FIVE SUBSETS OF PERMUTATIONS ENUMERATED AS WEAK SORTING PERMUTATIONS

DAVID CALLAN AND TOUFIK MANSOUR

Abstract. We show that the number of permutations of $\{1, 2, \ldots, n\}$ that avoid any one of five specific triples of 4-letter patterns is given by sequence A111279 in OEIS, which is known to count weak sorting permutations. By numerical evidence, there are no other (non-trivial) triples of 4-letter patterns giving rise to this sequence. We make use of a variety of methods in proving our result, including recurrences, the kernel method, direct counting, and bijections.

Keywords: pattern avoidance, Wilf-equivalence, kernel method, weak sorting permutations

2010 Mathematics Subject Classification: 05A15, 05A05

1. Introduction

Let $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ and $\tau \in S_k$ be two permutations. We say that π *contains* τ if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ is order-isomorphic to τ; in such a context τ is usually called a *pattern*. We say that π *avoids* τ, or is τ-*avoiding*, if no such subsequence exists. The set of all τ-avoiding permutations in S_n is denoted $S_n(\tau)$. For an arbitrary finite collection of patterns T, we say that π *avoids* T if π avoids every $\tau \in T$; the corresponding subset of S_n is denoted $S_n(T)$. Two sets of patterns T and T' are said to be *Wilf-equivalent* if their avoiders have the same counting sequence, that is, if $|S_n(T)| = |S_n(T')|$ for all $n \ge 0$. In the context of pattern avoidance, a *symmetry class* refers to an orbit of the dihedral group of order eight generated by the operations reverse, complement, and inverse acting entrywise on sets of patterns. Two pattern sets in the same symmetry class obviously have equinumerous avoiders, that is, are trivially Wilf-equivalent.

The weak sorting permutations are those that avoid 3241, 3421 and 4321 [1], counted by sequence A111279 in [2]. We will show that there are precisely five symmetry classes of triples of 4-letter patterns counted as the weak sorting permutations. Representatives Π_j, $1 \le j \le 5$, of these five classes are listed in Theorem 1 below. The weak sorting triple 3241, 3421, 4321 is in the same symmetry class as Π_1. (Our proof for Π_1 is different from that in [1] for the weak sorting triple and is included because similar methods are used for Π_2 and Π_3.) A computer check of initial terms shows that no other symmetry class of triples of 4-letter patterns has this counting sequence.

Theorem 1 (Main Theorem). *Define*

$$\Pi_1 = \{1234, 1243, 1342\}, \qquad \Pi_2 = \{1243, 1324, 1342\}, \qquad \Pi_3 = \{1324, 1342, 1432\}$$
$$\Pi_4 = \{2314, 3214, 4213\}, \qquad \Pi_5 = \{3214, 3241, 4213\}\,.$$

Then, for all $j = 1, 2, 3, 4, 5$,

$$\sum_{n\geq 0} \#S_n(\Pi_j)x^n = \frac{1-5x+(1+x)\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}}\,. \tag{1}$$

2. Proof of main theorem

To prove our main theorem, we find an explicit formula for the generating function $\sum_{n\geq 0}\#S_n(\Pi_j)x^n$, where $j = 1, 2, 3, 4, 5$. Furthermore, for the fifth class, Π_5, we give an explicit formula for the number of members of the set $S_n(\Pi_5)$.

2.1. **Class 1.** $\Pi_1 = \{1234, 1243, 1342\}$. Let $A_n = S_n(\Pi_1)$. Define $a_n = \#A_n$ and $a_n(i_1, \dots, i_s)$ to be the number of permutations $\pi = \pi_1 \cdots \pi_n \in A_n$ such that $\pi_1 \cdots \pi_s = i_1 \cdots i_s$. Then we have the following recurrence.

Lemma 2. *Define $b_n(i) = a_n(i, n-1)$. For all $1 \leq i \leq n-3$,*

$$\begin{aligned} a_n(i) &= a_{n-1}(i) + \cdots + a_{n-1}(1) + b_n(i),\\ b_n(i) &= b_{n-1}(i) + \cdots + b_{n-1}(1) \end{aligned}$$

with $a_n(n-2) = a_n(n-1) = a_n(n) = a_{n-1}$, $b_n(n-1) = 0$ and $b_n(n-2) = b_n(n) = a_{n-2}$.

