ON ERROR SUMS FORMED BY RATIONAL APPROXIMATIONS WITH SPLIT DENOMINATORS

Thomas Baruchel and Carsten Elsner*

Abstract

In this paper we consider error sums of the form

$$\sum_{m=0}^{\infty} \varepsilon_m \Big(b_m \alpha - \frac{a_m}{c_m} \Big) ,$$

where α is a real number, a_m, b_m, c_m are integers, and $\varepsilon_m = 1$ or $\varepsilon_m = (-1)^m$. In particular, we investigate such sums for

$$\alpha \in \big\{ \pi, e, e^{1/2}, e^{1/3}, \ldots, \log(1+t), \zeta(2), \zeta(3) \big\}$$

and exhibit some connections between rational coefficients occurring in error sums for Apéry's continued fraction for $\zeta(2)$ and well-known integer sequences. The concept of the paper generalizes the theory of ordinary error sums, which are given by $b_m = q_m$ and $a_m/c_m = p_m$ with the convergents p_m/q_m from the continued fraction expansion of α.

Keywords: Errorr sums, continued fractions
AMS Subject Classification: 11J70, 11J04, 33B10, 33C05, 11M06.

*Lycée naval, Centre d'instruction naval, Brest, France, e-mail: baruchel@riseup.net
Fachhochschule für die Wirtschaft, University of Applied Sciences, Freundallee 15, D-30173 Hannover, Germany, e-mail: carsten.elsner@fhdw.de

1 Introduction

Let α be a real number. We assume that there is a sequence $B := (b_n)_{n\geq 0}$ of integers, a sequence $R := (r_n)_{n\geq 0}$ of rationals $r_n = a_n/c_n$, say, with $a_n \in \mathbb{Z}$ and $c_n \in \mathbb{N}$, and a real number $\omega > 1$ satisfying

$$\left|b_n c_n \alpha - a_n\right| \ll \frac{c_n}{\omega^n} \qquad (n \geq 0)\,. \tag{1.1}$$

This is equivalent with

$$\left|b_n \alpha - r_n\right| = \left|b_n \alpha - \frac{a_n}{c_n}\right| \ll \frac{1}{\omega^n} \qquad (n \geq 0)\,. \tag{1.2}$$

We consider the fraction $a_n/b_n c_n$ as a rational approximation of α with split denominator $b_n c_n$. Since $\omega > 1$, the error sums

$$\mathcal{E}^*\big(B,R,\alpha\big) := \sum_{m=0}^{\infty} \big(b_m \alpha - r_m\big) = \sum_{m=0}^{\infty} \Big(b_m \alpha - \frac{a_m}{c_m}\Big)\,, \tag{1.3}$$

$$\mathcal{E}\big(B,R,\alpha\big) := \sum_{m=0}^{\infty} \big|b_m \alpha - r_m\big| = \sum_{m=0}^{\infty} \Big|b_m \alpha - \frac{a_m}{c_m}\Big| \tag{1.4}$$

exist. Let $(p_n/q_n)_{n\geq 0}$ be the sequence of convergents of α defined by $p_n/q_n = \langle a_0; a_1, a_2, \dots a_n\rangle$ from the regular continued fraction expansion

$$\alpha = \langle a_0; a_1, a_2, \dots\rangle = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

of α. The error sums of α for $B = (q_n)_{n\geq 0}$ and $R = (p_n)_{n\geq 0}$, namely

$$\mathcal{E}^*(\alpha) := \mathcal{E}^*\big(B,R,\alpha\big) = \sum_{m=0}^{\infty} \big(q_m \alpha - p_m\big)\,,$$

$$\mathcal{E}(\alpha) := \mathcal{E}\big(B,R,\alpha\big) = \sum_{m=0}^{\infty} \big|q_m \alpha - p_m\big|\,,$$

were already studied in some papers [6, 7, 8, 9]. We call $\mathcal{E}^*(\alpha)$ and $\mathcal{E}(\alpha)$ *ordinary error sums*. Conversely, for $B = (1)_{n\geq 0}$ and $R = (p_n/q_n)_{n\geq 0}$, until now nobody has found any remarkable approach to the error sums

$$\mathcal{E}^*(B,R,\alpha) = \sum_{m=0}^{\infty} \Big(\alpha - \frac{p_m}{q_m}\Big)\,,$$

$$\mathcal{E}(B,R,\alpha) = \sum_{m=0}^{\infty} \Big|\alpha - \frac{p_m}{q_m}\Big|\,.$$

In this paper we focus our interest on the series in (1.3) and (1.4) in the case of particular values of α and well-known rational approximations of the form

$$0 < \Big|b_n \alpha - \frac{a_n}{c_n}\Big| \ll \frac{1}{\omega^n} \qquad (n \geq 0)\,. \tag{1.5}$$

Among others we are going to study the numbers

$$\alpha \in \left\{ \pi, \, e^{1/l}, \, \frac{\log \rho}{\sqrt{5}}, \, \log(1+t), \, \zeta(2), \, \zeta(3) \right\},$$

where $l = 1, 2, \ldots$, $e = \exp(1)$, $\rho = (1+\sqrt{5})/2$, and $-1 < t \leq 1$, and we shall investigate extraordinary properties of corresponding error sums (1.3) and (1.4).

2 Ordinary error sums for values of the exponential function

Ordinary error sums connected with the exponential function are studied in [1, 10]. Here, our goal is to express this usual error sums itselves by a non-regular continued fraction. For this purpose we express the error integral

$$\text{erf}\,(z) \;=\; \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt$$

by a hypergeometric series, which again can be transformed into a Gauss-type continued fraction.

Theorem 2.1. *Let $l \geq 2$ be an integer, and let p_n/q_n denote the convergents of $e^{1/l}$. Then we have*

$$\begin{aligned}
\mathcal{E}(e^{1/l}) \;&=\; \sum_{n\geq 0} \left| e^{1/l} q_n - p_n \right| \;=\; e^{1/l} \sqrt{\frac{\pi}{l}}\, \mathit{erf}\Big(\frac{1}{\sqrt{l}}\Big) \;=\; \frac{2e^{1/l}}{\sqrt{l}} \int_0^{1/\sqrt{l}} e^{-t^2}\,dt \\
&=\; \cfrac{1/l}{1/2 - \cfrac{1/2l}{3/2 + \cfrac{2/2l}{5/2 - \cfrac{3/2l}{7/2 + \cfrac{4/2l}{9/2 - \cfrac{5/2l}{11/2 + \ddots \cfrac{(-1)^m m/2l}{(2m+1)/2 + \ddots}}}}}}} \qquad (m \geq 1)\,.
\end{aligned}$$

Proof: The first identity of the theorem expressing $\mathcal{E}(e^{1/l})$ by an error integral is already known from [1, 10]. In order to prove the continued fraction expansion, we set

$$f(z) \;:=\; \frac{\sqrt{\pi}}{2} z e^{z^2} \text{erf}\,(z) \;=\; z e^{z^2} \int_0^z e^{-t^2}\,dt\,.$$

We express $f(z)$ in terms of a hypergeometric function ${}_1F_1(\alpha,\beta;z^2)$.

$$\begin{aligned} f(z) &= ze^{z^2}\int_0^z \sum_{\nu=0}^{\infty}\frac{(-1)^\nu t^{2\nu}}{\nu!}\,dt \;=\; ze^{z^2}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu z^{2\nu+1}}{(2\nu+1)\nu!} \\ &= z^2\left(\sum_{\mu=0}^{\infty}\frac{z^{2\mu}}{\mu!}\right)\left(\sum_{\nu=0}^{\infty}\frac{(-1)^\nu z^{2\nu}}{(2\nu+1)\nu!}\right) \\ &= z^2\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu z^{2(\nu+\mu)}}{(2\nu+1)\nu!\mu!} \;=\; z^2\sum_{k=0}^{\infty}\Big(\sum_{\substack{\mu=0\\ \mu+\nu=k}}^{\infty}\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(2\nu+1)\nu!\mu!}\Big)z^{2k} \\ &= z^2\sum_{k=0}^{\infty}\Big(\sum_{\nu=0}^{k}\frac{(-1)^\nu}{(2\nu+1)\nu!(k-\nu)!}\Big)z^{2k} \;=\; z^2\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\sum_{\nu=0}^{k}\frac{(-1)^\nu\binom{k}{\nu}}{2\nu+1}\Big)z^{2k}\,. \end{aligned}$$

