A formula for the partition function of the $\beta\gamma$ system on the cone pure spinors

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Abstract

In this note, we propose a closed formula for the partition function $Z(t, q)$ of the $\beta\gamma$ system on the cone of pure spinors. We give the answer in terms of theta functions, q-Pochhammer symbols and Eisenstein series.

1 Introduction

The $\beta\gamma$ system on the cone of pure spinors $\mathcal{C}$ is an integral part of the version of the string theory invented by N. Berkovits [5]. $\mathcal{C}$ is an eleven-dimensional subvariety in 16-dimensional linear space with coordinates $\lambda, p_i, w_{ij}, \quad 1 \leq i, j \leq 5, w_{ij} = -w_{ji}$ defined by the equations

$$\begin{aligned} &\lambda p_i - \mathrm{Pf}_i(w) = 0 \quad i = 1, \dots, 5, \\ &pw = 0. \end{aligned} \tag{1}$$

$\mathrm{Pf}_i(w), 1 \leq i \leq 5$ are the principal Pfaffians of w. The action is common to all $\beta\gamma$ systems:

$$S(\beta, \gamma) = \int_\Sigma \langle \bar{\partial}\beta, \gamma \rangle.$$

The field β is a smooth map $\beta : \Sigma \to \mathcal{C}$, where Σ is a Riemann surface, γ is a smooth section of the pullback $\beta^* T^*_{\mathcal{C}} \otimes T^*_\Sigma$. We advise the reader to consult [7] for the notation and discussion of issues related to definition of a $\beta\gamma$ systems on the nonsmooth $\mathcal{C}$.

In [7], a geometric construction of the space of states $H^{i+\frac{\infty}{2}}, i = 0, \dots, 3$ of this system was presented. It is a properly regularized space of the semi-infinite local cohomology of the space of polynomial maps $\mathcal{M}aps(\mathbb{C}^\times, \mathcal{C})$. The support of the local cohomology lies at $\mathcal{M}aps(\mathbb{C}, \mathcal{C})$. The space $\mathcal{C}$ is an affine cone over $\mathrm{OGr}(5, 10)$. $\mathbb{C}^\times \times \mathrm{Spin}(10)$ is the groups of symmetries $\mathcal{C}$, where $\mathbb{C}^\times$ acts by dilations. The group $\mathbb{C}^\times \times \mathbf{T} \times \mathrm{Spin}(10)$ acts by the symmetries of the pair $\mathcal{M}aps(\mathbb{C}, \mathcal{C}) \subset \mathcal{M}aps(\mathbb{C}^\times, \mathcal{C})$. The factor $\mathbf{T} \cong \mathbb{C}^\times$ corresponds to loop rotations. The action of $\mathbb{C}^\times \times \mathbf{T} \times \mathrm{Spin}(10)$ survive the regularization and continue to act on $H^{i+\frac{\infty}{2}}$. It turns out (see [7]) that the formal character

$$Z(t, q, z) = \sum_{i=0}^{3} (-1)^i \chi_{H^{i+\frac{\infty}{2}}}(t, q, z) \tag{2}$$

is well defined as an element in $\mathbb{Z}((t, z_1, \ldots, z_5))((q)) \cap \mathbb{Q}(t, z_1, \ldots, z_5)((q))$, where $z_1, \ldots, z_5$ are the coordinates on the Cartan subgroup $\mathbf{T}^5 \subset \mathrm{Spin}(10)$. More precisely, $Z(t, q, z)$ is a limit of coefficients of an infinite matrix product [7]. The matrix product simplifies when $z = 1$ (10). We use it to derive the formula for $Z(t, q) := Z(t, q, 1)$.

It is necessary to state from the outset that the analysis presented in this note is based on experimentations with the formula for $Z_N^{N'}(t, q)$ using *Mathematica* for finite N, N' and extrapolation of the found structures to infinite Ns. Though somewhat loose in justification, the result looks convincing because it passes a number of consistency checks.

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2 The formula

As it is mentioned in the abstract, $Z(t, q)$ will be expressed in therms of some standard special functions. We start with reviewing their definitions.

The functions used in the formula Recall that the q-Pochhammer symbol is an infinite product

$$(t; q)_\infty := \prod_{n \geq 0} (1 - tq^n).$$

It is used to write concisely the three-term identity

$$\theta(t, q) = (1 - t^{-1})(q, q)_\infty (q/t, q)_\infty (qt, q)_\infty \tag{3}$$

for the theta function

$$\theta(q, t) := \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n(n+1)}{2}} t^n.$$

Another ingredient of the formula are theta functions with characteristics:

$$\epsilon_k(t, q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{7n(n+1)}{2} + kn} t^{7n+k} = t^k \left(1 - q^{-k} t^{-7}\right) \left(q^7, q^7\right)_\infty \left(q^{7-k} t^{-7}, q^7\right)_\infty \left(q^{k+7} t^7, q^7\right)_\infty, k = 0, \ldots, 6. \tag{4}$$

