THE BETTI NUMBERS OF REAL TORIC VARIETIES ASSOCIATED TO WEYL CHAMBERS OF TYPE B

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Abstract. We compute the (rational) Betti number of real toric varieties associated to Weyl chambers of type B. Furthermore, we show that their integral cohomology is p-torsion free for all odd primes p.

1. Introduction

A *toric variety* of complex dimension n is a normal algebraic variety over $\mathbb{C}$ with an effective algebraic action of $(\mathbb{C} \setminus \{O\})^n$ having an open dense orbit. A compact smooth toric variety is called a *toric manifold*. One of the most important facts on toric geometry is that there is a 1-1 correspondence between the class of toric varieties of complex dimension n and the class of fans in $\mathbb{R}^n$. This fact is called the *fundamental theorem of toric geometry*. In particular, a toric manifold X of complex dimension n corresponds to a complete regular fan Σ_X in $\mathbb{R}^n$.

Among toric manifolds, the class of toric manifolds associated to Weyl chambers has been considered since it is introduced by Procesi [12]. A classical construction associates to each root system a toric manifold whose fan corresponds to the reflecting hyperplanes of the root system and its weight lattice. It is natural to ask about the topology of the corresponding toric manifold. Note that the integral cohomology of a toric manifold is well-established by Jurkiwicz [11] for the projective cases and by Danilov [7] for general cases. For a coefficient field $\mathbf{k}$, the *ith* $\mathbf{k}$*-Betti number* of a topological space X is the rank of $H^i(X;\mathbf{k})$ over $\mathbf{k}$, and it is denoted by $\beta^i(X;\mathbf{k})$. One remarkable fact is that the Betti numbers of a toric manifold X depend only on the face numbers of its associated fan Σ_X. Especially, the structures of the cohomology of toric manifolds associated to Weyl chambers have been studied by [12], [15], [9] and [1].

On the other hand, the subset consisting of points with real coordinates of a toric manifold is called a *real toric manifold*. Unlike toric manifolds, little is known about the topology of real toric manifolds. Let X be a toric manifold and $X^{\mathbb{R}}$ its real toric manifold. By Davis and Januszkiewicz [8], the ith $\mathbb{Z}_2$-Betti number of $X^{\mathbb{R}}$ is equal to the $2i$th $\mathbb{Z}$-Betti number of X, and, hence, it depends only on the face numbers. However, the Betti numbers with rational coefficients are not only determined by the face numbers. For instance, both the torus and the Klein bottle are real toric manifolds and their corresponding fans have the face structure combinatorially equivalent to the 4-gon. Hence, their $\mathbb{Z}_2$-Betti numbers are the same while their

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$\mathbb{Q}$-Betti numbers are different. From this sense, the computation of the rational Betti number of real toric manifolds is difficult, and only a few examples have been computed so far. One known example is the real toric manifolds associated to Weyl chambers of type A_n due to Henderson [10]. Interestingly, their rational Betti numbers are the Euler zigzag numbers. Arnol′d [2] has defined the notion of snake numbers as a generalization of the Euler zigzag numbers as follows: a *snake of type* A_n (respectively B_n), or an A_n*-snake* (respectively, B_n*-snake*), is a sequence of integers x_i satisfying the conditions:

- for $A_n : x_0 < x_1 > x_2 < \cdots x_n$, $x_i \neq x_j$ for $i \neq j$;
- for $B_n : 0 < x_1 > x_2 < \cdots x_n$, $x_i \neq \pm x_j$ for $i \neq j$;

where $0 \leq x_i \leq n$ for all i for A_n, and $1 \leq |x_i| \leq n$ for all i for B_n. Denote by a_n (respectively, b_n) the number of A_n-snakes (respectively, B_n-snakes). The number a_n is also known as the Euler zigzag number (see A000111 of [13]), and the number b_n is also known as the generalized Euler number or the Springer number (see A001586 of [13]).

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
a_n	1	1	1	2	5	16	61	272	1385	7936	$\cdots$
b_n	1	1	3	11	57	361	2763	24611	250737	2873041	$\cdots$

TABLE 1. The list of a_n and b_n for small n

The formula of the rational Betti number by Henderson was recovered by Suciu [16] later using the general formula for rational Betti numbers of real toric manifolds established by Suciu and Trevisan [17].

Theorem 1.1 ([10], [16]). *Denote by* $X^{\mathbb{R}}_{A_n}$ *the real toric manifold associated to the Weyl chambers of type* A_n*. The* k*th* $\mathbb{Q}$*-Betti number of* $X^{\mathbb{R}}_{A_n}$ *is*

$$\beta^k(X^{\mathbb{R}}_{A_n};\mathbb{Q}) = \binom{n+1}{2k} a_{2k}.$$

We note that Choi and Park [5] showed that the formula used in [17] works for not only $\mathbb{Q}$ coefficient but also arbitrary field $\mathbf{k}$ coefficient whose characteristic is not equal to 2. Combining it with [16], we obtain the following.

