,k} H_{n-k} = \sum_{k=1}^{n} C_{2n,k} H_k = H_n\binom{2n-1}{n} - \frac{1}{2n}\sum_{k=1}^{n}\binom{2n}{k} = \frac{2nH_n - 1}{4n}\binom{2n}{n} - \frac{2^{2n-1}-1}{2n},$$

and

$$\sum_{k=1}^{n} A_{n,k} H_{n-k+1} = \sum_{k=1}^{n} C_{2n+1,k} H_k = H_n\binom{2n}{n} - \frac{1}{2n+1}\sum_{k=1}^{n}\binom{2n+1}{k} = H_n\binom{2n}{n} - \frac{2^{2n}-1}{2n+1}.$$

□

Remark. By Corollary 5.2 (i), we have

$$\sum_{k=1}^{n}(n-2k)H_k\binom{n}{k} = 1 - 2^n,$$

which was shown in [13, Formula (13)].

6. New conjectures, final comments and conclusions

In this last section, we present two conjectures about new identities in Catalan triangle numbers. We have directly checked that these identities hold for first values of n and m. Although analytic proofs are not yet available, alternative proofs as to apply WZ-theory ([13, 22]) or some mathematical software, indicate us that these equalities hold. Note that an analytic proof will give us some extra information about these nature of the sums. To conclude the paper, we present some final comments and conclusions.

Conjecture 6.1. For $m > n \geq 1$ and an odd integer p, the factor $\binom{m-1}{n}$ divides $\sum_{k=0}^{n} C_{m,k}^p$. Note that the conjecture holds for $p = 1$ and $p = 3$, see Theorem 2.2 (i) and Theorem 4.1 respectively. Now we present two important cases of this conjecture.

(i) Taking into count that $B_{n,k} = C_{2n,n-k}$, a positive answer of the conjecture 6.1 would imply that the factor $\frac{n+1}{2}C_n$ divides $\sum_{k=1}^{n} B_{n,k}^p$ for $n \geq 1$ and an odd integer p. In the case that $p = 1$ and $p = 3$ the sums are explicitly given in (2.11) and Corollary 4.2(i) respectively. We have directly checked that the factor $\frac{n+1}{2}C_n$ divides $\sum_{k=1}^{n} B_{n,k}^5$ for first values of n.

(ii) Now we considerer that $A_{n,k} = C_{2n+1,n+1-k}$. A positive answer of the conjecture 6.1 would imply that the factor $(n+1)C_n$ divides $\sum_{k=1}^{n+1} A_{n,k}^p$ for $n \geq 1$ and an odd integer p. For $p = 1$ and $p = 3$ these sums are given in (2.11) and Corollary 4.2 (ii) respectively. We have also checked that $(n+1)C_n$ divide $\sum_{k=1}^{n+1} A_{n,k}^5$ for first values of n.

Conjecture 6.2. For $n, m \in \mathbb{N}$, the identity

$$\sum_{k=1}^{r} B_{n,k}^2 B_{m,k} = \frac{1}{2}\binom{2n}{n}^2\binom{2m}{m}\left[1 - \frac{n+2m}{r}\binom{n+m}{n}^{-1}\binom{n+r}{n}^{-1}\sum_{j=0}^{r-1}\binom{s+j}{s}\binom{n+j}{n-1}\right],$$

holds where $r = \min(n,m)$ and $s = \max(n,m)$. In the particular case, $m = n$, we recover the identity given in Corollary 4.2 (i). Note that the nature of this formula is different than the formula (2.6).

Final comments and conclusions. In this paper we have presented a unified study of two families of Catalan triangle numbers. We have considered finite sums of powers (linear, squares and cubes) of these numbers to show original (and nice) identities involving Catalan numbers (section 2-4). Some of these equalities solve some open problems and connect Catalan sequences with other some known sequences, see for example Theorem 4.3. Note that we have not considered moments on these sums of powers as in other papers in the literature, see for example [3, 11, 23]. We have also presented a natural connection between harmonic numbers and Catalan triangle numbers which seems to be new and may be completed in later studies. Finally some conjectures about other sums of Catalan triangle numbers are posed.

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