Proof. By the definitions, $a_n(n) = a_n(n-1) = a_n(n-2) = a_{n-1}$, $b_n(n-1) = 0$ and $b_n(n-2) = b_n(n) = a_{n-2}$. If $1 \leq i \leq n-2$, then

$$\begin{aligned} a_n(i) &= \sum_{j=1}^{i-1} a_n(i,j) + \sum_{j=i+1}^{n} a_n(i,j) = \sum_{j=1}^{i-1} a_{n-1}(i) + a_n(i,n) + b_n(i)\\ &= \sum_{j=1}^{i} a_{n-1}(i) + b_n(i). \end{aligned}$$

Also,

$$b_n(i) = \sum_{j=1}^{i-1} a_n(i, n-1, j) + \sum_{j=i+1}^{n-2} a_n(i, n-1, j) + a_n(i, n-1, n).$$

By the definitions, $a_n(i, n-1, n) = 0$ (the permutations in the question have subsequence $i, n-1, n, n-2$ which is order isomorphic to 1342. Let $\pi = i(n-1)j\pi' \in A_n$ with $i+1 \leq j \leq n-3$, since π avoids 1234 and 1342, we see that $a_n(i, n-1, j) = 0$. Clearly, $a_n(i, n-1, n-2) = a_{n-1}(i, n-2) = b_{n-1}(i)$. Thus,

$$b_n(i) = \sum_{j=1}^{i-1} a_n(i, n-1, j) + b_{n-1}(i).$$

Note that $\pi = i(n-1)j\pi' \in A_n$ with $1 \leq j \leq i$ if and only if $j(n-2)\pi'' \in A_{n-1}$, where π'' is a word obtained from π' by decreasing each letter greater than i by 1. Hence, $a_n(i, n-1, j) = a_{n-1}(j, n-2)$, for all $j = 1, 2, \dots, i-1$. In other words, $b_n(i) = \sum_{j=1}^{i} b_{n-1}(j)$, as required. $\square$

Define $A_n(v)=\sum_{i=1}^{n}a_n(i)v^{i-1}$ and $B_n(v)=\sum_{i=1}^{n}b_n(i)v^{i-1}$. Then by multiplying the recurrence relations in Lemma 2 by v^{i-1}, we obtain

$$\begin{aligned}\sum_{i=1}^{n-3}a_n(i)v^{i-1}&=\sum_{i=1}^{n-3}\sum_{j=1}^{i}a_{n-1}(j)v^{i-1}+\sum_{i=1}^{n-3}b_n(i)v^{i},\\ \sum_{i=1}^{n-3}b_n(i)v^{i-1}&=\sum_{i=1}^{n-3}\sum_{j=1}^{i}b_{n-1}(j)v^{i-1},\end{aligned}$$

which, by the initial conditions, gives that for $n\geq 3$,

$$\begin{aligned}A_n(v)&=\frac{1}{1-v}(A_{n-1}(v)-v^nA_{n-1}(1))+B_n(v)-v^{n-1}A_{n-2}(1),\\ B_n(v)&=\frac{1}{1-v}(B_{n-1}(v)-v^{n-3}B_{n-1}(1))+v^{n-3}A_{n-3}(1)+v^{n-1}A_{n-2}(1)+v^{n-3}A_{n-2}(1).\end{aligned}$$

By direct calculations, we have $A_0(v)=A_1(v)=1$, $A_2(v)=1+v$, $B_0(v)=B_1(v)=0$ and $B_2(v)=v$.

Let $A(x,v)=\sum_{n\geq 0}A_n(v)x^n$ and $B(x,v)=\sum_{n\geq 0}B_n(v)x^n$ be the generating functions for the sequences $A_n(v)$ and $B_n(v)$, respectively. By multiplying by x^n and summing over $n\geq 3$, we obtain

$$\begin{aligned}&A(x,v)-1-x-(1+v)x^2\\ &\quad=\frac{x}{1-v}(A(x,v)-1-x-vA(xv,1)+v+xv^2)+B(x,v)-vx^2A(xv,1),\end{aligned}\tag{2}$$

$$\begin{aligned}&B(x,v)-vx^2\\ &\quad=\frac{x}{1-v}(B(x,v)-v^{-3}B(xv,1))+x^3A(xv,1)+(vx^2+x^2v^{-1})(A(xv,1)-1).\end{aligned}\tag{3}$$

Hence, (2) and (3) can be written as

$$\begin{aligned}\left(1-\frac{x}{v(1-v)}\right)A(x/v,v)&=1-\frac{x}{1-v}A(x,1)+B(x/v,v)-\frac{x^2}{v}A(x,1),\\ \left(1-\frac{x}{v(1-v)}\right)B(x/v,v)&=\frac{-x}{v^4(1-v)}B(x,1)+\left(\frac{x^3}{v^3}+\frac{x^2}{v}+\frac{x^2}{v^3}\right)A(x,1)-\frac{x^2}{v^3}.\end{aligned}$$

By substituting $v=\frac{1+\sqrt{1-4x}}{2}$ (the zero of the kernel $1-\frac{x}{v(1-v)}$, see [3]) into the second equation, we obtain

$$B(x,1)=\frac{x(\sqrt{1-4x}-1)}{2}+\frac{2x^2+x-x\sqrt{1-4x}}{2}A(x,1).\tag{4}$$

