# Two-log-convexity of the Catalan-Larcombe-French sequence

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**Abstract.** The Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 0}$ arises in a series expansion of the complete elliptic integral of the first kind. It has been proved that the sequence is log-balanced. In the paper, by exploring a criterion due to Chen and Xia for testing 2-log-convexity of a sequence satisfying three-term recurrence relation, we prove that the new sequence $\{P_n^2 - P_{n-1}P_{n+1}\}_{n\geq 1}$ are strictly log-convex and hence the Catalan-Larcombe-French sequence is strictly 2-log-convex.

**Keywords:** log-balanced sequence; log-convex sequence; log-concave sequence; the Catalan-Larcombe-French sequence; three-term recurrence

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## 1 Introduction

This paper is concerned with the log-behavior of the Catalan-Larcombe-French sequence. To begin with, let us recall that a sequence $\{z_n\}_{n\geq 0}$ is said to be log-concave if

$$z_n^2 \geq z_{n+1}z_{n-1}, \quad for \;\; n \geq 1, \tag{1.1}$$

and it is log-convex if

$$z_n^2 \leq z_{n+1}z_{n-1}, \quad for \;\; n \geq 1. \tag{1.2}$$

Meanwhile, the sequence $\{z_n\}_{n\geq 0}$ is called strictly log-concave(resp. log-convex) if the inequality in (1.1)(resp. (1.2)) is strict for all $n \geq 1$. We call $\{z_n\}_{n\geq 0}$ log-balanced if the sequence itself is log-convex while $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave.

Given a sequence $A = \{z_n\}_{n\geq 0}$, define the operator $\mathcal{L}$ by

$$\mathcal{L}(A) = \{s_n\}_{n\geq 0},$$

where $s_n = z_{n-1}z_{n+1} - z_n^2$ for $n \geq 1$. We say that $\{z_n\}_{n\geq 0}$ is $k$-log-convex(*resp. $k$-log-concave*) if $\mathcal{L}^j(A)$ is log-convex(*resp. log-concave*) for all $j = 0, 1, \cdots, k-1$, and that $A = \{z_n\}_{n\geq 0}$ is $\infty$-log-convex(*resp. $\infty$-log-concave*) if $\mathcal{L}^k(A)$ is log-convex(*resp. log-concave*) for any $k \geq 0$. Similarly, we can define strict $k$-log-concavity or strict $k$-log-convexity of a sequence.

It is worthy to mention that besides that they are fertile sources of inequalities, log-convexity and log-concavity have many applications in some different mathematical disciplines, such as geometry, probability theory, combinatorics and so on. See the surveys due to Brenti [1] and Stanley [10] for more details. Additionally, it is clear that the log-balancedness implies the log-convexity and a sequence $\{z_n\}_{n\geq 0}$ is log-convex(resp. log-concave) if and only if its quotient sequence $\{\frac{z_n}{z_{n-1}}\}_{n\geq 1}$ is nondecreasing(resp. nonincreasing). It is also known that the quotient sequence of a log-balanced sequence does not grow too fast. Therefore, log-behavior are important properties of combinatorial sequences and they are instrumental in obtaining the growth rate of a sequence. Hence the log-behaviors of a sequence deserves to be investigated.

In this paper, we investigate the 2-log-behavior of the Catalan-Larcombe-French sequence, denoted by $\{P_n\}_{n\geq 0}$, which arises in connection with series expansions of the complete elliptic integrals of the first kind [6, 12]. To be precise, for $0 < |c| < 1$,

$$\int_0^{\pi/2} \frac{1}{\sqrt{1 - c^2 \sin^2\theta}} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \left( \frac{1 - \sqrt{1 - c^2}}{16} \right)^n P_n.$$

Furthermore, the numbers $P_n$ can be written as the following sum:

$$P_n = 2^n \sum_{k=0}^{n} (-4)^k \binom{n-k}{k} \binom{2n-2k}{n-k}^2,$$

see [11, A05317]. Besides, the number $P_n$ satisfies three-term recurrence relations [12] as follows:

$$(n+1)^2 P_{n+1} = 8(3n^2 + 3n + 1)P_n - 128n^2 P_{n-1}, \ for \quad n \geq 1, \tag{1.3}$$

with the initial values $P_0 = 1$ and $P_1 = 8$.