The proposed answer will also depend on the Eisenstein series

$$E_2(q) := 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \quad E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}.$$

Define linear combinations of $\epsilon_k(q, t)$:

$$\theta_1(t, q) := -q^3 \epsilon_3(q, t) - q^4 \epsilon_4(q, t),$$

$$\theta_2(t, q) := q^2 \epsilon_2(q, t) + q^5 \epsilon_5(q, t), \text{ and}$$

$$\theta_3(t, q) := -q \epsilon_1(q, t) - q^6 \epsilon_6(q, t)$$

and introduce abbreviation $\theta_k(q) := \theta_k(1, q), k = 1, 2, 3$.

The following conjecture contains the promised formula.

Conjecture 1 *The partition function $Z(t,q)$ has the form*

$$Z(t,q)=\frac{a(q)\theta_1(t,q)+b(q)\theta_2(t,q)+c(q)\theta_3(t,q)}{t^6\theta(t,q)^{11}}=:\Psi(t,q) \tag{5}$$

where the string functions a,b,c are

$$\begin{aligned}a:=\frac{1}{768(q,q)_\infty^4}&(2744q^2E_2(q)\theta_3(q)\theta_2''(q)-2744q^2E_2(q)\theta_2(q)\theta_3''(q)-343qE_2(q)^2\theta_3(q)\theta_2'(q)+3626qE_2(q)\theta_3(q)\theta_2'(q)\\&+343qE_2(q)^2\theta_2(q)\theta_3'(q)-5194qE_2(q)\theta_2(q)\theta_3'(q)+98E_2(q)^2\theta_2(q)\theta_3(q)-476E_2(q)\theta_2(q)\theta_3(q)\\&+42qE_4(q)\theta_3(q)\theta_2'(q)-42qE_4(q)\theta_2(q)\theta_3'(q)-12E_4(q)\theta_2(q)\theta_3(q)-65856q^3\theta_2''(q)\theta_3'(q)\\&+65856q^3\theta_2'(q)\theta_3''(q)-29400q^2\theta_3(q)\theta_2''(q)+37632q^2\theta_2'(q)\theta_3'(q)+10584q^2\theta_2(q)\theta_3''(q)\\&-25725q\theta_3(q)\theta_2'(q)+18333q\theta_2(q)\theta_3'(q)+1350\theta_2(q)\theta_3(q)),\end{aligned} \tag{6}$$

$$\begin{aligned}b:=-\frac{1}{768(q,q)_\infty^4}&(2744q^2E_2(q)\theta_3(q)\theta_1''(q)-2744q^2E_2(q)\theta_1(q)\theta_3''(q)-343qE_2(q)^2\theta_3(q)\theta_1'(q)+2842qE_2(q)\theta_3(q)\theta_1'(q)\\&+343qE_2(q)^2\theta_1(q)\theta_3'(q)-5194qE_2(q)\theta_1(q)\theta_3'(q)+147E_2(q)^2\theta_1(q)\theta_3(q)-546E_2(q)\theta_1(q)\theta_3(q)\\&+42qE_4(q)\theta_3(q)\theta_1'(q)-42qE_4(q)\theta_1(q)\theta_3'(q)-18E_4(q)\theta_1(q)\theta_3(q)-65856q^3\theta_1''(q)\theta_3'(q)\\&+65856q^3\theta_1'(q)\theta_3''(q)-29400q^2\theta_3(q)\theta_1''(q)+56448q^2\theta_1'(q)\theta_3'(q)+1176q^2\theta_1(q)\theta_3''(q)\\&-17325q\theta_3(q)\theta_1'(q)+2205q\theta_1(q)\theta_3'(q)+225\theta_1(q)\theta_3(q)),\end{aligned} \tag{7}$$

$$\begin{aligned}c:=\frac{1}{768(q,q)_\infty^4}&(2744q^2E_2(q)\theta_2(q)\theta_1''(q)-2744q^2E_2(q)\theta_1(q)\theta_2''(q)-343qE_2(q)^2\theta_2(q)\theta_1'(q)+2842qE_2(q)\theta_2(q)\theta_1'(q)\\&+343qE_2(q)^2\theta_1(q)\theta_2'(q)-3626qE_2(q)\theta_1(q)\theta_2'(q)+49E_2(q)^2\theta_1(q)\theta_2(q)-70E_2(q)\theta_1(q)\theta_2(q)\\&+42qE_4(q)\theta_2(q)\theta_1'(q)-42qE_4(q)\theta_1(q)\theta_2'(q)-6E_4(q)\theta_1(q)\theta_2(q)-65856q^3\theta_1''(q)\theta_2'(q)\\&+65856q^3\theta_1'(q)\theta_2''(q)-10584q^2\theta_2(q)\theta_1''(q)+18816q^2\theta_1'(q)\theta_2'(q)+1176q^2\theta_1(q)\theta_2''(q)\\&-9261q\theta_2(q)\theta_1'(q)+1533q\theta_1(q)\theta_2'(q)+27\theta_1(q)\theta_2(q)).\end{aligned} \tag{8}$$