From [15, p. 68], Remark 8.5, we have the following formula (with k replaced by ν and n replaced by k)

$$\frac{1}{d^k[k]_k}\sum_{\nu=0}^{k}\frac{(-1)^\nu\binom{k}{\nu}}{c+\nu d} \;=\; \frac{1}{c(c+d)(c+2d)\cdots(c+kd)}\,,$$

where $[k]_k = k!$. Setting $c=1$ and $d=2$, it follows that

$$\frac{1}{k!}\sum_{\nu=0}^{k}\frac{(-1)^\nu\binom{k}{\nu}}{2\nu+1} \;=\; \frac{2^k}{1\cdot 3\cdot 5\cdots(2k+1)} \;=\; \frac{1}{(3/2)_k}\,.$$

This gives

$$f(z) \;=\; z^2\sum_{k=0}^{\infty}\frac{z^{2k}}{(3/2)_k} \;=\; z^2\sum_{k=0}^{\infty}\frac{(1)_k}{k!(3/2)_k}z^{2k} \;=\; z^2\,{}_1F_1\big(1,3/2;z^2\big)\,.$$

The function ${}_1F_1\big(1,3/2;z^2\big)$ can be expressed by a Gauss-type continued fraction. Using formula (8) on page 123 in [16] with $\gamma=3/2$ and $x=z^2$, we have

$${}_1F_1\big(1,3/2;z^2\big) \;=\; \cfrac{1/2}{1/2-\cfrac{z^2/2}{3/2+\cfrac{2z^2/2}{5/2-\cfrac{3z^2/2}{7/2+\cfrac{4z^2/2}{9/2-\cfrac{5z^2/2}{11/2+\ddots\cfrac{(-1)^m mz^2/2}{(2m+1)/2+\ddots}}}}}}} \qquad (m\geq 1)$$

Hence the continued fraction expansion given by the theorem follows from

$$\sum_{n\geq 0}\big|e^{1/l}q_n-p_n\big| \;=\; 2\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{e^{1/l}}{\sqrt{l}}\cdot\mathrm{erf}\big(1/\sqrt{l}\,\big) \;=\; 2f\big(1/\sqrt{l}\,\big) \;=\; \frac{2}{l}\,{}_1F_1\big(1,3/2;1/l\big)\,.$$

□

We point out the particular case $z=1$.

Corollary 2.1. *We have*

$$\begin{aligned} {}_1F_1\big(1,3/2;1\big) &= e\int_0^1 e^{-t^2}\,dt \\ &= \mathcal{E}_{MC}(e) \,=\, e-2+\sum_{n=1}^{\infty}\sum_{b=1}^{a_n}\big|(bq_{n-1}+q_{n-2})e-(bp_{n-1}+p_{n-2})\big| \\ &= \cfrac{1/2}{1/2-\cfrac{1/2}{3/2+\cfrac{1}{5/2-\cfrac{3/2}{7/2+\cfrac{2}{9/2-\cfrac{5/2}{11/2+\ddots\cfrac{(-1)^m m/2}{(2m+1)/2+\ddots}}}}}}} \qquad (m\geq 1)\,, \end{aligned}$$

where $\mathcal{E}_{MC}(e)$ is the error sum of e taking into account all the minor convergents of

$$e \,=\, \langle 2;1,2,1,1,4,1,\dots\rangle \,=\, \langle 2;a_1,a_2,a_3,\dots\rangle\,. \tag{2.1}$$

Proof: The formula

$$\mathcal{E}_{MC}(e) \,=\, e\int_0^1 e^{-t^2}\,dt$$

follows by (2.1) using

$$\mathcal{E}_{MC}(e) \,=\, e-1+\sum_{\nu=0}^{\infty}(-1)^{\nu+1}\big(q_\nu e-p_\nu\big)\Big(\frac{1}{2}(1+a_{\nu+1})a_{\nu+1}-a_{\nu+2}\Big)$$

and the formulas

$$\begin{aligned} q_{3m-1}e-p_{3m-1} &= -\int_0^1 \frac{x^{m+1}(x-1)^m}{m!}e^x\,dx\,, \\ q_{3m}e-p_{3m} &= -\int_0^1 \frac{x^{m}(x-1)^{m+1}}{m!}e^x\,dx\,, \\ q_{3m+1}e-p_{3m+1} &= \int_0^1 \frac{x^{m+1}(x-1)^{m+1}}{(m+1)!}e^x\,dx \end{aligned}$$

due to H. Cohn [5]. □

Let $l\geq 1$. Then we know from [18, p. 193] that the numbers $e^{1/l}$ and $\int_0^{1/\sqrt{1/l}} e^{-t^2}\,dt$ are algebraically independent over $\mathbb{Q}$. This proves

Corollary 2.2. *Let $l\geq 2$ be an integer. Then the numbers $\mathcal{E}(e^{1/l})$ and $\mathcal{E}_{MC}(e)$ are transcendental.*

3 Error sums for π and $(\log\rho)/\sqrt{5}$

In [11], A.Klauke and the second-named author have found new continued fractions for $1/\pi$ and $(\log\rho)/\sqrt{5}$. In this section we are going to apply these results to compute the corresponding error sums and to decide on their algebraic character. We start with the continued fraction for $1/\pi$.

1.) From Theorem 8 in [11] and its proof we have the following results.

$$\frac{1}{\pi} = \frac{3}{10} - \frac{14}{25} - \frac{110}{171} - \ldots - \frac{\frac{1}{9}m(m-1)(2m-1)(2m+1)(4m-5)(4m+3)}{(4m+1)(4m^2+2m-1)} - \ldots$$
$$= \frac{p_0}{q_0} - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \ldots - \frac{p_m}{q_m} - \ldots \qquad (m \geq 2)\,.$$

Let $n = 0, 1, 2, \ldots$. Set

$$B_n := \frac{2\cdot 4^{n+1}}{n!}\sum_{k=0}^{n}\binom{n}{k}(2k+3)\big(k+5/2\big)_n\,,$$
$$A_n := \frac{2\cdot 4^{n+1}}{n!}\sum_{k=0}^{n}\sum_{\nu=0}^{k}(-1)^{k+\nu}\binom{n}{k}\frac{(2k+3)\big(k+5/2\big)_n}{2k-2\nu+1} + (-4)^{n+1}\,.$$

Here,

$$\big(k+5/2\big)_n = (k+5/2)(k+7/2)(k+9/2)\cdots(k+n+3/2)\,.$$

Note that A_n is a rational number, but no integer, while $B_n/4$ is an integer. Then, for $n \geq 0$, one has

$$\frac{p_0}{q_0} - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \ldots - \frac{p_n}{q_n} = \frac{B_n}{4A_n}\,,$$

and

$$0 < A_n - \frac{\pi B_n}{4} = 4(n+3/2)\int_0^1 \frac{t\sqrt{1-t}}{2-t}\Big(\frac{4t(1-t)}{2-t}\Big)^n\,dt\,.$$

For $0 \leq t \leq 1$ the rational function $4t(1-t)/(2-t)$ takes its maximum $2(6-4\sqrt{2})$ at the point $2-\sqrt{2}$. Therefore, it follows that

$$0 < A_n - \frac{\pi B_n}{4} < 8(n+3/2)(6-4\sqrt{2})^n\int_0^1\frac{t\sqrt{1-t}}{2-t}\,dt = 8(10/3-\pi)(n+3/2)(6-4\sqrt{2})^n\,.$$

The integral on the right-hand side is a Pochhammer integral of a certain hypergeometric function. We show the analogous details below in part 2.) which is devoted to the number $(\log\rho)/\sqrt{5}$.

For (1.3) and (1.4) we define the sequences $B := (b_n)_{n\geq 0}$ and $R := (r_n)_{n\geq 0}$ by $b_n := -B_n/4$ and $r_n = -A_n$. Then we have the error sums

$$\begin{aligned}\mathcal{E}^*(B,R,\pi) &= \sum_{m=0}^{\infty}(b_m\pi - r_m) = -\mathcal{E}(B,R,\pi)\\ &= -4\int_0^1\frac{t\sqrt{1-t}}{2-t}\sum_{m=0}^{\infty}(m+3/2)\Big(\frac{4t(1-t)}{2-t}\Big)^m\,dt\\ &= -4\int_0^1\frac{t\sqrt{1-t}}{2-t}\cdot\frac{(2-t)(4t^2-7t+6)}{2(4t^2-5t+2)^2}\,dt\\ &= -4\int_0^1\frac{u^2(1-u^2)(4u^4-u^2+3)}{(4u^4-3u^2+1)^2}\,du\,.\end{aligned}$$

Here we have introduced the new variable $u := \sqrt{1-t}$. Computing this integral, we have the following theorem.

