ARCHIMEDES' QUADRATURE OF THE PARABOLA AND MINIMAL COVERS

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ABSTRACT. The generalization of Archimedes strategy to obtain the area of a parabolic segment leads to combinatorial formulas involving minimal covers of sets. These, in turn, are conjecturally related to q-binomial coefficients.

1. Introduction

Archimedes [2, pp. 233-252] calculated the area any parabolic segment with supreme ingenuity via an exhaustion argument, successively adding vertices to a polygon inscribed in it (beginning with a triangle). He proved that each iteration contributed a fixed fraction of the area of the polygon of the previous step (namely, one fourth), and thus he could sum the resulting geometric series.

When I planned to teach this rather elementary method in the unit interval, I realized that I could not use the straightforward but convolved euclidean geometry, since my students were not familiar with it. Thus I recurred to the shoelace algorithm to compute areas. This yields the determinant of a simple Vandermonde matrix to obtain the area of a single triangle in the unit interval with 2^n equal subdivisions:

$$\frac{1}{2}\det\begin{pmatrix} \frac{k-1}{2^n} & \left(\frac{k-1}{2^n}\right)^2 & 1 \\ \frac{k}{2^n} & \left(\frac{k}{2^n}\right)^2 & 1 \\ \frac{k+1}{2^n} & \left(\frac{k+1}{2^n}\right)^2 & 1 \end{pmatrix} = \frac{1}{2}\left(\frac{2}{2^{3n}}\right) = \frac{1}{2^{3n}}.$$

As it is readily seen, the value of the determinant is independent of the three consecutive points chosen to build the triangle. Summing the area of the 2^n new triangles that appear in the n-th iteration, we obtain $\frac{2^n}{2^{3n}} = \frac{1}{2^{2n}} = \frac{1}{4^n}$, as expected.

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2. Generalization

What happens if we try the same trick with the function $f(x) = x^s$, with $s > 2$? We are led to the determinant

$$T(k,s,n) := \det \begin{pmatrix} \frac{k-1}{2^n} & \left(\frac{k-1}{2^n}\right)^s & 1 \\ \frac{k}{2^n} & \left(\frac{k}{2^n}\right)^s & 1 \\ \frac{k+1}{2^n} & \left(\frac{k+1}{2^n}\right)^s & 1 \end{pmatrix} = \frac{(k+1)^s - 2k^s + (k-1)^s}{2^{n(s+1)}}$$

In this case, the area $\frac{1}{2}T(k,s,n)$ of the triangle depends on the points selected. For instance, when $s = 3$, we have

$$\frac{1}{2} \cdot \frac{(k+1)^3 - 2k^3 + (k-1)^3}{2^{n(3+1)}} = \frac{3k}{2^{4n}}.$$

To calculate, for example,

$$\int_0^1 x^3\,dx$$

we first find the sum

$$\sum_{i=0}^{2^{n-1}-1} \tfrac{1}{2}T(2i+1,3,n) = \sum_{i=0}^{2^{n-1}-1} \frac{3(2i+1)}{2^{4n}}$$

that accounts for the area of the n-th iteration of the exhaustion. A virtue of this approach is that only with the knowledge of the formula for triangular numbers $\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$ we can find the area under a cubic parabola in the unit interval, which is more difficult with the approach using Riemann sums with arithmetic or even geometric subdivisions. More explicitly, we calculate

$$\sum_{i=0}^{2^{n-1}-1} \frac{3(2i+1)}{2^{4n}} = \frac{1}{2^{4n+1}}\left(6 \cdot \frac{2^{n-1}(2^{n-1}-1)}{2} + 3 \cdot 2^{n-1}\right) = \frac{3}{2^{2n+2}}$$

and thus

$$\sum_{n=1}^{\infty} \frac{3}{2^{2n+2}} = \frac{3}{4}\left(\frac{1}{4(1-\frac{1}{4})}\right) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}.$$

Keeping in mind that we have calculated the area between the curve and the identity function, we simply subtract this result to $\frac{1}{2}$, and we get the expected $\frac{1}{4}$. It is amusing to try by hand the case $s = 4$.

It is worthwhile to mention that the former is a purely symbolic recast of Kirfel's more geometric exposition in [3], where upper bounds for the integrals are also considered.