3 Supporting evidences

The matrix product presentation for $Z(t,q)$ It was established in [7] that $Z(t,q)$ is the limit in the sense of formal power series convergence of a certain infinite matrix product. To state the result let us fix some additional notations:

$$\begin{aligned}&B_0^1:=\frac{1+3t+t^2}{(1-t)^8(1-qt)},\quad A_0^0:=\frac{1+5t+5t^2+t^3}{(1-t)^{11}},\\&K(t,q):=\begin{pmatrix}\frac{t\left(t^2+3t+1\right)}{(t-1)^7(qt-1)} & \frac{\left(t^2+3t+1\right)(t^3+q^2)-5q(t+1)t^2}{q^2(t-1)^7(qt-1)}\\ \frac{t(t+1)\left(t^2+4t+1\right)}{(t-1)^{10}} & \frac{\left(t^3+5t^2+5t+1\right)(t^3+q^2)-q\left(5t^2+14t+5\right)t^2}{q^2(t-1)^{10}}\end{pmatrix},\end{aligned} \tag{9}$$

$$\begin{pmatrix} B_0^{r+1} \\ A_0^r \end{pmatrix} := K(q^r t, q) \cdots K(qt, q) \begin{pmatrix} B_0^1 \\ A_0^0 \end{pmatrix}, \tag{10}$$

$$A_N^{N'}(t,q) := A_0^{N'-N}(tq^N, q).$$

It was verified in [7] that the limit of

$$Z_N^{N'}(t,q) := A_N^{N'}(t,q) t^{4-4N} q^{-2+4N-2N^2}, N < 0 \tag{11}$$

$N \to -\infty, N' \to \infty$ coincides with $Z(t,q)$ (2).

Poles of $Z(t,q)$ The rational function $Z_N^{N'}(t,q)$ has a fairly complicated structure. Still, experiments with *Mathematica* show that $Z_N^{N'}(t,q)$ has poles of multiplicity $\dim_{\mathbb{C}} \mathcal{C} = 11$ precisely at $q^{-N}, \ldots, q^{-N'}$. It is natural to conjecture that in the limit $N \to -\infty, N' \to \infty$ this pattern persists and Z is a meromorphic function for $|q| < 1$ with poles at $t = q^n, n \in \mathbb{Z}$ of multiplicity 11.

$Z(t,q)$ and a line bundle of degree 7 If this conjecture is true, then the product

$$\Theta(t,q) := t^6 Z(t,q) \theta(t,q)^{11} \tag{12}$$

is an analytic function for $t \neq 0, |q| < 1$.

One of the results of [7] is that $Z(t,q,g)$ as a formal power series in q satisfies

$$\frac{Z(qt,q,g)}{Z(t,q,g)} = \frac{t^4}{q^2}, \tag{13}$$

$$\frac{Z(1/t,q,g^{-1})}{Z(t,q,g)} = -t^8. \tag{14}$$

It follows from the functional equations

$$\theta(qt,q) = -\theta(t,q)/(qt), \quad \theta(1/t,q) = -t\theta(t,q) \tag{15}$$

that $\Theta(t,q)$ obeys

$$\frac{\Theta(tq,q)}{\Theta(t,q)} = -\frac{1}{q^7 t^7}, \tag{16}$$

$$\frac{\Theta(1/t,q)}{\Theta(t,q)} = t^7 \tag{17}$$

Stated differently, equation (15) says that θ is a holomorphic section of a line bundle $\mathcal{L}$ of degree one on the elliptic curve $\mathbb{C}^\times/\{q^k\}$. To say that Θ satisfy (16) is equivalent to saying that Θ is a section of $\mathcal{L}^{\otimes 7}$. The space of global sections of $\mathcal{L}^{\otimes 7}$ has a basis $\epsilon_k, k = 0, \ldots, 6$ (4). Symmetry condition (17) determines a subspace in the span of ϵ_k of dimension three with a basis $\theta_i, i = 1, 2, 3$. As a consequence, we get

$$\Theta(t,q) = a(q)\theta_1(t,q) + b(q)\theta_2(t,q) + c(q)\theta_3(t,q), \tag{18}$$

which is equivalent to (5).