Corollary 1.2. *The integral cohomology of* $X^{\mathbb{R}}_{A_n}$ *is* p*-torsion free for all odd primes* p*.*

In this paper, we compute the rational Betti number of real toric manifolds associated to the Weyl chambers of type B_n, and show that their integral cohomologies are p-torsion free for all odd primes p. We prove the following:

Theorem 1.3. *Denote by* $X^{\mathbb{R}}_{B_n}$ *the real toric manifold associated to the Weyl chambers of type* B_n*. Then, we have the following:*

$$\beta^k(X^{\mathbb{R}}_{B_n};\mathbb{Q}) = \binom{n}{2k} b_{2k} + \binom{n}{2k-1} b_{2k-1}.$$

Furthermore, their integral cohomologies are p*-torsion free for all odd primes* p*.*

It is worthwhile to note that the same techniques to prove the above theorem do not directly apply to the case of type C and D, the other regular types. This is because the analogues for the shellability results like Lemma 3.4 fail for type C or D, making it hard to compute homology of the corresponding posets.

This paper is organized as follows. In Section 2, we introduce preliminary facts including the formula of Suciu-Trevisan to compute the Betti numbers of real toric manifolds and the way to define projective toric manifolds associated to Weyl chambers. In Section 3, we prove the main theorem, that is, we compute the Betti numbers of real toric manifolds of type B_n.

2. Preliminaries

2.1. **The Betti numbers of real toric manifolds.** In this subsection, we shall introduce a formula of the Betti numbers of real toric manifolds. From now on, we restrict our interests in the projective toric manifolds and its real toric manifolds. Let X be a projective toric manifold of complex dimension n and $X^{\mathbb{R}}$ its real toric manifold. We assume that the associated fan Σ_X of X has m rays $r_1, \ldots, r_m$. Then, Σ_X can be regarded as a pair of an $(n-1)$-dimensional polytopal simplicial sphere K with the vertex set $[m] = \{1, \ldots, m\}$ and a map $\lambda\colon [m] \to \mathbb{Z}^n$ such that

- $\sigma = \{i_1, \ldots, i_\ell\} \in K$ if and only if $\{r_{i_1}, \ldots, r_{i_\ell}\}$ forms a cone in Σ_X, and
- $\lambda(i)$ is the primitive vector in the direction of r_i.

We call K the *underlying simplical complex* of X and λ the *characteristic map* of X. Furthermore, since X is projective, there is a convex simple polytope P with m facets $F_1, \ldots, F_m$ whose face structure is isomorphic to K and the outward normal vector of F_i is $\lambda(i)$ for $i = 1, \ldots, m$.

Similarly to the fundamental theorem for toric geometry, it is known that as a $\mathbb{Z}_2$-space, a real toric manifold $X^{\mathbb{R}}$ is determined by the pair $(K, \lambda^{\mathbb{R}})$, where $\lambda^{\mathbb{R}}$ the composition map of λ and the canonical quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, i.e., $\lambda^{\mathbb{R}}\colon [m] \xrightarrow{\lambda} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. We call $\lambda^{\mathbb{R}}$ the $\mathbb{Z}_2$-*characteristic map*, and we note that $\lambda^{\mathbb{R}}$ can be represented as a $\mathbb{Z}_2$-matrix of size $n \times m$, called the $\mathbb{Z}_2$-*characteristic matrix.* For each subset S of $\{1, \ldots, n\}$, write $\lambda_S = \sum_{i \in S} \lambda_i$, where λ_i is the ith row of $\lambda^{\mathbb{R}}$. Let $[m]_S := \{j \in [m] \mid \text{the } j\text{th entry of } \lambda_S \text{ is nonzero}\} \subset [m]$. For such S we define $K_S := \{\sigma \in K | \sigma \subset [m]_S\}$, and, as dual, $P_S := \bigcup_{j \in [m]_S} F_j$. We note that the topological realization of K_S is homotopy equivalent to P_S. Throughout this paper, we denote by K the topological realization of a simplicial complex K if there is no danger of confusion.