Theorem 3.1. *For the sequences $B := (b_n)_{n\geq 0}$ and $R := (r_n)_{n\geq 0}$ defined by $b_n := -B_n/4$ and $r_n = -A_n$ we have*

$$\mathcal{E}^*(B,R,\pi) \;=\; -\mathcal{E}(B,R,\pi) \;=\; \frac{\sqrt{7}}{49}\log\Big(\frac{3-\sqrt{7}}{3+\sqrt{7}}\Big) - \frac{3\pi}{2} - \frac{4}{7} \;=\; -5.4333111067784\ldots.$$

Expressing π by $\pi = \frac{2\log i}{i}$, we see that $\mathcal{E}(B,R,\pi)$ is a nonvanishing linear form in logarithms with algebraic arguments and algebraic coefficients. Then, by Theorem 2.2 in [2], we have the following corollary.

Corollary 3.1. *For the sequences $B := (b_n)_{n\geq 0}$ and $R := (r_n)_{n\geq 0}$ defined by $b_n := -B_n/4$ and $r_n = -A_n$ the error sum $\mathcal{E}(B,R,\pi)$ is transcendental, and so is the error sum $\mathcal{E}^*(B,R,\pi)$.*

2.) From Theorem 6 in [11] and its proof we have the following results.

$$\frac{\sqrt{5}}{\log\rho} = \frac{60}{13} - \frac{7}{80} - \frac{110}{522} - \ldots - \frac{\frac{1}{9}m(m-1)(2m-1)(2m+1)(4m-5)(4m+3)}{2(4m+1)(6m^2+3m-1)} - \ldots$$
$$= \frac{p_0}{q_0} - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \ldots - \frac{p_m}{q_m} - \ldots \qquad (m \geq 2)\,.$$

Let $n = 0,1,2,\ldots$. Set (cf. (25) in [11] with $c = d = 1$)

$$D_n \;:=\; \frac{5\cdot 4^{n+1}}{n!}\sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}(2k+3)\big(k+5/2\big)_n 5^k\,,$$
$$C_n \;:=\; 4^n + \frac{4^{n+1}}{n!}\sum_{k=0}^{n}\sum_{\nu=0}^{k}(-1)^{n+k}\binom{n}{k}\frac{(2k+3)\big(k+5/2\big)_n 5^\nu}{2k-2\nu+1}\,.$$

Applying Lemma 6 in [11] with $x = \tau = 1$, we find that

$$\frac{p_0}{q_0} - \frac{p_1}{q_1} - \frac{p_2}{q_2} - \ldots - \frac{p_n}{q_n} = \frac{D_n}{C_n}\,,$$

and

$$0 \;<\; C_n - \frac{D_n\log\rho}{\sqrt{5}} \;=\; \frac{(5/2)_n(n+1)!}{4(5/2)_{2n+1}}\,{}_2F_1\left(\begin{matrix} n+1 \quad n+2 \\ 2n+7/2\end{matrix}\,\Big|\, -\frac{1}{4}\right).$$

We define the sequences $B := (b_n)_{n\geq 0}$ and $R := (r_n)_{n\geq 0}$ by $b_n := -D_n$ and $r_n = -C_n$. Then we have the error sum

$$\mathcal{E}^*\Big(B,R,\frac{\log\rho}{\sqrt{5}}\Big) \;=\; \sum_{m=0}^{\infty}\Big(b_m\frac{\log\rho}{\sqrt{5}} - r_m\Big) = -\frac{1}{4}\sum_{m=0}^{\infty}\frac{(5/2)_m(m+1)!}{4(5/2)_{2m+1}}\,{}_2F_1\left(\begin{matrix} m+1 \quad m+2 \\ 2m+7/2\end{matrix}\,\Big|\, -\frac{1}{4}\right).$$

To compute this error sum, the method is the same as used above for the error sum of $\mathcal{E}(\pi)$. First we express the hypergeometric function by Pochhammer's integral. Let a,b,c,z be complex numbers satisfying $|z| < 1$, $\Re(c-b) > 0$, and $\Re(b) > 0$. Then we have the identity

$${}_2F_1\left(\begin{matrix} a \quad b \\ c\end{matrix}\,\Big|\, z\right) \;=\; \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt\,,$$

cf. [19, p. 20]. The conditions are fulfilled for $a = n+1$, $b = n+2$, $c = 2n+7/2$, and $z = -1/4$, where $n = 0, 1, 2, \ldots$. Hence, it follows that

$$ {}_2F_1\left(\begin{matrix} n+1 & n+2 \\ 2n+7/2 & \end{matrix} \,\Big|\, -\frac{1}{4} \right) = \frac{\Gamma(2n+7/2)}{\Gamma(n+2)\Gamma(n+3/2)} \int_0^1 t^{n+1}(1-t)^{n+1/2}\Big(1+\frac{t}{4}\Big)^{-n-1} dt\,. $$

In order to simplify the above expressions we need two identities involving Pochhammer's symbol ([19, p. 239]).

$$ \begin{aligned} \frac{(5/2)_n}{(5/2)_{2n+1}} &= \frac{1}{(n+5/2)_{n+1}}\,, \\ \frac{\Gamma(2n+7/2)}{\Gamma(n+3/2)} &= (n+3/2)(n+5/2)_{n+1}\,. \end{aligned} $$

Collecting together all the above results, it follows that

$$ \begin{aligned} \mathcal{E}^*\Big(B, R, \frac{\log\rho}{\sqrt{5}}\Big) &= -\frac{1}{4}\sum_{m=0}^{\infty}(m+3/2)\int_0^1 \frac{t^{m+1}(1-t)^{m+1/2}}{(1+t/4)^{m+1}}\,dt \\ &= -\frac{1}{4}\int_0^1 \frac{4t\sqrt{1-t}}{4+t}\sum_{m=0}^{\infty}(m+3/2)\Big(\frac{4t(1-t)}{4+t}\Big)^m dt \\ &= -\frac{1}{2}\int_0^1 \frac{t(4t^2-t+12)\sqrt{1-t}}{(4t^2-3t+4)^2}\,dt \\ &= \int_0^1 \frac{u^2(1-u^2)(4u^4-7u^2+15)}{(4u^4-5u^2+5)^2}\,du\,, \end{aligned} $$

where $u = \sqrt{1-t}$. This proves

Theorem 3.2. *For the sequences $B := (b_n)_{n\geq 0}$ and $R := (r_n)_{n\geq 0}$ defined by $b_n := -D_n$ and $r_n = -C_n$ we have*

$$ \begin{aligned} &\mathcal{E}^*\Big(B, R, \frac{\log\rho}{\sqrt{5}}\Big) \\ =\;& \frac{\sqrt{124\sqrt{5}-265}\log\Big(1+\frac{\sqrt{5}}{2}-\frac{\sqrt{4\sqrt{5}+5}}{2}\Big) - \sqrt{124\sqrt{5}+265}\arccos\Big(\frac{\sqrt{5}}{2}-1\Big)}{55\sqrt{11}} + \frac{1}{11} \\ =\;& -0.1210649459927\ldots\,. \end{aligned} $$

Using $\arccos z = \frac{1}{i}\log(z+\sqrt{z^2-1})$, we obtain by Theorem 2.2 in [2] the following result.