Recently, Zhao [12] studied the log-behavior of the Catalan-Larcombe-French sequence and proved that the sequence $\{P_n\}_{n\geq 0}$ is log-balanced. What's more, the Catalan-Larcombe-French sequence has many interesting properties and the reader can refer [6, 7, 12]. In the sequel, we study the 2-log-behavior of the sequences and obtain the following result.

**Theorem 1.1.** *The Catalan-Larcombe-French sequence* $\{P_n\}_{n\geq 0}$ *is strictly 2-log-convex, that is,*

$$\mathcal{P}_n^2 < \mathcal{P}_{n-1}\mathcal{P}_{n+1}, \tag{1.4}$$

*where* $\mathcal{P}_n = P_n^2 - P_{n-1}P_{n+1}$.

We will give our proof of Theorem 1.1 in the third section by utilizing a testing criterion, which is proposed by Chen and Xia [4].

To make this paper self-contained, let us recall their criterion.

**Theorem 1.2** (Chen and Xia[4]). *Suppose* $\{z_n\}_{n\geq 0}$ *is a positive log-convex sequence that satisfies the following three-term recurrence relation*

$$z_n = a(n)z_{n-1} + b(n)z_{n-2}, \; for \;\; n \geq 2. \tag{1.5}$$

*Let*

$$\begin{aligned}
c_0(n) =& - b^2(n+1)[a^2(n+2) + b(n+1) - a(n+2)a(n+3) - b(n+3)];\\
c_1(n) =& b(n+1)[2a(n+2)b(n+1) + 2a(n+3)a(n+2)a(n+1)\\
&+ a(n+3)b(n+2) + 2a(n+1)b(n+3) - 2a^2(n+2)a(n+1)\\
&- 2a(n+2)b(n+2) - 3a(n+1)b(n+1)];\\
c_2(n) =& 4a(n+1)a(n+2)b(n+1) + 2b(n+1)b(n+2) + a^2(n+1)a(n+2)a(n+3)\\
&+ a(n+1)a(n+3)b(n+2) + a^2(n+1)b(n+3) - 3a^2(n+1)b(n+1)\\
&- a(n+3)a(n+2)b(n+1) - a^2(n+2)a^2(n+1) - b(n+3)b(n+1)\\
&- 2a(n+2)a(n+1)b(n+2) - b^2(n+2);\\
c_3(n) =& 2a^2(n+1)a(n+2) + 2a(n+1)b(n+2) - a(n+1)b(n+3) - a^3(n+1)\\
&- a(n+1)a(n+2)a(n+3) - a(n+3)b(n+2);
\end{aligned}$$

*and*

$$\Delta(n) = 4c_2^2(n) - 12c_1(n)c_3(n).$$

*Assume that* $c_3(n) < 0$ *and* $\Delta(n) \geq 0$ *for all* $n \geq N$, *where* $N$ *is a positive integer. If there exist* $f_n$ *and* $g_n$ *such that, for all* $n \geq N$,

*(I)* $f_n \leq \frac{z_n}{z_{n-1}} \leq g_n$;

*(II)* $f_n \geq \frac{-2c_2(n)-\sqrt{\Delta(n)}}{6c_3(n)}$;

*(III)* $c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n) \geq 0$,

*then we see that* $\{z_n\}_{n\geq N}$ *is 2-log-convex, that is, for* $n \geq N$,

$$(z_{n-1}z_{n+1} - z_n^2)(z_{n+1}z_{n+3} - z_{n+2}^2) > (z_nz_{n+2} - z_{n+1}^2)^2.$$

With respect to the theory in this field, it should be mentioned that the log-behavior of a sequence which satisfies a three-term recurrence has been extensively studied; see Liu and Wang[9], Chen et al. [2, 3], Liggett [8], Došlić[5], etc.

## 2 Bounds for $\frac{P_n}{P_{n-1}}$

Before proving Theorem 1.1, we need the following two lemmas.

