A note on a bijection for Schröder permutations

DAVID CALLAN
Department of Statistics
University of Wisconsin-Madison
Madison, WI 53706-1532
callan@stat.wisc.edu

Abstract

There is a bijection from Schröder paths to {4132, 4231}-avoiding permutations due to Bandlow, Egge, and Killpatrick that sends "area" to "inversion number". Here we give a concise description of this bijection.

1 Introduction

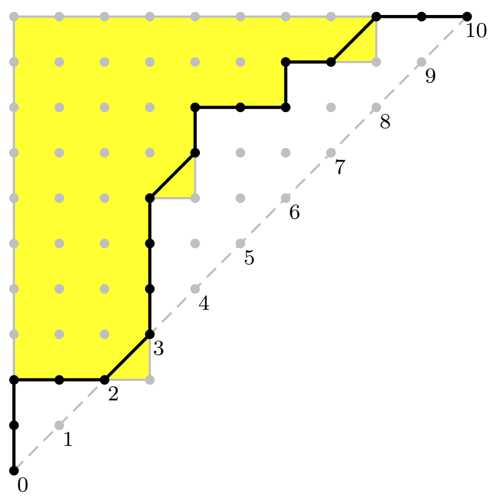
Permutations of length n avoiding the two 4-letter patterns 4132 and 4231 are known [1] to be counted by the large Schröder number r_{n-1} with generating function $\sum_{n\geq 0} r_n x^n = 1 + 2x + 6x^2 + 22x^3 + \cdots = \frac{1-x-\sqrt{1-6x+x^2}}{2x}$ (A006318). It is well known that r_n is the number of Schröder paths of size n where a *Schröder path* is a lattice path of north steps $N = (0,1)$, diagonal steps $D = (1,1)$ and east steps $E = (1,0)$ that starts at the origin, never drops below the diagonal $y = x$, and terminates on the diagonal. Its *size* is $\#\, N$ steps + $\#\, D$ steps, and a Schröder n-path is one of size n. Thus a Schröder n-path ends at (n,n).

Bandlow, Egge, and Killpatrick [2] construct a bijection from Schröder n-paths to {4132, 4231}-avoiding permutations of length $n+1$. They show that their bijection relates the area below the path to the number of inversions in the permutation. Here we give a concise, and possibly more transparent, formulation of their bijection.

It is convenient, and natural for present purposes, to take our permutations on $[0,n]$ rather than the standard $[1,n]$ and to define the *truncated coinversion table* of a permutation p on $[0,n]$ to be the list $c = (c_i)_{i=1}^n$ where c_i is the number of coinversions topped by i, that is, $c_i = \#\{j : 0 \leq j < i \text{ and } j \text{ precedes } i \text{ in the permutation}\}$. The table is "truncated" because we omit c_0 which would necessarily be 0. We say i is a fixed point of c if $c_i = i$ and denote the list of fixed points of c by FP(c). Thus, for $p = 350214$, we have $c = 11041$ and FP$(c) = (1,4)$. Clearly, the fixed points of c are right-to-left minima of p and conversely, every nonzero right-to-left minimum of p is a fixed point of c.

2 The Bandlow-Egge-Killpatrick bijection

In these terms, the bijection from {4132, 4231}-avoiders p on $[0, n]$ to Schröder n-paths has a short description: find the coinversion table c of p, then form the unique Schröder n-path whose D steps end at x coordinates in $\mathrm{FP}(c)$ and whose N steps, taken in order, have x coordinates given by the list $c \setminus \mathrm{FP}(c)$. For example, with $n = 10$ and $p = 2\ 1\ 0\ 7\ 9\ 6\ 10\ 5\ 3\ 4\ 8$, we have $c = 0\ 0\ 3\ 4\ 3\ 3\ 3\ 8\ 4\ 6$, $\mathrm{FP}(c) = 3\ 4\ 8$, and $c \setminus \mathrm{FP}(c) = 0\ 0\ 3\ 3\ 3\ 4\ 6$. The corresponding Schröder path is shown in Figure 1.



A Schröder 10-path

Figure 1

The map is both defined and reversible due to the following characterization of $\{4132, 4231\}$-avoiders whose straightforward proof is left to the interested reader.

Proposition. *Suppose p is a permutation on $[0, n]$ with truncated coinversion table c. Then p avoids $\{4132, 4231\}$ if and only if the list $c \setminus \mathrm{FP}(c)$ is weakly increasing.* □

The connection between number of inversions in the permutation ($= \binom{n+1}{2} - \#$ coinversions) and area of the path is clear: the number of unit squares in the first quadrant lying wholly or partially to the left of the path (in yellow in Figure 1) is the sum of the x coordinates of all the north and diagonal steps which, by the construction, is $\sum_{i=1}^{n} c_i$, the total number of coinversions in the permutation.

References

[1] Darla Kremer, Permutations with forbidden subsequences and a generalized Schröder number, *Discrete Mathematics* **218** (2000) 121–130.

[2] Jason Bandlow, Eric S. Egge, and Kendra Killpatrick, A weight-preserving bijection between Schröder paths and Schröder permutations, *Annals of Combinatorics* **6** (2002) 235–248.