Corollary 3.2. *For the sequences $B := (b_n)_{n\geq 0}$ and $R := (r_n)_{n\geq 0}$ defined by $b_n := -D_n$ and $r_n = -C_n$ the error sum $\mathcal{E}(B, R, \log\rho/\sqrt{5})$ is transcendental.*

4 An error sum for $\log(1+t)$

In this section we generalize a concept from the proof of Theorem 3 in [12], where a nonregular continued fraction for $\log 2$ is established. First we shall prove a continued fraction expansion for $\log(1+t)$ with $-1 < t \leq 1$, namely

$$\log(1+t) \;=\; \frac{2t}{2+t} \;-\; \frac{1^2t^2}{3(2+t)} \;-\; \frac{2^2t^2}{5(2+t)} \;-\; \frac{3^2t^2}{7(2+t)} \;-\cdots-\; \frac{m^2t^2}{(2m+1)(2+t)} \;-\ldots\quad, \tag{4.1}$$

where $m = 1, 2, \ldots$. O.Perron [16, p. 152] cites by equation (7) the continued fraction

$$\log(1+t) \;=\; \frac{t}{1} \;+\; \frac{1^2t}{2} \;+\; \frac{1^2t}{3} \;+\; \frac{2^2t}{4} \;+\; \frac{2^2t}{5} \;+\; \frac{3^2t}{6} \;+\; \frac{3^2t}{7} \;+\ldots\;.$$

Here we shall give full details of the proof, since a new argument is needed in H.Cohen's method [4] established for Apéry's irregular continued fractions of $\zeta(2)$ and $\zeta(3)$, and we need the details in order to compute the error sum. Similar to Apéry's approach we have to handle with combinatorial series.
In the sequel we fix a real number t with $-1 < t \leq 1$. Let $n \geq 0$ be an integer. We define two combinatorial series by

$$\begin{aligned} B_n \;&:=\; \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} t^{n-k}\,, \\ A_n \;&:=\; \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} t^{n-k} c_k\,, \end{aligned}$$

where

$$c_k \;:=\; \sum_{m=1}^{k} \frac{(-1)^{m-1}t^m}{m}\,.$$

By applying *Zeilberger's algorithm* [14, Ch. 7] (Algorithm 7.1) on a computer algebra system, it turns out that the numbers B_n satisfy the linear three-term recurrence formula

$$(n+1)X_{n+1} - (2n+1)(2+t)X_n + nt^2X_{n-1} \;=\; 0 \qquad (n \geq 1)\,. \tag{4.2}$$

In the sequel we prove this formula for $X_n = B_n$ without using a computer, since we need the details to show that even $X_n = A_n$ satisfy the recurrence. Let k, n denote integers. Set

$$\begin{aligned} \lambda_{n,k} \;&:=\; \binom{n}{k}\binom{n+k}{k} t^{n-k}\,, \\ B_{n,k} \;&:=\; -(4n+2)\lambda_{n,k}\,, \\ A_{n,k} \;&:=\; B_{n,k}c_k\,, \\ S_{n,k} \;&:=\; (n+1)\lambda_{n+1,k}c_k - (2n+1)(2+t)\lambda_{n,k}c_k + nt^2\lambda_{n-1,k}c_k\,. \end{aligned}$$

Note that $\binom{n}{k} = 0$ for $k < 0$ or $k > n$, which implies that $A_{n,n+1} = B_{n,n+1} = A_{n,-1} = B_{n,-1} = 0$. One easily verfies the identities[1]

$$\begin{aligned} \frac{\lambda_{n,k-1}}{\lambda_{n,k}} &= \frac{k^2 t}{(n+k)(n-k+1)}\,, \\ \frac{\lambda_{n+1,k}}{\lambda_{n,k}} &= \frac{(n+k+1)t}{n-k+1}\,, \\ \frac{\lambda_{n-1,k}}{\lambda_{n,k}} &= \frac{n-k}{(n+k)t}\,, \end{aligned}$$

which can be applied to prove the identity

$$B_{n,k} - B_{n,k-1} = (n+1)\lambda_{n+1,k} - (2n+1)(2+t)\lambda_{n,k} + nt^2\lambda_{n-1,k}\,. \tag{4.3}$$

Summing up on both sides of (4.3) from $k = 0$ to $k = n+1$, we obtain

$$0 = B_{n,n+1} - B_{n,-1} = \sum_{k=0}^{n+1} \big(B_{n,k} - B_{n,k-1}\big) = (n+1)B_{n+1} - (2n+1)(2+t)B_n + nt^2 B_{n-1}\,,$$

which proves (4.2) for $X_n = B_n$.
Multiplying (4.3) by c_k, we obtain $S_{n,k} = (B_{n,k} - B_{n,k-1})c_k$. Hence,

$$\begin{aligned} A_{n,k} - A_{n,k-1} &= B_{n,k}c_k - B_{n,k-1}c_{k-1} = \big(B_{n,k} - B_{n,k-1}\big)c_k + B_{n,k-1}\big(c_k - c_{k-1}\big) \\ &= S_{n,k} + B_{n,k-1}\frac{(-1)^{k-1}t^k}{k}\,. \end{aligned}$$

Again, we sum up from $k = 0$ to $k = n+1$. This gives

$$\begin{aligned} 0 &= A_{n,n+1} - A_{n,-1} = \sum_{k=0}^{n+1} \big(A_{n,k} - A_{n,k-1}\big) \\ &= \sum_{k=0}^{n+1} S_{n,k} - (4n+2)\sum_{k=1}^{n+1} \binom{n}{k-1}\binom{n+k-1}{k-1} t^{n-k+1}\frac{(-1)^{k-1}t^k}{k} \\ &= (n+1)A_{n+1} - (2n+1)(2+t)A_n + nt^2 A_{n-1} - (4n+2)t^{n+1}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k+1}\,. \end{aligned}$$

Finally Vandermonde's theorem for the hypergeometric series ${}_2F_1(n+1,-n,2;1)$ ([19, eq. (1.7.7)]) completes our proof of (4.2) for $X_n = A_n$ by

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k+1} = {}_2F_1(n+1,-n,2;1) = \frac{(1-n)_n}{(2)_n} = 0$$

for $n \geq 1$. In the next step we prove that

$$\lim_{n\to\infty}\frac{A_n}{B_n} = \log(1+t)\,. \tag{4.4}$$

[1]We should like to point out that there is a misprint in the formula for $\lambda_{n+1,k}/\lambda_{n,k}$ in [12].

For this purpose we shall prove that for every fixed integer $\nu \geq 0$ we have the limit

$$\lim_{n\to\infty} \frac{\binom{n}{\nu}\binom{n+\nu}{\nu} t^{n-\nu}}{\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} t^{n-k}} \;=\; 0\,. \tag{4.5}$$

Then, (4.5) implies (4.4) by a theorem of O.Toeplitz ([17, p. 10, no. 66]), since

$$\lim_{n\to\infty} c_n \;=\; \sum_{m=1}^{\infty} \frac{(-1)^{m-1}t^m}{m} \;=\; \log(1+t)\,.$$

There is nothing to show for $t = 0$, because $A_n = 0$ and $B_n = \binom{2n}{n} \neq 0$. Therefore, keep $\nu \in \mathbb{N}_0$ and $t \in (-1, 1] \setminus \{0\}$ fixed. We substitute $X_n = B_n$ into (4.2) and divide the equation by $(n+1)B_n$. Then we obtain

$$\frac{B_{n+1}}{B_n} - \frac{(2n+1)(2+t)}{n+1} + \frac{nt^2}{n+1} \cdot \frac{1}{B_n/B_{n-1}} \;=\; 0\,.$$

Let $\alpha := \lim_{n\to\infty} B_{n+1}/B_n$. By taking the limit $n \to \infty$, it follows that α satisfies the quadratic equation

$$\alpha - 2(2+t) + \frac{t^2}{\alpha} \;=\; 0\,,$$

which yields

$$\alpha \;=\; 2+t+2\sqrt{1+t} \;>\; |t| \qquad (-1 < t \leq 1)\,.$$

Put $\beta := (\alpha + |t|)/2$. Then,

$$0 \;<\; |t| \;<\; \beta \;<\; \alpha\,. \tag{4.6}$$

There is an integer $n_0 = n_0(t)$ satisfying

$$\frac{B_m}{B_{m-1}} \;>\; \beta \;>\; 0 \qquad (m \geq n_0)\,,$$

or

$$|B_m| \;>\; \beta |B_{m-1}| \qquad (m \geq n_0)\,.$$

Then, for $n \geq 2n_0$ and $k := n - n_0 \geq n_0$, we have

$$|B_n| \;>\; \beta|B_{n-1}| \;>\; \beta^2|B_{n-2}| \;> \cdots > \beta^k|B_{n-k}| \;=\; \beta^{n-n_0}|B_{n_0}|\,.$$