**Lemma 2.1.** *Let*

$$f_n = \frac{232n}{15(n+2)},$$

*and* $P_n$ *be the sequence defined by the recurrence relation* (1.3). *Then we have, for all* $n \geq 1$,

$$\frac{P_n}{P_{n-1}} > f_n. \tag{2.6}$$

*Proof.* We proceed the proof by induction. First note that, for $n = 1$ and $n = 2$, we have $\frac{P_1}{P_0} = 8 > \frac{232}{45}$ and $\frac{P_2}{P_1} = 10 > \frac{464}{60}$. Assume that the inequality (2.6) is valid for $n \leq k$. We will show that

$$\frac{P_{k+1}}{P_k} > f_{k+1}.$$

By the recurrence (1.3), we have

$$\begin{aligned}
\frac{P_{k+1}}{P_k} &= \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{P_{k-1}}{P_k} \\
&> \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{1}{f_k} \\
&= \frac{8(57k^2+27k+29)}{29(k+1)^2} \\
&> f_{k+1},
\end{aligned}$$

in which the last inequality follows by

$$\frac{8(57k^2+27k+29)}{29(k+1)^2} - f_{k+1} = \frac{8(14k^3+447k^2-873k+464)}{435(k+1)^2(k+3)} > 0,$$

for all $k \geq 1$. This completes the proof. ∎

**Lemma 2.2.** *Let*

$$g_n = 16 - \frac{16}{n} - \frac{16}{n^3},$$

*and* $P_n$ *be the sequence defined by the recurrence relation* (1.3). *Then we have, for all* $n \geq 6$,

$$\frac{P_n}{P_{n-1}} \leq g_n. \tag{2.7}$$

*Proof.* First note that, for $n = 6$, we have $\frac{P_6}{P_5} = \frac{3562}{269} < g_6 = \frac{358}{27}$. Assume that for $k \geq 6$, the inequality (2.7) is valid for $n \leq k$. We will show that

$$\frac{P_{k+1}}{P_k} < g_{k+1}.$$

By the recurrence (1.3), we have

$$\begin{aligned} \frac{P_{k+1}}{P_k} &= \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{P_{k-1}}{P_k} \\ &< \frac{8(3k^2+3k+1)}{(k+1)^2} - \frac{128k^2}{(k+1)^2}\frac{1}{g_k} \\ &= \frac{8(2k^5-2k^3-4k^2-3k-1)}{(k+1)^2(k^3-k^2-1)}. \end{aligned} \tag{2.8}$$

Consider

$$\frac{8(2k^5-2k^3-4k^2-3k-1)}{(k+1)^2(k^3-k^2-1)} - g_{k+1} = -\frac{8(5k^2+2k+3)}{(k+1)^3(k^3-k^2-1)} < 0, \tag{2.9}$$

for all $k \geq 2$. So we see that for all $n \geq 6$, the inequality (2.7) holds by induction.

∎

With the above lemmas in hand, we are now in a position to prove our main result in the next section.

# 3 Proof of Theorem 1.1

In this section, by using the criterion of Theorem 1.2, we can show that the Catalan-Larcombe-French sequence is strictly 2-log-convex.

To begin with, the following lemma, which is obtained by Zhao [12], is indispensable for us.

**Lemma 3.1** (Zhao[12])**.** *The Catalan-Larcombe-French sequence is log-balanced.*

By the definition of log-balanced sequence, we know that $\{P_n\}_{n\geq 0}$ is log-convex.

*Proof of Theorem 1.1.* By *Lemma 3.1*, it suffices for us to show that

$$(P_{n-1}P_{n+1} - P_n^2)(P_{n+1}P_{n+3} - P_{n+2}^2) - (P_nP_{n+2} - P_{n+1}^2)^2 > 0.$$

According to the recurrence relation (1.3), we see that

$$\begin{aligned} a(n) =& \frac{8(3n^2-3n+1)}{n^2}; \\ b(n) =& -\frac{128(n-1)^2}{n^2}. \end{aligned}$$