By multiplying the first equation by $1-\frac{x}{v(1-v)}$, and using the second equation, we obtain

$$\begin{aligned}\left(1-\frac{x}{v(1-v)}\right)^2A(x/v,v)&=1-\frac{x}{v(1-v)}-\frac{x^2}{v^3}-\left(\frac{x^2}{v}+\frac{x}{1-v}\right)\left(1-\frac{x}{v(1-v)}\right)A(x,1)\\ &\quad-\frac{x}{v^4(1-v)}B(x,1)+\left(\frac{x^3}{v^3}+\frac{x^2}{v}+\frac{x^2}{v^3}\right)A(x,1).\end{aligned}$$

After differentiating the above equation respect to v, substituting $v = \frac{1+\sqrt{1-4x}}{2}$ together with using (4), and several simple algebraic operations, we obtain an explicit formula for $A(x,1)$ as

$$A(x,1) = \frac{1-5x+(1+x)\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}},$$

which completes the proof of this case.

2.2. **Class 2.** $\Pi_2 = \{1243, 1324, 1342\}$. Let $A_n = S_n(\Pi_2)$. Define $a_n = \#A_n$ and $a_n(i_1,\dots,i_s)$ to be the number of permutations $\pi_1\cdots\pi_n \in A_n$ such that $\pi_1\cdots\pi_s = i_1\cdots i_s$.

Lemma 3. *Define* $b_n(i) = a_n(i,i+1)$. *For all* $1 \le i \le n-3$,

$$\begin{aligned} a_n(i) &= a_{n-1}(i) + \cdots + a_{n-1}(1) + b_n(i),\\ b_n(i) &= b_{n-1}(i) + \cdots + b_{n-1}(1) \end{aligned}$$

with $a_n(n-2) = a_n(n-1) = a_n(n) = a_{n-1}$, $b_n(n) = 0$ *and* $b_n(n-2) = b_n(n-1) = a_{n-2}$.

Proof. By the definitions, $a_n(n) = a_n(n-1) = a_n(n-2) = a_{n-1}$, $b_n(n) = 0$ and $b_n(n-2) = b_n(n-1) = a_{n-2}$. If $1 \le i \le n-2$, then

$$\begin{aligned} a_n(i) &= \sum_{j=1}^{i-1} a_n(i,j) + \sum_{j=i+1}^{n} a_n(i,j) = \sum_{j=1}^{i-1} a_{n-1}(i) + a_n(i,n) + b_n(i)\\ &= \sum_{j=1}^{i} a_{n-1}(i) + b_n(i). \end{aligned}$$

Also,

$$b_n(i) = \sum_{j=1}^{i-1} a_n(i,i+1,j) + \sum_{j=i+2}^{n} a_n(i,i+1,j).$$

By the definitions $a_n(i,i+1,j) = 0$ with $j > i+2$ (the permutations in the question have subsequence $i, i+1, j, i+2$ which is order isomorphic to 1243) and $a_n(i,i+1,i+2) = a_{n-1}(i,i+1) = b_{n-1}(i)$. Thus

$$b_n(i) = b_{n-1}(i) + \sum_{j=1}^{i-1} a_n(i,i+1,j).$$

Let $\pi = i(i+1)j\pi' \in A_n$ with $1 \le j \le i-1$. Then the letters $i, i+1, i+2, i+3, \dots, n$ creates an increasing subsequence in π. If j' with $j < j' < i$ appears on the right side of position of $i+2$ in π, then π contains either $j(i+2)j'(i+3)$ or $j(i+2)(i+3)j'$ which is order isomorphic to 1324 or 1342, respectively. Thus j' appears on the left side of the position of $i+2$ in π. Since π avoids 1324 then π contains the subsequence $j, j+1, \dots, i-1$. Thus $pi \in A_n$ if and only if $j(j+1)\pi'' \in A_{n-1}$, where π'' is a word obtained from π' by decreasing each letter greater than i by 1 and increasing the letters $j+1, j+2, \dots, i-1$ by 1. Hence, $a_n(i,i+1,j) = a_{n-1}(j,j+1)$, for all $j = 1, 2, \dots, i-1$. In other words, $b_n(i) = \sum_{j=1}^{i} b_{n-1}(j)$, as required. $\square$

By using the techniques that have been used in the proof of Class 1 and the similarity of Lemma 2 and Lemma 3, one can solve the recurrence relation in Lemma 3, and obtain that the generating function $A(x) = \sum_{n\geq 0} a_n x^n$ is given by

$$\frac{1-5x+(1+x)\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}},$$

as required.