Consequently, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\binom{n}{\nu}\binom{n+\nu}{\nu} t^{n-\nu}}{\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} t^{n-k}} \right| &= \lim_{n\to\infty} \frac{\binom{n}{\nu}\binom{n+\nu}{\nu} |t|^{n-\nu}}{|B_n|} \\
&\leq \lim_{n\to\infty} \frac{\binom{n}{\nu}\binom{n+\nu}{\nu} |t|^{n-\nu}}{\beta^{n-n_0}|B_{n_0}|} \;=\; \frac{\beta^{n_0}}{|t|^{\nu}|B_{n_0}|} \lim_{n\to\infty} \binom{n}{\nu}\binom{n+\nu}{\nu} \Big(\frac{|t|}{\beta}\Big)^n \\
&= 0\,,
\end{aligned}$$

since $0 < |t|\beta^{-1} < 1$ by (4.6), and $\binom{n}{\nu}\binom{n+\nu}{\nu}$ is a polynomial in n of degree 2ν. This completes the proof of (4.5) and, consequently, of (4.4).
We rewrite the recurrence formula (4.2) as

$$P(n+1)X_{n+1} - Q(n+1)X_n - R(n+1)X_{n-1} \,=\, 0\,,$$

where

$$\begin{aligned} P(n+1) &:= n+1\,, \\ Q(n+1) &:= (2n+1)(2+t)\,, \\ R(n+1) &:= -nt^2\,. \end{aligned}$$

Then, we obtain

$$\log(1+t) \,=\, b_0 + \frac{a_1}{b_1} \,+\, \frac{a_2}{b_2} \,+\, \frac{a_3}{b_3} \,+ \ldots$$

with

$$\begin{aligned} b_0 \;&=\; 0\,, \quad b_1 \;=\; 2+t\,, \quad b_{n+1} \;=\; \frac{Q(n+1)}{P(n+1)} = \frac{(2n+1)(2+t)}{n+1}\,, \\ a_1 \;&=\; 2t\,, \quad a_{n+1} \;=\; \frac{R(n+1)}{P(n+1)} = -\frac{nt^2}{n+1}\,. \end{aligned}$$

This gives the continued fraction

$$\log(1+t) \,=\, \frac{2t}{2+t} \,-\, \frac{t^2/2}{3(2+t)/2} \,-\, \frac{2t^2/3}{5(2+t)/3} \,-\, \frac{3t^2/4}{7(2+t)/4} \,- \ldots \quad ,$$

which is equivalent with (4.1).
Next, we compute the error sum $\mathcal{E}^*(B,R,\log(1+t)) = \sum_{m=0}^{\infty}(B_m\log(1+t) - A_m)$ for $B := (B_n)_{n\geq 0}$ and $R := (A_n)_{n\geq 0}$.

Lemma 4.1. *Let $-1 < t \leq 1$. For every integer $n \geq 0$ we have*

$$B_n\log(1+t) - A_n \,=\, t^{2n+1}\int_0^1 \frac{x^n(1-x)^n}{(1+tx)^{n+1}}\,dx\,.$$

Proof. For A_n and B_n defined above, we obtain

$$\begin{aligned}
& B_n \log(1+t) - A_n \\
=\ & \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} t^{n-k} \sum_{m=k+1}^{\infty} \frac{(-1)^{m-1} t^m}{m} \\
=\ & \sum_{m=0}^{\infty} (-1)^m t^{m+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} \frac{t^{n-k} t^k}{m+k+1} \\
=\ & t^{n+1} \int_0^1 \sum_{m=0}^{\infty} (-1)^m (tx)^m \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} x^k \, dx \\
=\ & t^{n+1} \int_0^1 \sum_{m=0}^{\infty} (-tx)^m \frac{d^n}{dx^n}\Big(\frac{x^n(1-x)^n}{n!} \Big)\, dx \\
=\ & t^{n+1} \int_0^1 \frac{1}{1+tx} \cdot \frac{d^n}{dx^n}\Big(\frac{x^n(1-x)^n}{n!} \Big)\, dx \\
=\ & (-1)^n t^{n+1} \int_0^1 \frac{d^n}{dx^n}\Big(\frac{1}{1+tx} \Big) \cdot \frac{x^n(1-x)^n}{n!}\, dx \\
=\ & t^{2n+1} \int_0^1 \frac{x^n(1-x)^n}{(1+tx)^{n+1}}\, dx\,.
\end{aligned}$$

The antiderivative in the last but one line was obtained using n-fold integration by parts. The lemma is proven. □

A first consequence of Lemma 4.1 is an explicit formula for the error sum of $\log(1+t)$.

Corollary 4.1. *Let $-1 < t \leq 1$. For the sequences $B := (B_n)_{n\geq 0}$ and $R := (A_n)_{n\geq 0}$ we have*

$$\mathcal{E}^*\big(B, R, \log(1+t)\big) \;=\; \frac{2}{\sqrt{3+2t-t^2}} \left(\arctan\Big(\frac{1+t}{\sqrt{3+2t-t^2}} \Big) - \arctan\Big(\frac{1-t}{\sqrt{3+2t-t^2}} \Big) \right) .$$

In particular, $\mathcal{E}^*(B, R, \log 2) = \pi/4$.

Proof: From Lemma 4.1 we obtain

$$\begin{aligned}
& \mathcal{E}^*\big(B, R, \log(1+t)\big) \\
=\ & t \int_0^1 \sum_{m=0}^{\infty} \frac{t^{2m} x^m (1-x)^m}{(1+tx)^{m+1}}\, dx \\
=\ & t \int_0^1 \frac{1}{1+tx} \sum_{m=0}^{\infty} \Big(\frac{t^2 x(1-x)}{1+tx} \Big)^m dx \\
=\ & t \int_0^1 \frac{dx}{1+t(1-t)x+t^2x^2} \\
=\ & \frac{2}{\sqrt{3+2t-t^2}} \left(\arctan\Big(\frac{1+t}{\sqrt{3+2t-t^2}} \Big) - \arctan\Big(\frac{1-t}{\sqrt{3+2t-t^2}} \Big) \right) .
\end{aligned}$$

This proves the corollary. □

By straightforward computations it can be seen that the error sum from Corollary 4.1 satisfies a linear first order differential equation.

Corollary 4.2. *Let $-1 < t \leq 1$. For the sequences $B := (B_n)_{n\geq 0}$ and $R := (A_n)_{n\geq 0}$ the function $f(t) := \mathcal{E}^*(B, R, \log(1+t))$ satisfies the differential equation*

$$(3 + 2t - t^2)f' + (1-t)f - 3 \,=\, 0\,,$$

where $f' = df/dt$.

A second consequence of Lemma 4.1 is

$$\mathcal{E}^*\big(B, R, \log(1+t)\big) \,=\, \text{sign}(t)\mathcal{E}\big(B, R, \log(1+t)\big)\,.$$

Finally, the continued fraction (4.1) and Lemma 4.1 allow to prove the irrationality of $\log(1+t)$ for certain rationals $t := a/b$.

Corollary 4.3. *Let $0 < a/b \leq 1$ be a rational number with $ea^2 < 4b$. Then the number $\log(1 + a/b)$ is irrational. In particular, for every integer $k \geq 1$ the number $\log(1 + 1/k)$ is irrational.*

Proof: Let $d_n :=$ l.c.m.$(1, 2, 3, \ldots, n)$ denote the least common multiple of the integers $1, 2, 3, \ldots, n$. One knows by the prime number theorem that

$$\log d_n \,=\, \sum_{p\leq n} \Big[\frac{\log n}{\log p}\Big] \log p \,\sim\, n\,,$$

where p runs through all primes less than or equal to n ([13, Theorem 434]). By the hypothesis $ea^2 < 4b$ there is a positive real number ε such that $e^{1+\varepsilon}a^2 < 4^{1-\varepsilon}b$. Hence, for all sufficiently large numbers n, it follows that

$$d_n \,<\, e^{(1+\varepsilon)n}\,.$$

Let $t = a/b$. With $b^n d_n A_n \in \mathbb{Z}$ and $b^n d_n B_n \in \mathbb{Z}$ we know by Lemma 4.1 that

$$\begin{aligned}
0 &< \big|b^n d_n B_n \log(1+t) - b^n d_n A_n\big| \\
&= \frac{a}{b} b^n d_n \Big(\frac{a}{b}\Big)^{2n} \int_0^1 \frac{x^n(1-x)^n}{(1+ax/b)^{n+1}}\, dx \\
&< \frac{a}{b} \cdot \frac{e^{(1+\varepsilon)n} a^{2n}}{b^n} \int_0^1 x^n(1-x)^n\, dx \\
&< \frac{a}{b} \cdot 4^{(1-\varepsilon)n} \int_0^1 \frac{dx}{4^n}\, dx \\
&= \frac{t}{4^{\varepsilon n}} \to 0
\end{aligned}$$

for n tending to infinity. This completes the proof of Corollary 4.3. □

5 On error sums formed by Apéry's continued fractions for $\zeta(2)$ and $\zeta(3)$

Computing the error sums formed by the linear three term recurrences and continued fractions of $\zeta(2)$, $\zeta(3)$ introduced by R. Apéry, this leads unexpectedly into a wide field of connections between famous sequences of integers. For the needed results we refer to [4] and [3].