Theorem 2.1. [17, 5] *Let X be a toric manifold and $X^{\mathbb{R}}$ its real toric manifold. Let $\mathbf{k}$ be a ring where 2 is invertible in $\mathbf{k}$. Then the ith Betti number $\beta^i(X^{\mathbb{R}}; \mathbf{k})$ of $X^{\mathbb{R}}$ with coefficient $\mathbf{k}$ is given by*

$$\beta^i(X^{\mathbb{R}}; \mathbf{k}) = \sum_{S \subseteq [n]} \operatorname{rank}_{\mathbf{k}} \tilde{H}^{i-1}(P_S; \mathbf{k}) = \sum_{S \subseteq [n]} \operatorname{rank}_{\mathbf{k}} \tilde{H}^{i-1}(K_S; \mathbf{k}).$$

Suciu and Trevisan in their unpublished paper [17] have established the formula for the rational Betti numbers of real toric manifolds. Later, Choi and Park in [5] have also derived a cohomology formula of real toric manifolds with the coefficient ring G, where 2 is invertible in G. It should be noted that if the reduced cohomology of P_S is p-torsion free for all $S \subseteq [n]$ and all odd primes p, then so is the cohomology of $X^{\mathbb{R}}$. This formula also determines a stable homotopy decomposition of a wider class of spaces called real toric spaces, see [4].

2.2. **Projective toric manifold associated to Weyl chambers.** As mentioned in Introduction, we mainly deal with the class of (real) toric manifolds associated to the decomposition given by Weyl chambers. Let V be a finite dimensional real Euclidean space, $\Phi \subset V$ a root system, and W its Weyl group. In V we have the lattice $\Lambda = \{v \in V \mid (v, \alpha) \in \mathbb{Z} \text{ for any } \alpha \in \Phi\}$ which defines an integral structure, where $(-,-)$ is the natural inner product. For each set Δ of simple roots in Φ we consider the cone $C_\Delta = \{v \in V \mid (v, \alpha) > 0 \text{ for any } \alpha \in \Delta\}$. These cones provide the rational polyhedral decomposition of V, i.e., the set of cones is a fan in V. Hence, it defines a projective toric variety, which is in fact smooth.

From now on, let us consider the Weyl groups of regular types. Throughout this paper, for the Weyl group of type A_n, the corresponding toric variety, its fan, the underlying simplicial complex, the characteristic map, the corresponding real toric variety, and the $\mathbb{Z}_2$-characteristic map are denoted by X_{A_n}, Σ_{A_n}, K_{A_n}, λ_{A_n}, $X^{\mathbb{R}}_{A_n}$ and $\lambda^{\mathbb{R}}_{A_n}$, respectively. For the Weyl group of type B_n, the corresponding notions are similarly denoted by X_{B_n}, Σ_{B_n}, K_{B_n}, λ_{B_n}, $X^{\mathbb{R}}_{B_n}$ and $\lambda^{\mathbb{R}}_{B_n}$, respectively.

2.3. **Type A_n.** In this subsection, we shall review a sketch of proof of Theorem 1.1 and Corollary 1.2. The proof presented here is essentially the same with that by [16] or [6] for the special case when the corresponding graph is a complete graph. However, we enclose this subsection for the sake of self-contained readability. Furthermore, it introduces a lemma needed to prove the main result.

Let W be the Weyl group of type A_n. It is well-known that the vertices of K_{A_n} can be identified by the nonempty proper subsets I of $[n+1]$ and each $(\ell-1)$-dimensional simplex of K_{A_n} is related to a nested ℓ nonempty proper subsets of $[n+1]$, that is, $\{I_{i_1}, \ldots, I_{i_\ell}\} \in K_{A_n}$ if and only if there is a permutation σ on $[\ell]$ such that $I_{i_{\sigma(1)}} \subset \cdots \subset I_{i_{\sigma(\ell)}}$. In addition, the characteristic map λ_{A_n} is

$$\lambda_{A_n}(I) = \begin{cases} \sum_{k \in I} \varepsilon_k, & \text{if } \{n+1\} \not\in I; \\ \sum_{k \in I \setminus \{n+1\}} \varepsilon_k - \varepsilon_1 - \cdots - \varepsilon_n, & \text{if } \{n+1\} \in I, \end{cases}$$

where ε_i is the ith standard vector of $\mathbb{Z}^n$. As consequence,

$$\lambda^{\mathbb{R}}_{A_n}(I) = \begin{cases} \sum_{k \in I} \mathbf{e}_k, & \text{if } \{n+1\} \not\in I; \\ \sum_{k \not\in I} \mathbf{e}_k, & \text{if } \{n+1\} \in I, \end{cases}$$

where $\mathbf{e}_i$ is the ith standard vector of $\mathbb{Z}_2^n$.

From now on, let us compute the $\mathbb{Q}$-Betti number of $X^{\mathbb{R}}_{A_n}$. By Theorem 2.1, we have to consider $(K_{A_n})_S$ for all subsets $S \subset [n]$. Here are three important nontrivial steps.

For an odd number r, define $K^{odd}_{A_r}$ as

$$K^{odd}_{A_r} := (K_{A_r})_{[r]}.$$

(1) $K^{odd}_{A_r}$ is homotopy equivalent to the wedge of spheres of dimension $\frac{r-1}{2}$.
(2) The reduced Euler characteristic $\tilde{\chi}(K^{odd}_{A_r})$ is $(-1)^{\frac{r-1}{2}} a_{r+1}$.
(3) For $S \subset [n]$ with $|S| = r$ or $|S| = r+1$ for some odd number r, $(K_{A_n})_S$ is homotopy equivalent to $K^{odd}_{A_r}$.