2.3. **Class 3.** $\Pi_3 = \{1324, 1342, 1432\}$. Let $A_n = S_n(\Pi_3)$. Define $a_n = \#A_n$ and $a_n(i_1,\dots,i_s)$ to be the number of permutations $\pi_1\cdots\pi_n \in A_n$ such that $\pi_1\cdots\pi_s = i_1\cdots i_s$. By using similar arguments as in the proof of Lemmas 2 and 3, one can state the following recurrence.

Lemma 4. *Define $b_n(i) = a_n(i,n)$. For all $1 \leq i \leq n-3$,*

$$\begin{aligned} a_n(i) &= a_{n-1}(i) + \cdots + a_{n-1}(1) + b_n(i),\\ b_n(i) &= b_{n-1}(i) + \cdots + b_{n-1}(1) \end{aligned}$$

with $a_n(n-2) = a_n(n-1) = a_n(n) = a_{n-1}$, $b_n(n) = 0$ and $b_n(n-2) = b_n(n-1) = a_{n-2}$.

By comparing Lemma 3 and Lemma 4, we obtain that $\#S_n(\Pi_2) = \#S_n(\Pi_3)$, which implies that the generating function $A(x) = \sum_{n\geq 0} a_n x^n$ is given by

$$\frac{1-5x+(1+x)\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}},$$

as required.

2.4. **Class 4.** $\Pi_4 = \{2314, 3214, 4213\}$ We first give a bijection from permutations avoiding $\{3214, 4213\}$ to one-size-smaller Schröder paths.

Recall that a *Schröder path* is a lattice path of North steps $N = (0,1)$, diagonal steps $D = (1,1)$ and East steps $E = (1,0)$ that starts at the origin, never drops below the diagonal $y = x$, and terminates on the diagonal. Its *size* is # N steps + # D steps, and a Schröder n-path is one of size n. Thus a Schröder n-path ends at (n,n). The vertices on $y = x$ split a nonempty Schröder path into its *components*, and a Schröder path whose only vertices on $y = x$ are its endpoints (hence, is a one component path) is *indecomposable*. Thus all components of a Schröder path are indecomposable. The number of Schröder n-paths is the large Schröder number r_n, A006318. A *peak* is a pair of consecutive steps NE (consider the path rotated 45°).

Every permutation on $[n]$ has a bounding up-down staircase (Figure 1) determined by its left to right (LR) maxima and right to left (RL) maxima.

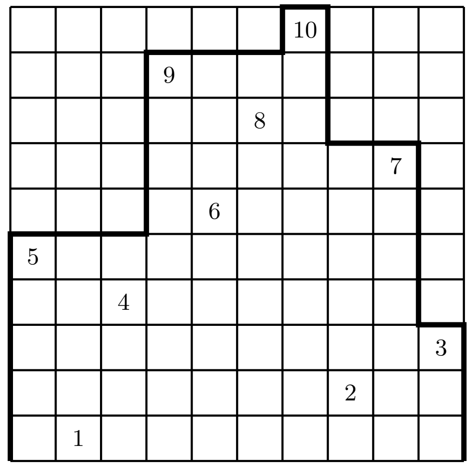
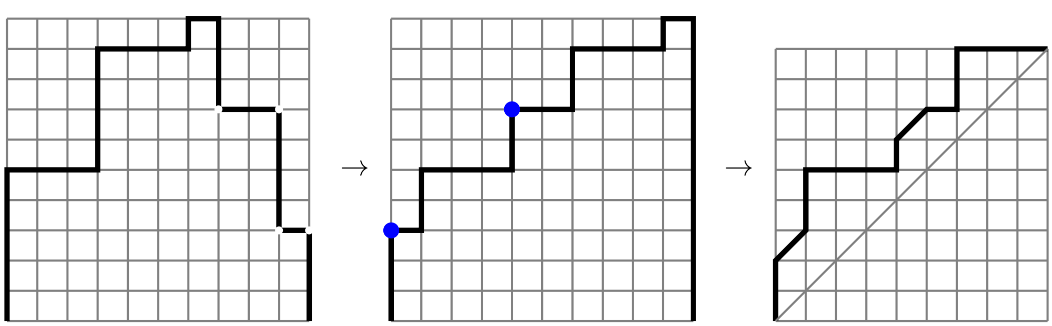


Figure 1. Bounding staircase of a permutation.

Bounding staircases of size n are lattice paths from the origin consisting of n steps each North $N = (0, 1)$, East $E = (1, 0)$, and South $S = (-1, 0)$, that are characterized by the following properties:

(1) all N steps precede all S steps,
(2) East runs (maximal sequence of contiguous E steps) are at different heights,
(3) measuring from the top, the i-th pair of matching N/S steps are at least i units apart (to make room for the permutation entries above them), and the first pair are just 1 unit apart (they bracket the entry n)

Proposition 5. *There is a bijection from bounding staircases to one-size-smaller Schröder paths.*



a) b) c)

Bijection from bounding staircases to one-size-smaller Schröder paths

Figure 2.