Equations on coefficients a, b, c It remains to determine a, b, c. Denote the right-hand side in (5) by $\Psi(t, q)$. Identity

$$\lim_{t\to 1} \partial_t^k Z(t,q)(1-t)^{11} = \lim_{t\to 1} \partial_t^k \Psi(t,q)(1-t)^{11}, \quad k = k_1, k_2, k_3 \tag{19}$$

produces three linear equations for $a(q), b(q), c(q)$. Denote by x the vector (a, b, c). We can write the system of equations on x in the matrix form

$$z^t = Ax^t. \tag{20}$$

The coefficients of the matrix A depend only on the functions θ_i and can be explicitly computed. The vector z is more complicated because it depends on the unknown function Z. But if we manage to compute three Taylor coefficients of $Z(t, q)$ with respect to the variable t, we can easily find the functions a, b, c.

Coefficients of the matrix A Let us find the matrix A first. The convenient choice for k_i in (19) is $0, 2, 4$. The function $\theta(t, q)$ has a zero of order one at $t = 1$ so that $\frac{\theta(t,q)}{t-1}$ is regular. Denote

$$\theta_0(q) := \lim_{t\to 1} \frac{\theta(t,q)}{1-t}.$$

The right-hand-side of fist equation Eq_0 (19), the $k = 0$ case, becomes

$$\lim_{t\to 1} \Psi(t,q)(1-t)^{11} = \frac{a\theta_1 + b\theta_2 + c\theta_3}{\theta_0^{11}}. \tag{21}$$

The second equation Eq_2, corresponding to $k = 2$ [1], contain t-partial derivatives of functions at $t = 1$. Note that θ and ϵ_k satisfy the heat equations:

$$t^2\partial_t^2\theta + t\partial_t\theta = 2q\partial_q\theta, \quad t^2\partial_t^2\epsilon_k + 8t\partial_t\epsilon_k - k(k+7)\epsilon_k = 14q\partial_q\epsilon_k.$$

In addition, linear combinations θ_i of ϵ_k satisfy (17). These equations allow to express t-derivatives of θ_i at $t = 1$ in terms of q-derivatives of θ_i. Thus equations 3 and the results of [4] imply that

$$\theta_0(q) = -(q,q)_\infty^3, \quad \lim_{t\to 1} \partial_t \frac{\theta(t,q)}{1-t} = -\theta_0(q), \quad \lim_{t\to 1} \partial_t^2 \frac{\theta(t,q)}{1-t} = \theta_0 \times \left(\frac{23}{12} + \frac{E_2}{12}\right),$$

$$\lim_{t\to 1} \partial_t^3 \frac{\theta(t,q)}{1-t} = \theta_0 \times \left(-\frac{11}{2} - \frac{1}{2}E_2\right),$$

$$\lim_{t\to 1} \partial_t^4 \frac{\theta(t,q)}{1-t} = \theta_0 \times \left(\frac{1689}{80} + \frac{23}{8}E_2 + \frac{1}{48}E_2^2 - \frac{1}{120}E_4\right),$$

$$\lim_{t\to 1} \partial_t^5 \frac{\theta(t,q)}{1-t} = \theta_0 \times \left(-\frac{1627}{16} - \frac{145}{8}E_2 - \frac{5}{16}E_2^2 + \frac{1}{8}E_4\right).$$

Similarly

$$\theta_1(q) := \theta_1(1,q) = 2(q^3,q^7)_\infty(q^4,q^7)_\infty(q^7,q^7)_\infty, \quad \partial_t\theta_1(1,q) = -7/2\theta_1(q), \quad \partial_t^2\theta_1(1,q) = 16\theta_1(q) + 14q\partial_q\theta_1(q),$$

[1] Eq_1 is proportional to Eq_0

$$\partial_t^3\theta_1(1,q) = -90\theta_1 - 189q\partial_q\theta_1, \quad \partial_t^4\theta_1(1,q) = 600\theta_1 + 2268q\partial_q\theta_1 + 196q^2\partial_q^2\theta_1,$$

$$\theta_2(q) := \theta_2(1,q) = -2(q^2,q^7)_\infty(q^5,q^7)_\infty(q^7,q^7)_\infty, \quad \partial_t\theta_2(1,q) = -7/2\theta_2, \quad \partial_t^2\theta_2(1,q) = 18\theta_2 + 14q\partial_q\theta_2,$$

$$\partial_t^3\theta_2(1,q) = -117\theta_2 - 189q\partial_q\theta_2, \quad \partial_t^4\theta_2(1,q) = 900\theta_2 + 2324q\partial_q\theta_2 + 196q^2\partial_q^2\theta_2,$$

$$\theta_3(q) := \theta_3(1,q) = 2(q,q^7)_\infty(q^6,q^7)_\infty(q^7,q^7)_\infty, \quad \partial_t\theta_3(1,q) = -7/2\theta_3, \quad \partial_t^2\theta_3(1,q) = 22\theta_3 + 14q\partial_q\theta_3,$$

$$\partial_t^3\theta_3(1,q) = -171\theta_3 - 189q\partial_q\theta_3, \quad \partial_t^4\theta_3(1,q) = 1524\theta_3 + 2436q\partial_q\theta_3 + 196q^2\partial_q^2\theta_3.$$