By (1)–(3) together with Theorem 2.1, both Theorem 1.1 and Corollary 1.2 are immediately proved.

3. Type B_n

Let Φ be a root system of type B_n. It consists of $2n^2$ roots

$$\pm\varepsilon_i \ (1 \le i \le n) \quad \text{and} \quad \pm \varepsilon_i \pm \varepsilon_j \ (1 \le i < j \le n),$$

where ε_i is the ith standard vector of $\mathbb{R}^n = V$. One can see that the lattice Λ consists of all integral vectors in $\mathbb{R}^n$. We note that a line containing a ray of Σ_{B_n} is the intersection of $n-1$ hyperplanes normal to $\Delta \setminus \{\alpha\}$, where Δ is a set of simple roots of type B_n and $\alpha \in \Delta$, and the direction of the ray is determined by α. A set of simple roots of type B_n forms

$$\Delta = \{\mu_1\varepsilon_{\sigma(1)} - \mu_2\varepsilon_{\sigma(2)}, \mu_2\varepsilon_{\sigma(2)} - \mu_3\varepsilon_{\sigma(3)}, \ldots, \mu_{n-1}\varepsilon_{\sigma(n-1)} - \mu_n\varepsilon_{\sigma(n)}, \mu_n\varepsilon_{\sigma(n)}\},$$

where $\mu_j = \pm 1$ and $\sigma\colon [n] \to [n]$ is a permutation. For $\alpha \in \Delta$, there exists a unique primitive integral vector $\beta = (b_1, \ldots, b_n)$ such that $(\beta, \alpha') = 0$ for all $\alpha' \in \Delta \setminus \{\alpha\}$ and $(\beta, \alpha) > 0$. We note that each component b_j of β is either ± 1 or 0. Then, we label the ray of Σ_{B_n} corresponding to $\alpha \in \Delta$ by the set $I = \{jb_j \mid j = 1, \ldots, n\} \subset [\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. More precisely, by putting $x_i = \mu_i\varepsilon_{\sigma(i)} - \mu_{i+1}\varepsilon_{\sigma(i+1)}$ for $i = 1, \ldots, n-1$ and $x_n = \mu_n\varepsilon_{\sigma(n)}$, if $\alpha = x_i$, then $\beta = \displaystyle\sum_{k=1}^{i} \mu_k\varepsilon_{\sigma(k)}$, and, hence, the corresponding label is $\{\mu_1\sigma(1), \ldots, \mu_i\sigma(i)\}$. Therefore, the vertices of K_{B_n} can be labelled by the nonempty subsets I of $[\pm n]$ satisfying

$$(*) \qquad \text{if } i \in I, \text{ then } -i \notin I,$$

and the characteristic map λ_{B_n} is

$$\lambda_{B_n}(I) = \sum_{k \in I \cap [n]} \varepsilon_k - \sum_{k \in I \setminus [n]} \varepsilon_{-k}.$$

As consequence,

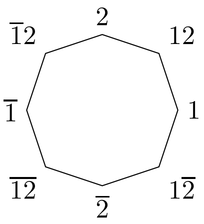
$$\lambda^{\mathbb{R}}_{B_n}(I) = \sum_{k \in (I \cup -I) \cap [n]} \mathbf{e}_k,$$

where $\mathbf{e}_i$ is the ith standard vector of $\mathbb{Z}_2^n$.

Furthermore, one can see that each n-dimensional cone C_Δ in Σ_{B_n} corresponds to n subsets $I_1, \ldots, I_n$ satisfying $(*)$ such that $I_1 \subsetneq \cdots \subsetneq I_n$ and vice versa. This implies that each $(\ell-1)$-dimensional simplex of K_{B_n} is labelled by a nested ℓ subsets of $[\pm n]$ satisfying $(*)$, that is, $\{I_{i_1}, \ldots, I_{i_\ell}\} \in K_{B_n}$ if and only if there is a permutation σ on $[\ell]$ such that $I_{i_{\sigma(1)}} \subset \cdots \subset I_{i_{\sigma(\ell)}}$.