3. A combinatorial excursion

When $s \geq 2$, $T(2x+1, s, 0)$ is a polynomial in x of degree $s-2$. If we arrange them in a triangle, we get

$$\begin{array}{lllllll} T(2x+1,2,0) = & 2, & & & & \\ T(2x+1,3,0) = & 6+ & 12x, & & & \\ T(2x+1,4,0) = & 14+ & 48x+ & 48x^2, & & \\ T(2x+1,5,0) = & 30+ & 140x+ & 240x^2+ & 160x^3, & \\ T(2x+1,6,0) = & 62+ & 360x+ & 840x^2+ & 960x^3+ & 480x^4, \end{array}$$

whose coefficients can be rewritten as

$$\begin{array}{ccccc} 1\cdot 2 & & & & \\ 3\cdot 2 & 3\cdot 4 & & & \\ 7\cdot 2 & 12\cdot 4 & 6\cdot 8 & & \\ 15\cdot 2 & 35\cdot 4 & 30\cdot 8 & 10\cdot 16 & \\ 31\cdot 2 & 90\cdot 4 & 105\cdot 8 & 60\cdot 16 & 15\cdot 32. \end{array}$$

Disregarding[1] the powers of 2 that we factorized, we get $\frac{1}{2}T(x+1, s, 0)$. Now note the following

$$\begin{aligned} T(x+1,s,0) &= (x+2)^s - 2(x+1)^s + x^s \\ &= \sum_{k=0}^{s} \binom{s}{k} 2^{s-k} x^i - 2\sum_{k=0}^{s}\binom{s}{k} x^k + x^s \\ &= 2\sum_{k=0}^{s}\binom{s}{k}(2^{s-k-1}-1)x^k + x^s \\ &= 2\sum_{k=0}^{s}\binom{s}{k}(2^{k-1}-1)x^{s-k} + x^s \\ &= 2\sum_{k=2}^{s}\binom{s}{k}(2^{k-1}-1)x^s \end{aligned}$$

The coefficients $\binom{s}{k}(2^{k-1}-1)$ of the polynomial correspond to the number $M(s,2,k)$ of minimal[2] 2-covers of a labeled s-set that cover k points uniquely (where $k \geq 2$), listed as sequence A05763 in the OEIS

[1]The act of overlooking the powers of two may seem artificial, and indeed it is done here for the sake of the flow of ideas of the exposition. The combinatorial significance of the result was found as an accident, caused precisely by failing to consider only the triangles with odd indices during the limit process.

[2] A *minimal* cover is a cover such that the elimination of any of its members results in a family of sets that fails to cover the original set.

[4]. Hence

$$T(x+1,s,0) = 2\sum_{k=2}^{s} M(s,2,k)x^{s-k}$$

and

$$T(x+1,s,1) = \frac{1}{2^s}\sum_{k=2}^{s} M(s,2,k)x^{s-k}.$$

In general

$$T(x+1,s,n) = \frac{1}{2^{n(s+1)}}\sum_{k=2}^{s} M(s,2,k)x^{s-k},$$

and therefore

$$\begin{aligned}\frac{s-1}{s+1} &= 1-\frac{2}{s+1}\\ &= 1-2\int_0^1 x^s\,dx = \sum_{n=1}^{\infty}\sum_{x=1}^{2^{n-1}-1} T(2x+1,s,n)\\ &= \sum_{n=1}^{\infty}\frac{1}{2^{(n-1)(s+1)}}\sum_{x=1}^{2^{n-1}-1}\sum_{k=2}^{s}\frac{M(s,2,k)}{2^k}x^{s-k}. \qquad (1)\end{aligned}$$

Hearne and Wagner proved [1] that if $M(s,j,k)$ is the number of j-member minimal covers of an s-set that cover k elements uniquely, then

$$M(s,j,k) = \binom{s}{k}(2^j-j-1)^{s-k}S(k,j)$$

where $S(k,j)$ denotes the Stirling numbers of the second kind. This formula allows us to extend (1) for $j>2$, and the following Maxima code (for the particular case when $s=4$ and $j=2$, which should be changed accordingly), valid for $3\le j\le s-1$, automatizes the calculations.

```
sumando: subst([s=4,j=2],(1/2^((n-1)*(s+1)))*
 sum(sum(binomial(s,k)*(2^j-j-1)^(s-k)*stirling2(k,j)*
 x^(s-k)/2^k,k,2,s),x,0,2^(n-1)-1)), simpsum;
sumando: ratexpand(sumando);
acumulado:0;
for k:1 thru nterms(sumando)step 1 do
 acumulado:acumulado+sum(part(sumando,k),n,1,inf),simpsum;
print(acumulado);
```

We can now compile the following triangle of rational numbers.