Proof. Delete each run of East steps bounded by two S steps (Fig. 2a), insert it between the matching N steps, and color the newly introduced NE corner blue (Fig. 2b). Then delete the last $n+2$ steps (necessarily $N\,E\,S^n$) and replace each blue NE corner with a diagonal step $D=(1,1)$ to get the desired Schröder path (Fig. 2c). $\square$

Lemma 6. *A permutation p avoids $\{3214, 4213\}$ if and only if it is lexicographically least among all permutations with the same bounding staircase as p.*

Proof. If either offending pattern is present in p, then there is also a subsequence $xbay$ with x a LR max, y a RL max, $b>a$ and x,y both $>b$. Switching the a and b gives a lexicographically smaller permutation with the same LR max/RL max, both in value and position, and hence the same bounding staircase. Conversely, if p is not lexicographically least, then a ba is present with $b>a$ and neither a nor b a LR max or RL max, implying that ba is the "21" of an offending pattern. $\square$

Remark. To construct this lexicographically least permutation, use the bounding staircase to fill the LR max and RL max slots in the permutation, then fill the remaining slots right to left in turn with the largest available entry that will not create a new RL max.

Corollary 7. *The map "permutation $\rightarrow$ bounding staircase" is a bijection from $S_n(3214, 4213)$ to bounding staircases of size n.*

Combining this bijection with that of Prop. 5, we have a bijection $\phi : S_n(3214, 4213) \rightarrow$ Schröder $(n-1)$-paths.

Corollary 8. [4] $|S_n(3214, 4213)| = r_{n-1}$*, the large Schröder number.*

Proposition 9. *The restriction $\phi_{|S_n(\Pi_4)}$ is a bijection from $S_n(\Pi_4)$ to Schröder $(n-1)$-paths in which each component has at most one peak.*

Proof. In a 2314 pattern in a $\{3214, 4213\}$-avoider p, the "2" and "3" must be LR maxima of p, and LR maxima in the permutation correspond to peaks in the Schröder path. Now consider the insertion of two dividers in p, one just before a LR max and the other just after a RL max, to split p into three segments A, B, C. Necessarily, $n \in B$ while A, C may be empty. Returns to $y=x$ in the Schröder path correspond to such insertions for which $A \cup C$ is a nonempty initial segment of the positive integers. The shortest AC thus corresponds to the first component of the Schröder path. The "2" and "3" of the 2314 pattern either both lie in A or both lie in B. If they lie in A, the "1" cannot lie in B. These observations are the basis for an inductive proof and allow us to assume that, in addition to AC being shortest, B is the singleton n, and so the Schröder path has just one component. If a 2314 is present, the "2" and "3" produce two peaks. On the other hand, if there are two peaks, they produce a "2" and "3", and there must also be present a "1" and "4" to make a 2314 for otherwise AC would not be shortest. $\square$

We have the following elementary counts for Schröder paths.

Lemma 10. *For $n \geq 1$,*
(i) [5, Ex. 45] *The number of Schröder n-paths with no peaks is the Catalan number C_n.*
(ii) [See A060693] *The number of Schröder n-paths with exactly 1 peak is $\binom{2n-1}{n-1}$.*

An indecomposable Schröder path of size $n \geq 2$ has the form NPE with P a Schröder path of size $n-1$; hence we have

Corollary 11. (i) *The number of indecomposable Schröder n-paths with no peaks is* 2 *for* $n=1$ *and* C_{n-1} *for* $n \geq 2$.
(ii) *The number of indecomposable Schröder n-paths with exactly* 1 *peak is* 0 *for* $n=1$ *and* $\binom{2n-3}{n-2}$ *for* $n \geq 2$.

Proposition 12. *The generating function for indecomposable Schröder paths with at most* 1 *peak is*

$$\frac{1}{2}\left(1+x+\frac{x}{\sqrt{1-4x}}-\sqrt{1-4x}\right).$$

Proof. Immediately by Corollary 11. □

Corollary 13. *The generating function for Schröder paths with at most* 1 *peak in each component is*

$$\frac{2\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}}.$$

Proof. This generating function is the Invert transform of the generating function in Proposition 12. □