After simplifications Eq_2 and Eq_4 become

$$\begin{aligned}&\lim_{t\to1}\partial_t^2\Psi(t,q)(1-t)^{11} = -\frac{1}{12\theta_0^{11}}\times \\ &\times((11E_2\theta_1 - 168q\theta_1' - 23\theta_1)a + (11E_2\theta_2 - 168q\theta_2' - 47\theta_2)b + (11E_2\theta_3 - 168q\theta_3' - 95\theta_3)c),\end{aligned} \tag{22}$$

$$\begin{aligned}&\lim_{t\to1}\partial_t^4\Psi(t,q)(1-t)^{11} = \frac{1}{240\theta_0^{11}}\times \\ &\times((-18480qE_2\theta_1' + 605E_2^2\theta_1 - 990E_2\theta_1 + 22E_4\theta_1 + 47040q^2\theta_1'' + 58800q\theta_1' + 363\theta_1)a \\ &(-18480qE_2\theta_2' + 605E_2^2\theta_2 - 3630E_2\theta_2 + 22E_4\theta_2 + 47040q^2\theta_2'' + 72240q\theta_2' + 3003\theta_2)b \\ &(-18480qE_2\theta_3' + 605E_2^2\theta_3 - 8910E_2\theta_3 + 22E_4\theta_3 + 47040q^2\theta_3'' + 99120q\theta_3' + 14043\theta_3)c).\end{aligned} \tag{23}$$

The matrix A consists of coefficients of a, b, c from equations (21,22,23).

The vector z components Computation of the vector z from (20) relies on the extrapolation of the results obtained with *Mathematica*. We use that $Z(t,q)$ has presentation as a limit of (11), which makes it possible computation of the q-series $\lim_{t\to1}\partial_t^i Z(t,q)(1-t)^{11}$ with arbitrary precision. Due to integrality of the coefficients, $\lim_{t\to1}\partial_t^i Z_N^{N'}(t,q)(1-t)^{11} \mod q^r$ stabilizes for sufficiently large N and N'. The first computation shows that

$$\begin{aligned}&\frac{1}{12}\lim_{t\to1}Z(t,q)(1-t)^{11} = \\ &= 1 + 22q + 275q^2 + 2530q^3 + 18975q^4 + 122430q^5 + 702328q^6 + 3661900q^7 + 17627775q^8 + \ldots O(q^{20}).\end{aligned}$$

The database http://oeis.org hints that these series is the generating function for the sequence A023020 . a_n is the number of partitions of n into parts of 22 kinds. The generating function coincides with the Taylor expansion of $\frac{1}{(q,q)_\infty^{22}}$.

Another computation with *Mathematica* shows that

$$\frac{1}{98}(q,q)_\infty^{22}\lim_{t\to1}\partial_t^2 Z(t,q)(1-t)^{11} = 1/6 + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + 12q^6 + 8q^7 + 15q^8 + \cdots O(q^{20}).$$

If we drop the term 1/6, the sequence a_n of Taylor coefficients of the remaining series is A000203 . $a(n) = \sigma(n)$, the sum of the divisors of n. The generating function for a_n is $(1 - E_2(q))/24$. The same way we find that

$$(q,q)_\infty^{22}\lim_{t\to1}\partial_t^4 Z(t,q)(1-t)^{11} = \frac{42}{5} - 12E_2(q) + 4E_2^2(q) - \frac{2}{5}E_4 + O(q^{20}).$$

To summarize, the vector z in (20) is conjecturally equal

$$\left(\frac{12}{(q,q)_\infty^{22}}, \frac{20-4E_2(q)}{(q,q)_\infty^{22}}, \frac{\frac{42}{5}-12E_2(q)+4E_2^2(q)-\frac{2}{5}E_4}{(q,q)_\infty^{22}}\right)$$

The matrix A has the determinant

$$\Delta = \frac{8\tilde{\Delta}}{\theta_0^{33}}$$

$$\begin{aligned}\tilde{\Delta} = (&-343q^3\theta_3\theta_1''\theta_2' + 343q^3\theta_2\theta_1''\theta_3' + 343q^3\theta_3\theta_1'\theta_2'' \\ &- 343q^3\theta_2\theta_1'\theta_3'' - 343q^3\theta_1\theta_2''\theta_3' + 343q^3\theta_1\theta_2'\theta_3'' + 98q^2\theta_2\theta_3\theta_1'' \\ &+ 98q^2\theta_3\theta_1'\theta_2' - 294q^2\theta_2\theta_1'\theta_3' - 147q^2\theta_1\theta_3\theta_2'' + 196q^2\theta_1\theta_2'\theta_3' \\ &+ 49q^2\theta_1\theta_2\theta_3'' + 42q\theta_2\theta_3\theta_1' - 126q\theta_1\theta_3\theta_2' + 84q\theta_1\theta_2\theta_3' + 6\theta_1\theta_2\theta_3).\end{aligned}$$

It is not hard to check with *Mathematica* that

$$\tilde{\Delta} = -48(q,q)_\infty^{15} + O(q^{300}).$$

We conjecture that this is an exact equality. We use it and the Kramer's rule to derive from (20) the formulas (6,7,8).