By taking $a(n), b(n)$ in $c_0, \cdots, c_3$, we can obtain

$$
\begin{aligned}
c_3(n) = & -\frac{512}{(n+1)^6(n+2)^2(n+3)^2}\big(3n^8+5n^7-27n^6-32n^5+112n^4\\
& +234n^3+177n^2+63n+9\big)\\
& < 0,
\end{aligned}
$$

for all $n \geq 1$. Besides, we have to verify that for some positive integer $N$, the conditions $(II)$ and $(III)$ in *Theorem 1.2* hold for all $n \geq N$. That is,

$$
f_n \geq \frac{-2c_2(n) - \sqrt{\Delta(n)}}{6c_3(n)}; \tag{3.10}
$$

$$
c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n) \geq 0. \tag{3.11}
$$

Let

$$
\delta(n) = -6c_3(n)f_n - 2c_2(n)
$$

and

$$
f(g_n) = c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n).
$$

To show (3.10), it is equivalent to show that, for some positive integers $N$, $\delta(n) \geq 0$ and $\delta^2(n) \geq \Delta(n)$. By calculating, we easily find that, for all $n \geq 1$,

$$
\begin{aligned}
\delta(n) = & \frac{8192}{5(n+1)^6(n+2)^4(n+3)^2}\big(32n^{10}+129n^9+472n^8+3556n^7+12157n^6\\
& +17632n^5+10550n^4+1293n^3-1500n^2-798n-135\big)\\
& \geq 0,
\end{aligned}
$$

and for all $n \geq 3$,

$$
\begin{aligned}
\delta^2(n) - \Delta(n) = & \frac{67108864n}{25(n+3)^4(n+2)^7(n+1)^{12}}\big(699n^{18}+2158n^{17}+6983n^{16}\\
& +97994n^{15}+155517n^{14}-1256916n^{13}-3302168n^{12}+\\
& 5191280n^{11}+25505142n^{10}+14486584n^9-63005002n^8\\
& -153766236n^7-178037517n^6-131841558n^5-68012397n^4\\
& -24910146n^3-6269211n^2-975888n-70470\big)\\
& \geq 0.
\end{aligned}
$$

Thus, take $N = 3$ and for all $n \geq N$, we have $\delta(n) \geq 0, \delta^2(n) \geq \Delta(n)$, which follows from the inequality (3.10). We show the inequality (3.11) for some positive integer $M$. Note that, by *Lemma 2.2* and some calculations, we have

$$
\begin{aligned}
f(g_n) = & c_3(n)g_n^3 + c_2(n)g_n^2 + c_1(n)g_n + c_0(n)\\
= & \frac{1048576}{n^9(n+1)^6(n+2)^4(n+3)^2}\big(54n^{15}+378n^{14}+916n^{13}+644n^{12}-1529n^{11}\\
& -5340n^{10}-8383n^9-7416n^8-2284n^7+4156n^6+7969n^5+7688n^4\\
& +4953n^3+2154n^2+576n+72\big).
\end{aligned}
$$

Take $M = 6$, it is not difficult to verify that for all $n \geq M$,

$$f(g_n) > 0.$$

Let $N_0 = max\{N, M\} = 6$, then for all $n \geq 6$, all of the above inequalities hold. By *Lemma 3.1 and Theorem 1.2*, the Catalan-Larcombe-French sequence $\{P_n\}_{n\geq 6}$ is strictly 2-log-convex for all $n \geq 6$. What is more, one can easily test that these numbers $\{P_n\}_{0\leq n\leq 8}$ also satisfy the property of 2-log-convexity by simple calculations. Therefore, the whole sequence $\{P_n\}_{n\geq 0}$ is strictly 2-log-convex. This completes the proof. ∎

It deserves to be mentioned that by considerable calculations and plenty of verifications, the following conjectures should be true.

**Conjecture 3.2.** *The Catalan-Larcombe-French sequence is $\infty$-log-convex.*

**Conjecture 3.3.** *The quotient sequence $\{\frac{P_n}{P_{n-1}}\}_{n\geq 1}$ of the Catalan-Larcombe-French sequence is log-concave, equivalently, for all $n \geq 2$,*

$$P_{n-2}P_n^3 \geq P_{n+1}P_{n-1}^3.$$

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