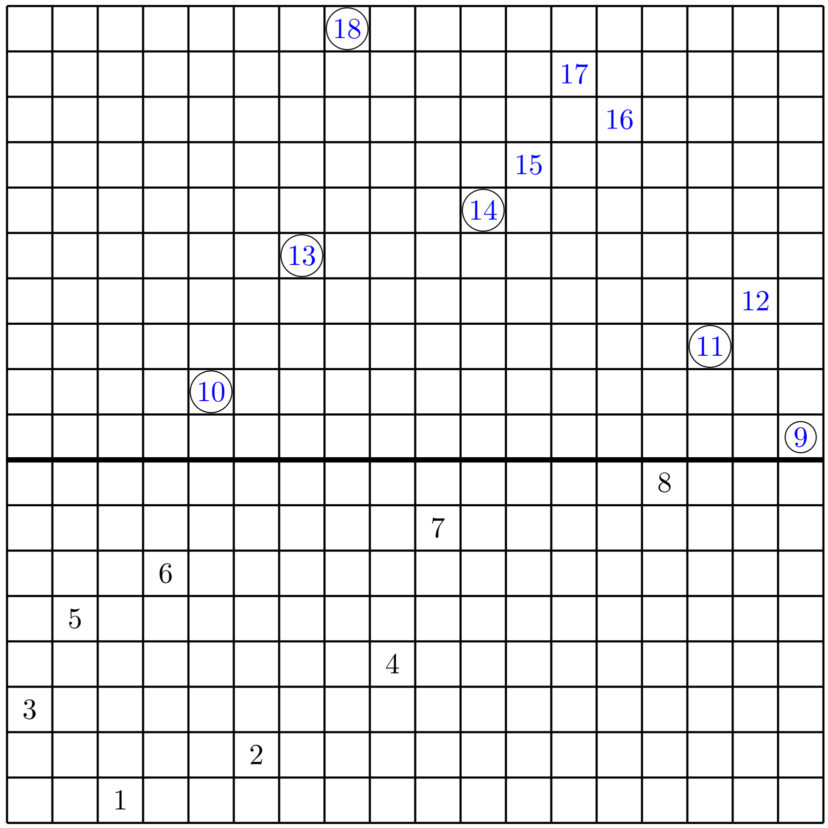
Corollary 14. *The generating function for nonempty π_4-avoiders is*

$$\frac{2x\sqrt{1-4x}}{1-5x+(1-x)\sqrt{1-4x}}. \tag{5}$$

Proof. Immediately by Proposition 9 and Corollary 13. □

Adding 1 to (5) to include the empty permutation gives (1).

2.5. **Class 5.** $\Pi_5=\{3214, 3241, 4213\}$. To characterize Π_5-avoiders, draw a horizontal line just below the last entry of a permutation p as in Figure 3 to obtain two subpermutations, A above the line (in blue) and B below the line (in black). Split A into two segments, A_1 consisting of the entries weakly left of n and A_2 consisting of the remaining entries. Here, $A_1=(10,13,18)$, $A_2=(14,15,17,16,11,12,9)$. Say an entry in p is *key* if it either lies in A_1 or is a LR min in A_2 (key entries are circled in Figure 3 and we use the terms "key" and "circled" interchangeably below). Let B_2 denote the terminal segment of B consisting of the entries that lie (in p) after the first entry of A. Here $B_2=(2,4,7,8)$.



A Π_5-avoider with $n = 18$

Figure 3

Here are some properties of a Π_5-avoider $p = (p_1, \dots, p_n)$. Let f and l denote the first and last entries of A respectively.

(1) A, and hence $\mathrm{St}(A)$, the standardization of A, is 213-avoiding, for if bac is a 213 pattern in A, then each of a, b, c is $> l$ and $bacl$ is a forbidden 3241 in p.
(2) B is 321-avoiding, for if cba is a 321 pattern in B, then $cbal$ is a forbidden 3214 in p.
(3) B_2 is increasing, for if ba is a 21 in B then $f \neq l$ and $fbal$ is either a 3214 or 4213 in p, both forbidden.
(4) For every $x \in B$, the right neighbor y of x in p (it always has one) is either also in B or is circled, for otherwise y is in A_2 but not a LR min of A_2, and so there is $z \in A_2$ lying to the left of both x and y in p with $z < y$. Then $nzxy$ is a forbidden 4213 in p.

(Note that item 4 says that if B is divided into blocks of entries that are contiguous in p, then each block lies immediately to the left of a circled entry in p.) Conversely, if these 4 conditions are met, the reader may check that p is a Π_5-avoider.

Now, to count Π_5-avoiders, we first dispose of the cases where A has length $1, 2$ or n.

Lemma 15. *Suppose $n \geq 3$. Then for each of $a = 1, 2$ and n, we have* $|\{p \in S_n(\Pi_5) : \text{length}(A) = a\}| = C_{n-1}$.