4 Some consistency checks

Note that by construction the function Ψ (5) satisfies equations (13,14). $Z(t,q)$ is the solution of the same set of equations . The q-expansion of Ψ is

$$\Psi(t,q) = -\frac{t^3+5t^2+5t+1}{(t-1)^{11}}$$

$$-\frac{q\left(46t^3+86t^2+86t+46\right)}{(t-1)^{11}}$$

$$+\frac{q^2\left(t^{11}-11t^{10}+55t^9-181t^8-567t^7-947t^6-947t^5-567t^4-181t^3+55t^2-11t+1\right)}{(t-1)^{11}t^4}$$

$$+\frac{2q^3 (8t^{13}-65t^{12}+195t^{11}-143t^{10}-1011t^9-2657t^8-3917t^7-3917t^6-2657t^5-1011t^4-143t^3+195t^2-65t+}{(t-1)^{11}t^5}$$

$$-\frac{q^4 (-126t^{15}+794t^{14}-1491t^{13}-559t^{12}+3597t^{11}+18745t^{10}+38767t^9+54123t^8+54123t^7+38767t^6+18745t^5+}{(t-1)^{11}t^6}$$

$$\frac{2q^5 (336t^{17}-1662t^{16}+1810t^{15}+2173t^{14}+1337t^{13}-21131t^{12}-65159t^{11}-122387t^{10}-162607t^9-162607t^8-12238}{(t-1)^{11}t^7}$$

$$+O(q^6)$$

It agrees with the expansion from [2].

Another interesting consistency check gives comparison of the functions $Z(t,q)$ and $\Psi(t,q)$ at $t=-1$. Literal comparison is not very fruitful because by virtue of (17) $Z(-1,q)=\Psi(-1,q)=0$. To get a nonzero result, we used *Mathematica* to compute the derivatives $\partial_t Z(-1,q)$ and $\partial_t \Psi(-1,q)$. Two values agree by giving

$$-1024(q,q)_\infty^{22}\partial_t Z(-1,q) = -1024(q,q)_\infty^{22}\partial_t \Psi(-1,q) = 1-48q+1104q^2-16192q^3+170064q^4-1362336q^5+8662720q^6-44981376q^7+195082320q^8+O(q^9). \tag{24}$$

The coefficients a_n (up to a sign) coincide with the sequence A000156 . a_n, according to the database, is the number of ways of writing n as a sum of 24 squares. This is why it is very plausible that the series (24) is the expansion of

$$\left(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}\right)^{24} = (1-q)^{24}(q,q^2)_\infty^{24}(q^2,q^2)_\infty^{24}(q^3,q^2)_\infty^{24}.$$

5 Functions θ_k

$\theta_k, k=1,2,3$ are closely related to Rogers-Selberg functions (see e.g. [8],[3][6]). They satisfy

$$A(q) := \sum_{n\geq 0}\frac{q^{2n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})(1+q)(1+q^2)\cdots(1+q^{2n})} = \frac{(q^3,q^7)_\infty(q^4,q^7)_\infty(q^7,q^7)_\infty}{(q^2,q^2)_\infty} = \frac{\theta_1}{2(q^2,q^2)_\infty},$$

$$B(q) := \sum_{n\geq 0}\frac{q^{2n^2+2n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})(1+q)(1+q^2)\cdots(1+q^{2n})} = \frac{(q^2,q^7)_\infty(q^5,q^7)_\infty(q^7,q^7)_\infty}{(q^2,q^2)_\infty} = -\frac{\theta_2}{2(q^2,q^2)_\infty},$$

$$C(q) := \sum_{n\geq 0}\frac{q^{2n^2+2n}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})(1+q)(1+q^2)\cdots(1+q^{2n+1})} = \frac{(q,q^7)_\infty(q^6,q^7)_\infty(q^7,q^7)_\infty}{(q^2,q^2)_\infty} = \frac{\theta_3}{2(q^2,q^2)_\infty}.$$

For a list known identities these functions obey see [6].