Example 3.1. Let us consider Σ_{B_2}. The corresponding toric variety X_{B_2} is $\mathbb{CP}^2 \sharp 5\overline{\mathbb{CP}}^2$, and the corresponding real toric variety $X^{\mathbb{R}}_{B_2}$ is the connected sum of six $\mathbb{RP}^2$s. Let us compute the Betti number of $X^{\mathbb{R}}_{B_2}$ using Theorem 2.1. We express $\lambda^{\mathbb{R}}_{B_2}$ by a matrix and draw the geometric realization of K_{B_2} as below, respectively:

$$\left(\begin{array}{cccccccc}
1 & 2 & \bar{1} & \bar{2} & 12 & 1\bar{2} & \bar{1}2 & \bar{1}\bar{2} \\
\hline
1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right)$$



where the numbers over the horizontal lines are indicators for vertices of K_{B_2}. Then, $(K_{B_2})_{\{1\}} \simeq S^1 \setminus \{\{2\}, \{\overline{2}\}\}$ is homotopy equivalent to S^0, and similarly, we have $(K_{B_2})_{\{2\}} \simeq S^0$ and $(K_{B_2})_{\{1,2\}} \simeq \bigvee^3 S^0$. Therefore, the Betti number of $X^{\mathbb{R}}_{B_2}$ is

$$\beta^i(X^{\mathbb{R}}_{B_2};\mathbb{Q}) = \begin{cases} 1, & i = 0; \\ 5, & i = 1; \\ 0, & \text{otherwise.} \end{cases}$$

From now on, let us compute the $\mathbb{Q}$-Betti number of $X^{\mathbb{R}}_{B_n}$. By Theorem 2.1, we have to consider $(K_{B_n})_S$ for all subsets $S \subset [n]$. Given a subset $S \subset [n]$, then $(K_{B_n})_S$ is the restriction of K_{B_n} by $\{I \in V(K_{B_n}) \mid |S \cap I^{\pm}| \text{ is odd}\}$, where $V(K)$ is the vertex set of a simplicial complex K.

Now let us consider the case where $S = [n]$. Define $K^{odd}_{B_n}$ as

$$K^{odd}_{B_n} := (K_{B_n})_{[n]} = \{\sigma \in K_{B_n} \mid \sigma \text{ consists of } I \text{ such that } |I| \text{ is odd}\}.$$

We define the poset $S^{odd}_{B_n}$ whose vertices are the vertices of $K^{odd}_{B_n}$ and the partial order is given by inclusion, and define another poset $\tilde{S}^{odd}_{B_n} := S^{odd}_{B_n} \cup \{\emptyset, [\pm n]\}$ with inclusion. Note that the order complex of $S^{odd}_{B_n}$ is $K^{odd}_{B_n}$, and hence, $\tilde{\chi}(K^{odd}_{B_n}) = \mu(\emptyset, [\pm n])$ where μ is the Möbius function of $\tilde{S}^{odd}_{B_n}$ (see Section 3 of [14] for details), that is,

$$\mu(\rho,\tau) = \begin{cases} 1, & \text{if } \rho = \tau; \\ -\sum_{\rho \le \sigma < \tau} \mu(\rho,\sigma), & \text{if } \rho \subset \tau \text{ in } \tilde{S}^{odd}_{B_n}. \end{cases}$$

Lemma 3.2. *The absolute value of $\mu(\emptyset, [\pm n])$ is b_n. More precisely,*

$$\mu(\emptyset,[\pm n]) = \begin{cases} b_n, & \text{if } n \equiv 0, 1 \ (mod\ 4); \\ -b_n, & \text{if } n \equiv 2, 3 \ (mod\ 4). \end{cases}$$

Proof. In this proof, we use i to denote the imaginary unit such that $i^2 = -1$.

For a vertex I in $\tilde{S}^{odd}_{B_n}$ such that I is neither $\emptyset$ nor $[\pm n]$, put $|I| = 2k+1$. Note that the Möbius function $\mu(\emptyset, I)$ depends only on $|I|$ and $\mu(\emptyset, I) = (-1)^{k+1} a_{2k+1}$ (see the proof of Theorem 2.9 of [6]). Hence, we have

$$\begin{aligned} -\mu(\emptyset,[\pm n]) &= 1 + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{k+1} a_{2k+1} 2^{2k+1} \binom{n}{2k+1} \\ &= 1 + i \sum_{k=0}^{\infty} a_{2k+1} (2i)^{2k+1} \binom{n}{2k+1}. \end{aligned}$$

Recall that the exponential generating functions of a_n and b_n are

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = \sec x + \tan x$$

and

$$B(x) := \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} = \frac{1}{\cos x - \sin x} = (\cos x + \sin x) \sec 2x$$

respectively. Since $e^x = \cos(-ix) + i\sin(-ix)$, we have

$$\begin{aligned} M(x) := -\sum_{n=0}^{\infty} \mu(\emptyset, [\pm n]) \frac{x^n}{n!} &= e^x(1 + i\tan(2ix)) \\ &= e^x \frac{\cos(2ix) + i\sin(2ix)}{\cos(2ix)} \\ &= (\cos(ix) + i\sin(ix))\sec(2ix). \end{aligned}$$

Therefore, $M(ix) = (\cos x - i\sin x)\sec 2x$. Since the exponential generating function of $\sec x$ has only even degree terms, $\cos x \sec 2x$ contributes the even degree terms of $M(ix)$ and $\sin x \sec 2x$ contributes the odd degree terms of $-iM(ix)$. Therefore, the lemma immediately follows from that the odd degree term of $B(x)$ is equal to that of $M(ix)$ and the even degree term of $B(x)$ is equal to that of $-iM(ix)$. $\square$

We will use the following well-known lemma in [3]. This can be regarded as an alternative definition of shellability. Recall that a simplicial complex is called shellable if it admits a shelling.