Proof. Recall that both 321-avoiders and 213-avoiders on $[n]$ are counted by C_n We have $a = 1$ if and only of n is the last entry of p. Avoidance of 3214 then implies $p\backslash\{n\}$ avoids 321. Conversely, if $p\backslash\{n\}$ avoids 321 then, a fortiori, $p\backslash\{n\}$ avoids Π_5 and so does p. Next, $a = 2$ if and only of $n-1$ is the last entry of p. Suppose $n-1$ is the last entry of p and p is a Π_5-avoider. If cba were a 321 pattern in p, then $cba\,(n-1)$ would be a 4213 if $c = n$ and a 3214 if $c < n-1$, both of which are forbidden. So p must avoid 321. Conversely, if $p\backslash\{n-1\}$ avoids 321 then, again, p avoids Π_5. Lastly, $a = n$ if and only of 1 is the last entry of p and then p is a Π_5-avoider if and only of p avoids 213 (else a 3241 terminating at the last entry is present) and the result follows. □

For the remaining cases, we have $3 \leq a \leq n-1$ and so $n \geq 4$. Then $k \geq 3$ as follows. Since $p_n \leq n-2$ by the proof of Lemma 15, the three entries n, the successor of n in A, and p_n are all key and all distinct unless n is the second to last entry of A, but in that case $n-1$ occurs before n and so is a key entry, and $p_n, n-1, n$ are distinct. So $3 \leq k \leq a$.

The following elementary counting results will be useful; we omit the proofs. We use $C_{n,k}$ for the generalized Catalan number $\frac{k+1}{2n+k+1}\binom{2n+k+1}{n}$. Recall that $(C_{n,k})_{n\geq 0}$ is the $(k+1)$-fold convolution of the Catalan numbers $(C_n)_{n\geq 0} = (C_{n,0})_{n\geq 0}$ and so the generating function $\sum_{n\geq 0} C_{n,k}x^n$ is given by $C(x)^{k+1}$ where $C(x) := \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. It is convenient below to use the convention $C_{0,-1} := 1$.

Proposition 16.
(i) *The number of 213-avoiding permutations on $[n]$ whose last entry is 1 with n in first position and k key entries is $C_{n-k,k-3}$ for $2 \leq k \leq n$.*
(ii) *The number of 213-avoiding permutations on $[n]$ whose last entry is 1 with n in position j and k key entries is $C_{n-k,k-2-j}$ for $1 \leq j \leq k-1,\ k \leq n$.*

Corollary 17. *The number of 213-avoiding permutations on $[n]$ whose last entry is 1 with k key entries is $w(n,k) := \sum_{j=1}^{n-1}\binom{k-2}{j-1}C_{n-k,k-2-j}$ for $n \geq 2,\ 1 \leq k \leq n$.*

Lemma 18. *The number of 321-avoiding permutations on $[n]$ in which the last i entries are increasing is $C_{n-i,i}$ for $0 \leq i \leq n$.*

We are now ready to count permutations p in $S_n(\Pi_5)$ by $a :=$ length(A), $k :=$ number of key entries, $i :=$ number of entries of B after the first circled entry in p. The cases $a = 1, 2$ or n have been treated already. So suppose given n, a, k, i, with $3 \leq k \leq a \leq n-1$ and $0 \leq i \leq b := n-a$. By Cor. 17, there are $w(a,k)$ 213-avoiding permutations A_1 of length a that end with 1 and have k key entries. By Lemma 18, there are $C_{b-i,i}$ 321-avoiding permutations of length b such that the last i entries are increasing. There are $\binom{i+k-2}{i}$ ways to distribute these last i entries into $k-1$ blocks to be placed just before the $k-1$ non-first key entries of $A = A_1 + b$. (Of course, the initial block of $b-i$ entries of B lies before the first key entry.) These choices uniquely determine a Π_5-avoider of length n.

Hence, summing over a, k, i, we have for $n \geq 3$,

$$
\begin{aligned}
(6) \qquad |S_n(\Pi_5)| &= 3C_{n-1} + \sum_{a=3}^{n-1}\sum_{k=3}^{a}\sum_{i=0}^{b} w(a,k) C_{b-i,i}\binom{i+k-2}{i} \\
&= 3C_{n-1} + \sum_{a=3}^{n-1}\sum_{k=3}^{a}\sum_{i=0}^{b}\sum_{j=1}^{a-1}\binom{k-2}{j-1} C_{a-k,k-j-2}\, C_{b-i,i}\binom{i+k-2}{i} \\
&= 3C_{n-1} + \sum_{a=3}^{n-1}\sum_{k=3}^{a}\sum_{j=1}^{a-1}\binom{k-2}{j-1} C_{a-k,k-j-2}\, C_{n-a,k-1}\,.
\end{aligned}
$$