6 $Z(t,q,g)$ in the general case

Some of the above arguments extend to $Z(t,q,g)$. In particular the divisor of poles of $Z(t,q,g)$ satisfies equations

$$1-q^n t v_\alpha(z) = 0$$

where $v_{\hat\alpha}(z)$ are the weights of the spinor representation. In order to be more explicit recall (see see e.g. [7]) that coordinates λ^α on the spinor representation of Spin(10) can be labelled by elements of the set

$$\alpha \in \mathrm{E} := \{(0),(ij),(k) | 1\leq i<j\leq 5, 1\leq k\leq 5\}$$

Let $\widetilde{\mathbf{T}^5}$ be the two sheeted cover of the maximal torus $\mathbf{T}^5 \subset \mathrm{SO}(10)$ and $z = (z_1, \ldots, z_5)$ be the image of g under projection $\widetilde{\mathbf{T}^5} \to \mathbf{T}^5$. The action Π of $g \in \widetilde{\mathbf{T}^5}$ on λ^α is given by the formula

$$\begin{aligned}
&\Pi(g)\lambda^\alpha v_\alpha(z)\lambda^\alpha, \\
&\text{or in more details:} \\
&\Pi(g)\lambda^{(0)} = \det{}^{-\frac{1}{2}}(z)\lambda^{(0)}, \\
&\Pi(g)\lambda^{(ij)} = \det{}^{-\frac{1}{2}}(z) z_i z_j \lambda^{(ij)}, \\
&\Pi(g)\lambda^{(k)} = \det{}^{\frac{1}{2}}(z) z_k^{-1}\lambda^{(k)}, \\
&\det{}^{\frac{1}{2}}(z) = \sqrt{z_1 \cdots z_5}.
\end{aligned} \tag{25}$$

Introduce the product

$$\theta_{\mathrm{E}}(t, q, z) := \prod_{\alpha \in \mathrm{E}} \theta(t v_\alpha(z), q)$$

If the conjecture about the structure of the poles of $Z(t, q, z)$ is correct, then

$$\Xi(t, q, z) := t^6 Z(t, q, z)\theta_{\mathrm{E}}(t, q, z)$$

is an analytic function for $|q| < 1$ and $t, z \in \mathbb{C}^\times \times \widetilde{\mathbf{T}^5}$. Equations (13) (15) imply that $\Theta(t, q, z)$ satisfies

$$\Xi(qt, q, z) = \frac{1}{t^{12} q^{12}} \Xi(t, q, z)$$

That is $\Xi(t, q, z)$ is a section of $\mathcal{L}^{\otimes 12}$. Let us fix a basis

$$\eta_k(t, q) = \sum_{n \in \mathbb{Z}} q^{\frac{12n(n+1)}{2} + kn} t^{12n+k}, k = 0, \ldots, 11$$

in the space of global sections of $\mathcal{L}^{\otimes 12}$. Functions η_k satisfy

$$\eta_k(qt, q) = 1/(tq)^{12}\eta_k(t, q), \quad \eta_k(1/t, q) = q^{12-2k} t^{12} \eta_{12-k}(t, q), \quad \eta_k(t, q) = q^{k+12}\eta_{k+12}(t, q)$$

The function $\Xi(t, q, z)$ is a linear combination

$$\Xi(t, q, z) = \sum_{k=0}^{11} c_k(q, z)\eta_k(t, q).$$

(17) implies that

$$\begin{gathered}
c_k(q, z) = -q^{12-2k} c_{12-k}(q, z^{-1}), k \neq 0, 6, \\
c_0(q, z) = -c_0(q, z^{-1}), \quad c_6(q, z) = -c_6(q, z^{-1})
\end{gathered}$$

Determination of the coefficients $c_i(q, z)$ is more difficult than in the case $z = 1$ and will be postponed for the future publications.

Note that after specialization $g = 1$ $\Xi(t, q, 1) = \Theta(t, q)\theta(t, q)^5$ and $\theta_{\mathrm{E}}(t, q, 1) = \theta(t, q)^{16}$ giving as a fraction the function $Z(t, q)$.

7 Concluding remarks

Partition function $Z_{\mathcal{X}}$ for $\mathcal{X}=\mathcal{Q}$ a smooth affine quadric of dimension $n-1$ is known [1]:

$$Z_{\mathcal{Q}}(t,q)=\frac{1-t^2}{(1-t)^n}\frac{(qt^2,q)_\infty(qt^{-2},q)_\infty}{(qt,q)_\infty^n(qt^{-1},q)_\infty^n}$$

It satisfies

$$\frac{Z_{\mathcal{Q}}(qt,q)}{Z_{\mathcal{Q}}(t,q)}=(-1)^n t^{n-4}q^{-1}, \tag{26}$$

$$\frac{Z_{\mathcal{Q}}(t^{-1},q)}{Z_{\mathcal{Q}}(t,q)}=-(-t)^{n-2}. \tag{27}$$

Functions $Z_{\mathcal{C}}(t,q), Z_{\mathcal{Q}}(t,q)$ have some common features. In both cases $\lim_{t\to 1} Z_{\mathcal{X}}(t,q)(1-t)^{\dim\mathcal{X}}=c_{\mathcal{X}}\frac{1}{(q,q)^{\dim\mathcal{X}}}$, where $c_{\mathcal{X}}$ is some constant. Functions $Z_{\mathcal{X}}(t,q)$ have poles of multiplicity $\dim\mathcal{X}$ at points $\{q^k\}$.