Lemma 3.3. [3, Lemma 2.3] *An order $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_t$ of the facets of a simplicial complex is a shelling if and only if for every i and k with $1 \le i < k \le t$ there is a j with $1 \le j < k$ such that $\mathcal{F}_i \cap \mathcal{F}_k \subseteq \mathcal{F}_j \cap \mathcal{F}_k$ and $|\mathcal{F}_j \cap \mathcal{F}_k| = |\mathcal{F}_k| - 1$.*

Lemma 3.4. *For any integer n, $K^{odd}_{B_n}$ is shellable.*

Proof. Note that since K_{B_n} bounds for a convex polytope, it is shellable. Choose a shelling $\sigma\colon F_1, \ldots, F_t$ of K_{B_n}. For each $m \in [t]$, let F'_m be the face obtained from F_m by deleting all vertices of F_m corresponding to even subsets of $[\pm n]$. Note that for any $m \in [t]$, F'_m is a facet of $K^{odd}_{B_n}$. Then consider an ordering $\sigma' : F'_1, \ldots, F'_t$ of the facets of $K^{odd}_{B_n}$, and then we delete F'_m whenever $F'_m = F'_\ell$ for some ℓ such that $\ell < m$. Let $\sigma^*\colon F^*_1, F^*_2, \ldots, F^*_s$ be the resulting ordering, that is, the ordering obtained from σ' by dropping all facets of $K^{odd}_{B_n}$ not firstly appeared in σ'. Clearly, σ^* is an ordering of the facets of $K^{odd}_{B_n}$. We will show that σ^* is a shelling of $K^{odd}_{B_n}$. By Lemma 3.3, it is enough to show that, for every i and k with $1 \le i < k \le s$, there is j with $1 \le j < k$ such that

(1) $F^*_i \cap F^*_k \subseteq F^*_j \cap F^*_k$, and
(2) $|F^*_j \cap F^*_k| = |F^*_k| - 1$.

For each $m \in [s]$, let $d_m \in [t]$ be the smallest integer such that $F^*_m \subset F_{d_m}$, i.e., F_{d_m} is the first facet in σ containing F^*_m. Note that for all $\ell, m \in [s]$,

$$d_\ell < d_m \text{ if and only if } \ell < m. \tag{3.1}$$

Take i and k with $1 \le i < k \le s$. Then $F^*_i \subset F_{d_i}$ and $F^*_k \subset F_{d_k}$. Since $d_i < d_k$ by (3.1), by considering two facets F_{d_i} and F_{d_k} of K_{B_n} together with Lemma 3.3, there is J with $1 \le J < d_k$ such that $F_{d_i} \cap F_{d_k} \subseteq F_J \cap F_{d_k}$ and $|F_J \cap F_{d_k}| = |F_{d_k}| - 1$. Then we consider a facet F'_J of $K^{odd}_{B_n}$. Let j be the smallest integer such that $F^*_j = F'_J$. Then $d_j \le J$ by definition, and so we have $d_j \le J < d_k$. Thus $j < k$ by (3.1), and it indeed satisfies the conditions (1) and (2) as follows.

Let V be the set of vertices of $K^{odd}_{B_n}$. Note that $F^*_i \cap F^*_k = F_{d_i} \cap F_{d_k} \cap V$ and $F^*_j \cap F^*_k = F_J \cap F_{d_k} \cap V$. Therefore (1) follows from the fact that $F_{d_i} \cap F_{d_k} \subseteq F_J \cap F_{d_k}$. Moreover, since $F^*_j \cap F^*_k = F_J \cap F_{d_k} \cap V$ and $|F_J \cap F_{d_k}| = |F_{d_k}| - 1$, we have $|F^*_j \cap F^*_k| \ge |F^*_k| - 1$. Since $j \ne k$ implies that $F^*_j \ne F^*_k$, (2) is proved. $\square$