1.) Error sums for $\zeta(2)$. We have

$$\begin{aligned}\zeta(2) &= \frac{\pi^2}{6} = \frac{5}{3} + \frac{1^4}{25} + \frac{2^4}{69} + \dots + \frac{n^4}{11n^2+11n+3} + \dots \\ &= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots\end{aligned}$$

with

$$b_0 = 0\,, \quad b_1 = 3\,, \quad b_{n+1} = 11n^2+11n+3 \quad (n \ge 1)\,,$$
$$a_1 = 5\,, \quad a_{n+1} = n^4 \quad (n \ge 1)\,.$$

A recurrence formula for both sequences

$$\begin{aligned}B_n &:= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}\,, \\ A_n &:= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \left(2 \sum_{m=1}^{n} \frac{(-1)^{m-1}}{m^2} + \sum_{m=1}^{k} \frac{(-1)^{n+m-1}}{m^2 \binom{n}{m}\binom{n+m}{m}} \right)\,,\end{aligned}$$

is

$$0 \,=\, (n+1)^2 X_{n+1} - (11n^2+11n+3)X_n - n^2 X_{n-1}\,.$$

Then,

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots + \frac{a_n}{b_n} \,=\, \frac{A_n}{B_n}\,.$$

We obtain from [3, eq. (5)] for the sequences $B_2 := (B_n)_{n\ge 0}$ and $R_2 := (A_n)_{n\ge 0}$,

$$\begin{aligned}\mathcal{E}^*\big(B_2, R_2, \zeta(2)\big) &= \sum_{n=0}^{\infty} \Big(B_n \zeta(2) - A_n \Big) \,=\, \sum_{n=0}^{\infty} (-1)^n \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dx\, dy \\ &= \int_0^1 \int_0^1 \frac{dx\, dy}{1+x^2y^2-xy^2-yx^2} \,=\, 1.5832522167\dots\,. \qquad (5.1)\end{aligned}$$

Similarly, one has

$$\begin{aligned}\mathcal{E}\big(B_2, R_2, \zeta(2)\big) &= \sum_{n=0}^{\infty} \Big| B_n \zeta(2) - A_n \Big| \,=\, \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}}\, dx\, dy \\ &= \int_0^1 \int_0^1 \frac{dx\, dy}{1-x^2y^2-2xy+xy^2+yx^2} \,=\, 1.7141459142\dots\,. \qquad (5.2)\end{aligned}$$

2.) Error sums for $\zeta(3)$. Here,

$$\begin{aligned}\zeta(3) &= \frac{6}{5} - \frac{1^6}{117} - \frac{2^6}{535} - \dots - \frac{n^6}{34n^3+51n^2+27n+5} - \dots \\ &= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots\end{aligned}$$

with

$$b_0 = 0\,, \quad b_1 = 5\,, \quad b_{n+1} = 34n^3 + 51n^2 + 27n + 5 \quad (n \geq 1)\,,$$

$$a_1 = 6\,, \quad a_{n+1} = -n^6 \quad (n \geq 1)\,.$$

A recurrence formula for both sequences,

$$\begin{aligned} D_n &:= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 , \\ C_n &:= \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}} \right) , \end{aligned}$$

is

$$\begin{aligned} 0 &= P(n+1)X_{n+1} - Q(n+1)X_n - R(n+1)X_{n-1} \\ &= (n+1)^3 X_{n+1} - (34n^3 + 51n^2 + 27n + 5)X_n + n^3 X_{n-1}\,. \end{aligned}$$

The construction of C_n and D_n leads to the identity

$$b_0 + \frac{a_1}{b_1} \; + \; \frac{a_2}{b_2} \; + \; \frac{a_3}{b_3} \; + \cdots + \; \frac{a_n}{b_n} \; = \; \frac{C_n}{D_n}\,.$$

We obtain from [3, eq. (7)] for the sequences $B_3 := (D_n)_{n \geq 0}$ and $R_3 := (C_n)_{n \geq 0}$,

$$\begin{aligned} & \mathcal{E}^*\big(B_3, R_3, \zeta(3)\big) \\ = \; & \sum_{n=0}^{\infty} \Big(D_n \zeta(3) - C_n \Big) \; = \; \sum_{n=0}^{\infty} \Big| D_n \zeta(3) - C_n \Big| \; = \; \mathcal{E}\big(B_3, R_3, \zeta(3)\big) \\ = \; & \sum_{n=0}^{\infty} \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{x^n (1-x)^n y^n (1-y)^n w^n (1-w)^n}{\big(1 - (1-xy)w\big)^{n+1}} \, dx \, dy \, dw \\ = \; & \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 \frac{dx \, dy \, dw}{1 + x^2y^2w^2 - xy^2w^2 - x^2yw^2 - x^2y^2w + xyw^2 + xy^2w + x^2yw - w} \\ = \; & 1.2124982529\ldots\,. \end{aligned} \tag{5.3}$$

3.) Now we focus our interest on various methods in order to express the multiple integrals in (5.1), (5.2), and (5.3), by series with rational terms. A first approach to this subject involves the hypergeometric function.

Theorem 5.1. *For the sequences B_i, R_i $(i = 2, 3)$ defined above for $\zeta(2)$ and $\zeta(3)$, respectively, we have*

$$
\begin{aligned}
\mathcal{E}\big(B_2, R_2, \zeta(2)\big) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\binom{n+k}{n}}{(2n+k+1)^2 \binom{2n+k}{n}^2} \\
&= \sum_{n=0}^{\infty} \frac{{}_3F_2\left(\begin{matrix} n+1 & n+1 & n+1 \\ 2n+2 & 2n+2 & \end{matrix} \Big| 1\right)}{(2n+1)^2 \binom{2n}{n}^2}, \\
\mathcal{E}^*\big(B_2, R_2, \zeta(2)\big) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n \binom{n+k}{n}}{(2n+k+1)^2 \binom{2n+k}{n}^2} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{{}_3F_2\left(\begin{matrix} n+1 & n+1 & n+1 \\ 2n+2 & 2n+2 & \end{matrix} \Big| 1\right)}{(2n+1)^2 \binom{2n}{n}^2}, \\
\mathcal{E}\big(B_3, R_3, \zeta(3)\big) &= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{(-1)^l \binom{k}{l} \binom{n+l}{n}}{(2n+k+1)(2n+l+1)^2 \binom{2n+k}{n} \binom{2n+l}{n}^2} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{{}_4F_3\left(\begin{matrix} n+1 & n+1 & n+1 & -k \\ 2n+2 & 2n+2 & 1 & \end{matrix} \Big| 1\right)}{(2n+1)^2 (2n+k+1) \binom{2n}{n}^2 \binom{2n+k}{n}}.
\end{aligned}
$$

Note that the hypergeometric function

$$
{}_4F_3\left(\begin{matrix} n+1 & n+1 & n+1 & -k \\ 2n+2 & 2n+2 & 1 & \end{matrix} \Big| 1\right)
$$

takes rational values for all $0 \le k, n < \infty$.