(2)

$s\backslash j$	1	2	3	4	5	6	7	8
2	$\frac{1}{\mathbf{3}}$	$\frac{1}{\mathbf{3}}$						
3	$\frac{1}{\mathbf{7}}$	$\frac{1}{2}$	$\frac{1}{7}$					
4	$\frac{1}{\mathbf{15}}$	$\frac{3}{5}$	$\frac{10}{21}$	$\frac{1}{15}$				
5	$\frac{1}{\mathbf{31}}$	$\frac{2}{3}$	$\frac{865}{651}$	$\frac{71}{186}$	$\frac{1}{31}$			
6	$\frac{1}{\mathbf{63}}$	$\frac{5}{7}$	$\frac{2630}{651}$	$\frac{1427}{\mathbf{651}}$	$\frac{181}{651}$	$\frac{1}{63}$		
7	$\frac{1}{\mathbf{127}}$	$\frac{3}{4}$	$\frac{163133}{\mathbf{11811}}$	$\frac{306553}{15748}$	$\frac{36667}{\mathbf{11811}}$	$\frac{145}{762}$	$\frac{1}{127}$	
8	$\frac{1}{\mathbf{255}}$	$\frac{7}{9}$	$\frac{3368938}{66929}$	$\frac{129115655}{602361}$	$\frac{46958822}{602361}$	$\frac{43662}{\mathbf{10795}}$	$\frac{4036}{32385}$	$\frac{1}{255}$

Using the notation of Hearne and Wagner $M(s,k) = \sum_{j=0}^{k} M(s,j,k)$ and their formula for the generating function

$$M_s(x) = \sum_{k=0}^{s} M(s,k)x^k = \sum_{k=0}^{s}\sum_{j=0}^{k} M(s,j,k)x^k$$
$$= \sum_{j=0}^{s} \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell}\binom{j}{\ell}(2^j - j - 1 + \ell x)^s,$$

where $M(s,j,k)$ are defined as zero where convenient, we have

$$x^s M_s(\tfrac{1}{2x}) = \sum_{k=0}^{s}\sum_{j=0}^{s} \frac{M(s,j,k)}{2^k} x^{s-k}$$

and thus we get

$$\sum_{n=1}^{\infty} \frac{1}{2^{(n-1)(s+1)}} \sum_{x=1}^{2^{n-1}-1} \sum_{k=2}^{s}\sum_{j=0}^{s} \frac{M(s,j,k)}{2^k} x^{s-k}$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^{(n-1)(s+1)}} \sum_{x=1}^{2^{n-1}-1} x^s M_s(\tfrac{1}{2x})$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^{(n-1)(s+1)}} \sum_{x=1}^{2^{n-1}-1} \sum_{j=0}^{s} \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell}\binom{j}{\ell}((2^j - j - 1)x + \tfrac{\ell}{2})^s$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \sum_{x=1}^{2^{n-1}-1} \sum_{j=0}^{s} \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell}\binom{j}{\ell}((2^j - j - 1)\tfrac{x}{2^{n-1}} + \tfrac{\ell}{2})^s$$

whence we can deduce the following lower bound for the sums of the rows of the triangle

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{2^{(n-1)(s+1)}} \sum_{x=1}^{2^{n-1}-1} \sum_{k=2}^{s} \sum_{j=0}^{s} \frac{M(s,j,k)}{2^k} x^{s-k} \\
&\qquad \geq \int_0^1 \sum_{j=0}^{s} \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{j}{\ell} ((2^j - j - 1)x + \tfrac{\ell}{2})^s \, dx \\
&\qquad = \sum_{j=0}^{s} \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{j}{\ell} \left. \frac{((2^j - j - 1)x + \frac{\ell}{2})^{s+1}}{(s+1)(2^j - j - 1)} \right|_0^1 \\
&\qquad = \sum_{j=0}^{s} \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{j}{\ell} \frac{1}{s+1} \sum_{t=1}^{s+1} (2^j - j - 1 + \tfrac{\ell}{2})^{t-1} (\tfrac{\ell}{2})^{s-t+1}
\end{aligned}$$

4. A conjecture

The boldface denominators in (2) are the values when $q = 2$ of the q-analogues of the binomial coefficients

$$\binom{s}{m}_q = \frac{[s]_q!}{[m]_q![s-m]_q!},$$

where $[n]_q = \frac{q^n - 1}{q-1}$ and the q-analogue of the factorial is defined inductively by $[0]_q! = 1$ and $[n]_q! = [n]_q([n-1]_q!)$ (see [4, sequence A022166]). It is to be noted that some entries are divisors of the corresponding 2-binomial coefficient. This pattern persist for further rows of the triangle, but so far no explanation is evident to the author. The numerators are even more mysterious, since they do not seem to be listed in the OEIS.

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References

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