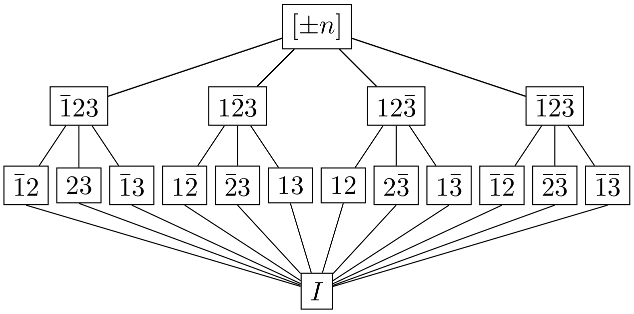


FIGURE 1. The interval $[I, [\pm n]]$ of the poset $\tilde{S}^{odd}_{C_n}$. Here, $\bar{i}$ means $-i$ and the label in each box means a vertex in $K^{odd}_{C_n}$ obtained by the union of I and the elements in the label. For example, $\bar{1}2$ and $\bar{2}3$ mean $I \cup \{-1, 2\}$ and $I \cup \{-2, 3\}$, respectively.

Note that $K^{odd}_{B_n}$ is homotopy equivalent to a wedge of uniform spheres S^d as it is shellable. One can easily see that the dimension of the sphere is $d = \lfloor \frac{n-1}{2} \rfloor$ by observing the dimension of the facets. Since the absolute value of the reduced Euler characteristic of $K^{odd}_{B_n}$ is b_n by Lemma 3.2, we conclude that $K^{odd}_{B_n} \simeq \bigvee^{b_n} S^{\lfloor \frac{n-1}{2} \rfloor}$.

Remark 3.5. Here we give an explicit shelling of $K^{odd}_{B_n}$. We define an ordering $\prec$ on $[\pm n]$,

$$1 \prec 2 \prec \cdots \prec n \prec -1 \prec \cdots \prec -n,$$

(just fix an ordering so that the positive integers proceed to the negative integers) and we define an order lexicographically induced by $\prec$ on the set of all maximal chains of $\tilde{S}^{odd}_{B_n}$ (comparing the smaller element). We also denote by the same symbol $\prec$ the order on the set of all maximal chains. More precisely, for two maximal chains σ and σ' such that

$$\sigma : \emptyset = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_r \subsetneq I_{r+1} = [\pm n]$$
$$\sigma' : \emptyset = I'_0 \subsetneq I'_1 \subsetneq I'_2 \subsetneq \cdots \subsetneq I'_r \subsetneq I'_{r+1} = [\pm n],$$

we say $\sigma' \prec \sigma$ if there exists $1 \leq i \leq r$ such that $I'_i <_{lexi} I_i$ (comparing lexicographically under the ordering $\prec$ on $[\pm n]$) and ${I'}_j = I_j$ for any $j < i$. Then it can be shown that this ordering on maximal chains gives a shelling of $\tilde{S}^{odd}_{B_n}$.

Remark 3.6. The proof of Lemma 3.4 is not extended naturally to the case for Weyl chambers of types C_n and D_n. One can check that $K^{odd}_{C_n}$ is a set of faces σ in K_{C_n} such that σ consists of I satisfying one of (1)∼(4):

1. $n \notin I^{\pm}$, and $|I|$ is odd;
2. $n \in I^{\pm}$, $|I| \neq n$, $n - |I|$ is odd;
3. $n \in I$, $|I| = n$, $|I \setminus [n]|$ is even;
4. $-n \in I$, $|I| = n$, $|I \cap [n]|$ is even,

where $I^{\pm}$ denotes $(I \cup -I) \cap [n]$. In general, $K^{odd}_{C_n}$ is not shellable. For an illustration, consider $\tilde{S}^{odd}_{C_n}$ when n is an even integer such that $n \geq 4$. Let $I = \{4, 5, \ldots, n\}$ and consider the interval $[I, [\pm n]]$ of the poset $\tilde{S}^{odd}_{C_n}$, see Figure 1. It is easy to see that the interval $[I, [\pm n]]$ is not shellable. Since $\tilde{S}^{odd}_{C_n}$ has an non-shellable interval, it is not shellable. It can be similarly shown that $K^{odd}_{D_n}$ is not shellable.

Now, let us return to the case where $S \neq [n]$. If S is an empty set, so is $(K_{B_n})_S$.

Lemma 3.7 (Lemma 5.2 of [6])**.** *Let I be a vertex of a simplicial complex K and suppose that the link of I, $\mathrm{Lk}I$, is contractible. Then K is homotopy equivalent to the complex $K \setminus \mathrm{St}I$, where $\mathrm{St}I$ is the star of I.*

Lemma 3.8. *For a positive integer $n \geq 3$, for $S \subset [n]$, $(K_{B_n})_S$ is homotopy equivalent to $(K_{B_n})'_S$, where $(K_{B_n})'_S$ is obtained from $(K_{B_n})_S$ by deleting vertices I in $(K_{B_n})_S$ such that $I^{\pm} \not\subset S$.*

Proof. For simplicity, we let $K = (K_{B_n})_S$ and $K' = (K_{B_n})'_S$. We will show that we can eliminate stars of vertices in $K \setminus K'$, one by one, from K to K', without changing the homotopy type. First, for any vertex I of K, $I \cap S \neq \emptyset$. In addition, two vertices I and J meet in K if and only if $I \subset J$ or $J \subset I$.