Proof: It suffices to prove the identities for $\mathcal{E}\big(B_2, R_2, \zeta(2)\big)$, since the arguments are the same for the remaining error sums. The basic idea is to use the expansion

$$
\frac{1}{(1-t)^{n+1}} = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} t^k .
$$

Then, (5.2) gives

$$\begin{aligned}
\mathcal{E}\big(B_2,R_2,\zeta(2)\big) &= \sum_{n=0}^{\infty}\int_0^1\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n}{(1-xy)^{n+1}}\,dx\,dy \\
&= \sum_{n=0}^{\infty}\int_0^1\int_0^1 \sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}(xy)^k x^n(1-x)^n y^n(1-y)^n\,dx\,dy \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}\int_0^1 x^{n+k}(1-x)^n\,dx\int_0^1 y^{n+k}(1-y)^n\,dy \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}\Big(\frac{\Gamma(n+1)\Gamma(n+k+1)}{\Gamma(2n+k+2)}\Big)^2 \qquad (5.4)\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{n!(n+k)!^3}{k!(2n+k)!^2(2n+k+1)^2} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\binom{n+k}{n}}{(2n+k+1)^2\binom{2n+k}{n}^2}.
\end{aligned}$$

The second identity for $\mathcal{E}\big(B_2,R_2,\zeta(2)\big)$ in Theorem 5.1 follows from (5.4) and from

$$\frac{\Gamma^2(n+1)\Gamma^2(n+k+1)}{\Gamma^2(2n+k+2)} = \frac{1}{(2n+1)^2\binom{2n}{n}^2}\cdot\frac{(n+1)_k(n+1)_k}{(2n+2)_k(2n+2)_k},$$

which can be verified by straightforward computations. □

Next, we define recursively a sequence $p_\nu(t)$ $(\nu=1,2,\dots)$ of polynomias in one variable t, namely

$$p_1(t) = t^2, \qquad (5.5)$$
$$p_2(t) = t^4-t^2+t, \qquad (5.6)$$
$$p_\nu(t) = t^2p_{\nu-1}(t)+t(1-t)p_{\nu-2}(t) \qquad (\nu=3,4,\dots). \qquad (5.7)$$

It is clear that $\deg p_\nu = 2\nu$, which follows easily by induction for ν with $\deg p_1 = 2$ and $\deg p_2 = 4$. The leading coefficient of p_ν is 1 for $\nu = 1,2,\dots$. Let

$$p_\nu(t) = \sum_{\mu=0}^{2\nu} a_{\nu,\mu}t^\mu .$$

Lemma 5.1. *For $\nu\geq 3$ we have*

$$p_\nu(t) = t^{2\nu}+(1-\nu)t^{2\nu-2}+\sum_{\mu=2}^{2\nu-3}\big(a_{\nu-1,\mu-2}+a_{\nu-2,\mu-1}-a_{\nu-2,\mu-2}\big)t^\mu$$

with

$$a_{\nu,\mu} = a_{\nu-1,\mu-2}+a_{\nu-2,\mu-1}-a_{\nu-2,\mu-2} \qquad (2\leq\mu\leq 2\nu-3).$$

Proof: Using the definition of $p_\nu(t)$ from (5.5) to (5.7) with $\nu \geq 3$, we obtain

$$
\begin{aligned}
p_\nu(t) &= \sum_{\mu=0}^{2\nu} a_{\nu,\mu} t^\mu \;=\; \sum_{\mu=0}^{2\nu-2} a_{\nu-1,\mu} t^{\mu+2} + \sum_{\mu=0}^{2\nu-4} a_{\nu-2,\mu} t^{\mu+1} - \sum_{\mu=0}^{2\nu-4} a_{\nu-2,\mu} t^{\mu+2} \\
&= \sum_{\mu=2}^{2\nu} a_{\nu-1,\mu-2} t^\mu + \sum_{\mu=1}^{2\nu-3} a_{\nu-2,\mu-1} t^\mu - \sum_{\mu=2}^{2\nu-2} a_{\nu-2,\mu-2} t^\mu \\
&= a_{\nu-1,2\nu-2} t^{2\nu} + a_{\nu-1,2\nu-3} t^{2\nu-1} + a_{\nu-1,2\nu-4} t^{2\nu-2} + a_{\nu-2,0} t - a_{\nu-2,2\nu-4} t^{2\nu-2} \\
&\quad + \sum_{\mu=2}^{2\nu-3} \big(a_{\nu-1,\mu-2} + a_{\nu-2,\mu-1} - a_{\nu-2,\mu-2} \big) t^\mu \\
&= t^{2\nu} + (1-\nu) t^{2\nu-2} + \sum_{\mu=2}^{2\nu-3} \big(a_{\nu-1,\mu-2} + a_{\nu-2,\mu-1} - a_{\nu-2,\mu-2} \big) t^\mu \,,
\end{aligned}
$$

since the four identities

$$
\begin{aligned}
a_{\nu-1,2\nu-2} &= 1 \,, \\
a_{\nu-1,2\nu-3} &= 0 \,, \\
a_{\nu-1,2\nu-4} - a_{\nu-2,2\nu-4} &= 1-\nu \,, \\
a_{\nu-2,0} &= 0
\end{aligned}
$$

follow easily from (5.5) to (5.7) by $a_{\nu,2\nu} = 1$, $a_{\nu,2\nu-1} = 0$, $a_{\nu,2\nu-2} = 1-\nu$, and $a_{\nu,0} = 0$ for $\nu \geq 1$. The lemma is proven. □

Theorem 5.2. *For the sequences* B_2, R_2 *defined above for* $\zeta(2)$ *we have*

$$
\mathcal{E}^*\big(B_2, R_2, \zeta(2)\big) \;=\; 1 + \sum_{\nu=1}^{\infty} \sum_{\mu=0}^{2\nu} \frac{a_{\nu,\mu}}{(\nu+1)(\mu+1)} \,.
$$

Proof: With (5.5) to (5.7) we obtain

$$\begin{aligned}
&(1+x^2y^2-xy^2-yx^2)\Big(1+\sum_{\nu=1}^{\infty}p_\nu(y)x^\nu\Big)\\
=\;& 1+x^2y^2-xy^2-yx^2+\sum_{\nu=1}^{\infty}p_\nu(y)x^\nu+\sum_{\nu=1}^{\infty}y^2p_\nu(y)x^{\nu+2}-\sum_{\nu=1}^{\infty}y^2p_\nu(y)x^{\nu+1}\\
&-\sum_{\nu=1}^{\infty}yp_\nu(y)x^{\nu+2}\\
=\;& 1+x^2y^2-xy^2-yx^2+\sum_{\nu=1}^{\infty}p_\nu(y)x^\nu-\sum_{\nu=3}^{\infty}y(1-y)p_{\nu-2}(y)x^\nu-\sum_{\nu=2}^{\infty}y^2p_{\nu-1}(y)x^\nu\\
=\;& 1-p_1(y)x-p_2(y)x^2+y^2p_1(y)x^2+\sum_{\nu=1}^{\infty}p_\nu(y)x^\nu-\sum_{\nu=3}^{\infty}y(1-y)p_{\nu-2}(y)x^\nu-\sum_{\nu=2}^{\infty}y^2p_{\nu-1}(y)x^\nu\\
=\;& 1+\sum_{\nu=3}^{\infty}p_\nu(y)x^\nu-\sum_{\nu=3}^{\infty}y(1-y)p_{\nu-2}(y)x^\nu-\sum_{\nu=3}^{\infty}y^2p_{\nu-1}(y)x^\nu\\
=\;& 1+\sum_{\nu=3}^{\infty}\Big[p_\nu(y)-\big(y^2p_{\nu-1}(y)+y(1-y)p_{\nu-2}(y)\big)\Big]x^\nu\\
=\;& 1\,.
\end{aligned}$$

Hence,

$$\frac{1}{1+x^2y^2-xy^2-yx^2}\;=\;1+\sum_{\nu=1}^{\infty}p_\nu(y)x^\nu\;=\;1+\sum_{\nu=1}^{\infty}\Big(\sum_{\mu=0}^{2\nu}a_{\nu,\mu}y^\mu\Big)x^\nu.$$

Now the theorem follows from (5.1) by two-fold integration with respect to x and y. □

We can proceed similarly in order to obtain similar results for $\mathcal{E}\big(B_2,R_2,\zeta(2)\big)$ and for $\mathcal{E}\big(B_3,R_3,\zeta(3)\big)$. Therefore, we state them without proofs. Again we define recursively a sequence $q_\nu(t)$ $(\nu=1,2,\dots)$ of integer polynomias in one variable t,

$$\begin{aligned}
q_1(t) &= 2t-t^2\,,\\
q_2(t) &= t^4-4t^3+5t^2-t\,,\\
q_\nu(t) &= t(2-t)q_{\nu-1}(t)+t(t-1)q_{\nu-2}(t) \qquad (\nu=3,4,\dots).
\end{aligned}$$

Let

$$q_\nu(t)\;=\;\sum_{\mu=0}^{2\nu}b_{\nu,\mu}t^\mu\,.$$

Here, we have

$$\frac{1}{1-x^2y^2-2xy+xy^2+yx^2}\;=\;\sum_{\nu=0}^{\infty}q_\nu(y)x^\nu\;=\;\sum_{\nu=0}^{\infty}\Big(\sum_{\mu=0}^{2\nu}b_{\nu,\mu}y^\mu\Big)x^\nu.$$