The last equality evaluates the sum over i using a generalized Catalan number identity. The generating function $F(x) := \sum_{n\geq 0} |S_n(\Pi_5)| x^n$ is easily deduced:

$$F(x) = 1 + x + 2x^2 + 3\sum_{n\geq 3} C_{n-1}x^n + G(x)\,,$$

where

$$
\begin{aligned}
G(x) &= \sum_{n\geq 4}\sum_{a=3}^{n-1}\sum_{k=3}^{a}\sum_{j=1}^{a-1}\binom{k-2}{j-1} C_{a-k,k-j-2} C_{n-a,k-1} x^n \\
&= \sum_{k\geq 3}\sum_{j=1}^{k-1}\binom{k-2}{j-1}\sum_{a\geq k} C_{a-k,k-j-2}\sum_{n\geq a+1} C_{n-a,k-1}x^n \\
&= \sum_{k\geq 3}\big(C(x)^k - 1\big)\sum_{j=1}^{k-1}\binom{k-2}{j-1}\sum_{a\geq k} C_{a-k,k-j-2}x^a \\
&= \sum_{k\geq 3} x^k\big(C(x)^k - 1\big)\sum_{j=1}^{k-1}\binom{k-2}{j-1} C(x)^{k-j-1} \\
&= \sum_{k\geq 3} x^k\big(C(x)^k - 1\big)\big(1 + C(x)\big)^{k-2},
\end{aligned}
$$

which is a difference of geometric sums. After evaluation and simplification, we find

$$F(x) = 1 + \frac{2x\sqrt{1-4x}}{1 - 5x + (1-x)\sqrt{1-4x}}\,,$$

agreeing with the expression in (1), or with rationalized denominator,

$$F(x) = 1 + \frac{2x^2 + x(1-5x)C(x)}{1 - 4x - x^2}\,.$$

In conclusion, we remark that the above characterization of Π_5-avoiders can easily be adapted to find the bivariate generating function for Π_5-avoiders by length and number of components. First, we count indecomposable Π_5-avoiders. For $n \geq 4$, the cases $a = 1, 2, n$ are counted by $0, C_{n-2}, C_{n-1}$ respectively. For $3 \leq a \leq n-1$, a Π_5-avoider is indecomposable iff B, in the notation above, in addition to being a 321-avoider whose last i entries are increasing, satisfies the property that for all $r = 1, 2, \ldots, b-i$, the first r entries of B, when sorted, do not form an initial segment of the positive integers (the property is vacuously satisfied when $i = b$). The number of such permutations is

$C_{b-i,i-1} = C_{n-a-i,i-1}$. Thus, in (6), the initial $3C_{n-1}$ term is replaced by $C_{n-2}+C_{n-1}$ and the $C_{b-i,i}$ factor in the sum is replaced by $C_{b-i,i-1}$. This modified sum leads to the counting sequence $(1, 1, 3, 11, 43, 173, 707, \dots)_{n\ge 1}$, A026671, for indecomposable Π_5-avoiders, with generating function $F_{\text{indec}}(x) := 1/(1 - x/\sqrt{1-4x})$. Further, a Π_5-avoider with $k \ge 2$ components has the form $p_1 \oplus \cdots \oplus p_{k-1} \oplus p_k$ where $p_1, \dots, p_{k-1}$ are all indecomposable 321-avoiders and p_k is an indecomposable Π_5-avoider. Here $\oplus$ is the direct sum defined on permutations π of length m and σ of length n by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } 1 \le i \le m, \\ \sigma(i-m)+m & \text{if } m+1 \le i \le m+n. \end{cases}$$

Since indecomposable 321-avoiders have the generating function $xC(x)$, the desired bivariate generating function, excluding the empty permutation, is

$$\frac{F_{\text{indec}}(x)y}{1-xyC(x)} = \frac{2xy\sqrt{1-4x}}{y-2x-3xy+(2-xy-y)\sqrt{1-4x}}.$$

References

[1] M.H. Albert, R.E.L. Aldred, M.D. Atkinson, C.C. Handley, D.A. Holton and D.J. McCaughan, Sorting Classes, *Elec. J. of Comb.* **12** (2005) #R31.

[2] The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2016.

[3] Q. Hou and T. Mansour, Kernel Method and Linear Recurrence System, *J. Computat. Appl. Math.* **261:1** (2008) 227–242.

[4] Darla Kremer, Permutations with forbidden subsequences and a generalized Schröder number, *Discr. Math.* **218** (2000) 121–130.

[5] R. H. Stanley, Catalan Numbers, *Cambridge University Press*, 2015.

Department of Statistics, University of Wisconsin, Madison, WI 53706

E-mail address: callan@stat.wisc.edu

Department of Mathematics, University of Haifa, 3498838 Haifa, Israel

E-mail address: tmansour@univ.haifa.ac.il