Let I be a vertex of $K \setminus K'$ such that $|I^{\pm} \cap S| = 1$, say $I^{\pm} \cap S = \{x\}$. Let J be a vertex in K such that $J^{\pm} = \{x\}$ and $J \subsetneq I$. Take any $L \in \mathrm{Lk}I$. If $I \subset L$, then $J \subset L$, and so L meets J. Suppose that $L \subset I$. Then $L^{\pm} \cap S$ is a subset of $I^{\pm} \cap S$. Since L is a vertex of K, $L^{\pm} \cap S \neq \emptyset$. Therefore $L^{\pm} \cap S = I^{\pm} \cap S = \{x\} = J^{\pm}$ and so $J^{\pm} \subset L^{\pm}$. Since $J^{\pm} \subset L^{\pm} \subset I^{\pm}$, $J \subset I$ and $L \subset I$, it follows that $J \subset L$, and so L meets J.

Hence, $\mathrm{Lk}I$ is contractible, and so K is homotopy equivalent to $K \setminus \mathrm{St}I$ by Lemma 3.7. By redefining $K := K \setminus \mathrm{St}I$ and repeating the argument, we can conclude that the star of any vertex I with $|I^{\pm} \cap S| = 1$ can be eliminated.

Inductively, assume that we could eliminate all vertices $I \in K \setminus K'$ such that $|I^{\pm} \cap S| < j$, and let K^* be the simplicial complex obtained by deleting stars of all those vertices, where $j \geq 2$. Take a smallest vertex $I \in K^* \setminus K'$ such that $|I^{\pm} \cap S| = j$. Let J be a vertex in K^* such that $I^{\pm} \cap S = J^{\pm}$ and $J \subset I$. (Note that $J \in K'$ and so J is in K^* and $I \neq J$.) Take any $L \in \mathrm{Lk}I$ in K^*. If $I \subset L$, then $J \subset L$, and so L meets J. Suppose that $L \subset I$. Then $L^{\pm} \cap S$ is a subset of $I^{\pm} \cap S = J^{\pm}$. If $|L^{\pm} \cap S| < j$, then such L should have already been deleted by our induction hypothesis. Thus $|L^{\pm} \cap S| = j$ and so $L^{\pm} \cap S = J^{\pm}$. Therefore $J^{\pm} \subset L^{\pm}$. Since $J \subset I$ and $L \subset I$, we have $J \subset L$, and so L meets J.

Hence, $\mathrm{Lk}I$ is contractible, and so K^* is homotopy equivalent to $K^* \setminus \mathrm{St}I$ by Lemma 3.7. By redefining $K^* := K^* \setminus \mathrm{St}I$ and repeating the argument, we can conclude that the star of any vertex I with $|I^{\pm} \cap S| = j$ can be eliminated in increasing order of the size $|I^{\pm} \cap S|$. □

Lemma 3.9. *Let $r = |S|$. Then, $(K_{B_n})_S$ is homotopy equivalent to $K^{odd}_{B_r}$.*

Proof. By Lemma 3.8, it clearly follows. □

Theorem 3.10. *The ith $\mathbb{Q}$-Betti number $\beta^i(X^{\mathbb{R}}_{B_n};\mathbb{Q})$ of $X^{\mathbb{R}}_{B_n}$ is*

$$\beta^i(X^{\mathbb{R}}_{B_n};\mathbb{Q}) = \binom{n}{2i} b_{2i} + \binom{n}{2i-1} b_{2i-1}.$$

Furthermore, their integral cohomologies of $X^{\mathbb{R}}_{B_n}$ are p-torsion free for all odd primes p.

Proof. Let $S \subset [n]$ and assume that $|S| = r$. By Lemma 3.9, $(K_{B_n})_S \cong K^{odd}_{B_r}$. We recall that $(K_{B_r})_S \cong \bigvee^{b_r} S^{\lfloor \frac{r-1}{2} \rfloor}$. Hence, the topology type of $(K_{B_n})_S$ only depends on the cardinality of S. For a fixed i and a field $\mathbf{k}$ with characteristic is not equal to 2, by Theorem 2.1,

$$\beta^i(X^{\mathbb{R}}_{B_n};\mathbf{k}) = \sum_S \beta^{i-1}\tilde{\beta}((K_{B_n})_S;\mathbf{k}) = \sum_{r=0}^{n} \delta_{i-1,\lfloor\frac{r-1}{2}\rfloor}\binom{n}{r} b_r,$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. It proves the theorem. $\square$

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