To further extend the analogy we need to digress. The spaces of polynomial maps $\mathbb{C}\to\mathcal{X}$ of degree N is a cone over the space of Drinfeld's quasimaps $QMaps_N(\mathcal{X})$ to projectivization of $\mathcal{X}$. The space $QMaps_N(\mathcal{X})$ is not smooth but still has a well defined line bundle of algebraic volume forms $\mathcal{K}$. $\mathcal{K}^*=\mathcal{O}(a(\mathcal{X})+Nb(\mathcal{X}))$ in for some constants $a(\mathcal{X}), b(\mathcal{X})$. As usual $\mathcal{O}(n)$ is the power of the tautological line bundle. Exponents of t in (13) and (26) coincide with $b(\mathcal{C})$ and $b(\mathcal{Q})$. Thus functions $\Theta_{\mathcal{C}}(t,q)$ and $\Theta_{\mathcal{Q}}(t,q)$ are sections of $\mathcal{L}^{\dim\mathcal{C}-a(\mathcal{C})}=\mathcal{L}^7$ and $\mathcal{L}^{\dim\mathcal{Q}-a(\mathcal{Q})}=\mathcal{L}^3$ respectively. Finally $P_{\mathcal{X}}(t)=\lim_{q\to 0} Z_{\mathcal{X}}(t,q)$ is the classical Poincaré series of the algebra of homogeneous functions on $\mathcal{X}$.

It is tempting to say that this is a common features of an elliptic generalization of Poincaré series of an algebra of functions $\mathcal{X}$ that should exist for a class of conical varieties whose members are $\mathcal{C}$ and $\mathcal{Q}$. This class contains in the class of local conical Calabi-Yao varieties $\mathcal{X}$, whose base $B(\mathcal{X})$ is Fano of sufficiently large index. In this generalization

$$Z_{\mathcal{X}}(t,q)=\frac{\Theta_{\mathcal{X}}(t,q)}{t^{l(\mathcal{X})}\theta(t,q)^{\dim\mathcal{X}}}.$$

$l(\mathcal{X})\in\mathbb{Z}^{>0}$, $\Theta_{\mathcal{X}}$ is a section of $\mathcal{L}^{\dim\mathcal{X}-a(\mathcal{X})}$. The denominator in this formula is similar to the denominator in the Kac formula for the character of an integrable representations of an affine Lie algebra $\hat{\mathfrak{g}}$ at a positive level.

It appears that the first nontrivial Laurent coefficients of $Z_{\mathcal{Q}_n}(t,q)$ for a quadric at $t=1$ can be expressed through algebraic combinations of $(q,q)_\infty, E_2, E_4$ and probably E_6 . The formulas are similar to the ones that have already appeared in Section 3:

$$\lim_{t\to 1} Z_{\mathcal{Q}_n}(t,q)(1-t)^{n-1}=2(q,q)_\infty^{2-2n}$$

$$\lim_{t\to 1}\partial_t(Z_{\mathcal{Q}_n}(t,q)(1-t)^{n-1})=(q,q)_\infty^{2-2n}$$

$$\lim_{t\to 1}\partial_t^2(Z_{\mathcal{Q}_n}(t,q)(1-t)^{n-1})=(q,q)_\infty^{2-2n}\frac{(n-4)(1-E_2)}{6}$$

$$\lim_{t\to 1}\partial_t^3(Z_{\mathcal{Q}_n}(t,q)(1-t)^{n-1})=(q,q)_\infty^{2-2n}\frac{(4-n)(1-E_2)}{4}$$

$$\lim_{t\to 1}\partial_t^4(Z_{\mathcal{Q}_n}(t,q)(1-t)^{n-1})=(q,q)_\infty^{2-2n}\left(\frac{(n-4)^2E_2^2}{24}-\frac{(n-4)(n+6)E_2}{12}+\frac{(n-16)E_4}{60}+\frac{5n^2+58n-288}{120}\right)$$

Introduce an increasing multiplicative filtration on the algebra of function generated by E_2, E_4, E_6. The n-th derivative of $Z_{\mathcal{Q}_n}(t,q)(1-t)^{n-1}$ at $t=1$ is a multiple of $(q,q)_\infty^{-2\dim\mathcal{Q}}$. The factor belongs to n filtration space of the algebra. This parallels between $Z_{\mathcal{Q}_n}$ and $Z_{\mathcal{C}}$ suggests that this also holds for a more general $\mathcal{X}$. It raises a question whether coefficients of the E_2, E_4, E_6-monomials in the above formulas can be computed in terms of the characteristic classes of some bundles on the base of the cone $\mathcal{X}$.

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