Lemma 5.2. *For $\nu \geq 3$ we have*

$$\begin{aligned} q_\nu(t) &= (-1)^\nu t^{2\nu} + 2(-1)^{\nu+1}\nu t^{2\nu-1} + (-1)^\nu(2\nu^2 - \nu - 1)t^{2\nu-2} \\ &\quad + \sum_{\mu=2}^{2\nu-3} \big(- b_{\nu-1,\mu-2} + 2b_{\nu-1,\mu-1} + b_{\nu-2,\mu-2} - b_{\nu-2,\mu-1}\big)t^\mu \end{aligned}$$

with

$$b_{\nu,\mu} \, = \, -b_{\nu-1,\mu-2} + 2b_{\nu-1,\mu-1} + b_{\nu-2,\mu-2} - b_{\nu-2,\mu-1} \qquad (2 \leq \mu \leq 2\nu - 3)\,.$$

Theorem 5.3. *For the sequences B_2, R_2 defined above for $\zeta(2)$ we have*

$$\mathcal{E}\big(B_2, R_2, \zeta(2)\big) \, = \, 1 + \sum_{\nu=1}^{\infty} \sum_{\mu=0}^{2\nu} \frac{b_{\nu,\mu}}{(\nu+1)(\mu+1)}\,.$$

The above method can be generalized such that it also works for $\mathcal{E}\big(B_3, R_3, \zeta(3)\big)$. Let

$$\begin{aligned} r_0(x,y) &= 1\,, \\ r_1(x,y) &= x^2y^2 - xy^2 - x^2y + 1\,, \\ r_2(x,y) &= x^4y^4 - 2x^3y^4 - 2x^4y^3 + x^2y^4 + x^4y^2 + 2x^3y^3 + x^2y^2 - xy^2 - x^2y - xy + 1\,, \\ r_\nu(x,y) &= \big(x^2y^2 - xy^2 - x^2y + 1\big)r_{\nu-1}(x,y) - \big(x^2y^2 - xy^2 - x^2y + xy\big)r_{\nu-2}(x,y)\,, \end{aligned}$$

where $\nu \geq 3$. Setting

$$r_\nu(x,y) \, = \, \sum_{\mu_1=0}^{2\nu} \sum_{\mu_2=0}^{2\nu} c_{\nu,\mu_1,\mu_2} x^{\mu_1} y^{\mu_2}\,,$$

it turns out that

$$\frac{1}{1 + x^2y^2w^2 - xy^2w^2 - x^2yw^2 - x^2y^2w + xyw^2 + xy^2w + x^2yw - w} \, = \, \sum_{\nu=0}^{\infty} r_\nu(x,y) \cdot w^\nu\,.$$

Then, (5.3) underlies the following result.

Theorem 5.4. *For the sequences B_3, R_3 defined above for $\zeta(3)$ we have*

$$\mathcal{E}\big(B_3, R_3, \zeta(3)\big) \, = \, \frac{1}{2} + \frac{1}{2} \sum_{\nu=1}^{\infty} \sum_{\mu_1=0}^{2\nu} \sum_{\mu_2=0}^{2\nu} \frac{c_{\nu,\mu_1,\mu_2}}{(\nu+1)(\mu_1+1)(\mu_2+1)}\,.$$

4.) As mentionned at the beginning of Section 5, some connections between rational coefficients involved in computing the error sums for Apéry's continued fraction and well-known integer sequences may be noticed. Furthermore some unproved identities have been empirically found for such coefficients.

The Theorems 5.2 and 5.3 rely on two triangles of integer coefficients, namely $a_{\nu,\mu}$ for the former and $b_{\nu,\mu}$ for the latter. Both can be expressed by binomial sums as follows.

$$\begin{aligned} a_{\nu,\mu} &= \sum_{k=0}^{\nu}\sum_{i=0}^{k}(-1)^{\nu+k}\binom{\mu-k}{2\mu-\nu-k-i}\binom{\mu-k}{i}\binom{\mu-i}{k-i}, \\ b_{\nu,\mu} &= \sum_{k=0}^{\nu}\sum_{i=0}^{k}(-1)^{\nu+\mu}\binom{\mu-k}{2\mu-\nu-k-i}\binom{\mu-k}{i}\binom{\mu-i}{k-i}, \end{aligned}$$

which both lead to non-recurrent formulas for the error sums as quadruple sums.
Several basic properties concerning the coefficients $a_{\nu,\mu}$ and $b_{\nu,\mu}$ can be noticed, including

$$\sum_{\mu=0}^{2\nu} a_{\mu,\nu} = 1 \qquad \text{and} \qquad \sum_{\mu=0}^{2\nu} b_{\mu,\nu} = 1 \qquad\qquad (\nu \in \mathbb{N})$$

and

$$a_{\nu,\mu} = a_{\mu,\nu} \qquad \text{and} \qquad b_{\nu,\mu} = b_{\mu,\nu}\,.$$

More unproved identities come from the theory of generating functions. Both coefficients $a_{\nu,\mu}$ and $b_{\nu,\mu}$ seem to be the coefficients of degree $2\nu-\mu$ in the MacLaurin series expansion of

$$\begin{cases} \dfrac{\left(\dfrac{x^2+1-\sqrt{x^4-4x^3+2x^2+1}}{2x^3}\right)^{\mu-\nu}}{\sqrt{x^4-4x^3+2x^2+1}} & \text{for } a_{\nu,\mu} \\[4ex] \dfrac{\left(\dfrac{x^2+2x-1+\sqrt{x^4+2x^2-4x+1}}{2x^3}\right)^{\mu-\nu}}{\sqrt{x^4+2x^2-4x+1}} & \text{for } b_{\nu,\mu} \end{cases}.$$

These generating functions actually allow to build the triangles of coefficients $a_{\nu,\mu}$ and $b_{\nu,\mu}$ by diagonals rather than by rows.
Summing these coefficients by rows according to Theorems 5.2 and 5.3, the results can be easely achieved by applying the following unproved recursive identities.

$$\begin{cases} \alpha_0 = 1, \quad \alpha_1 = 1/3, \quad \alpha_2 = 11/30, \quad \alpha_3 = 17/70 \\[2ex] \alpha_n = \dfrac{4n-1}{2n+1}\alpha_{n-1} - \dfrac{2n-2}{2n+1}\alpha_{n-2} \\[2ex] \qquad - \dfrac{n-1}{4n+2}\alpha_{n-3} + \dfrac{n-2}{4n+2}\alpha_{n-4} \end{cases}$$

and

$$\begin{cases} \beta_0 = 1, \quad \beta_1 = 2/3, \quad \beta_2 = 11/30, \quad \beta_3 = 47/210 \\[2ex] \beta_n = \dfrac{6n-1}{2n+1}\beta_{n-1} - \dfrac{6n-5}{2n+1}\beta_{n-2} \\[2ex] \qquad + \dfrac{5n-7}{4n+2}\beta_{n-3} - \dfrac{n-2}{4n+2}\beta_{n-4} \end{cases},$$

where

$$\alpha_\nu = \sum_{\mu=0}^{2\nu} \frac{a_{\nu,\mu}}{\mu+1} \qquad \text{and} \qquad \beta_\nu = \sum_{\mu=0}^{2\nu} \frac{b_{\nu,\mu}}{\mu+1}\,.$$

The special case $b_{n,n}$, which may be called the main diagonal in the triangle of coefficients $b_{\nu,\mu}$, leads to the following simplifications. We have

$$b_{n,n} = \sum_{k=0}^{n}\sum_{i=0}^{k} \binom{n-k}{i}^2 \binom{n-i}{k-i},$$

where the generating function of the $b_{n,n}$ is given by $1/\sqrt{x^4+2x^2-4x+1}$. This is the sequence A108626 from the *On-Line Encyclopedia of Integer Sequences*. This sequence gives the antidiagonal sums of the square array A108625 itself known to be highly related to the constant $\zeta(2)$.
$b_{\nu,\mu}$ is defined recursively by

$$b_{\nu,\mu} = 2b_{\nu-1,\mu-1} - b_{\nu-1,\mu-2} + b_{\nu-2,\mu-2} - b_{\nu-2,\mu-1}\,.$$

Assuming $b_{n,n+1} = b_{n+1,n}$ (unproved), a new recursive identity can be given concerning A108626:

$$\begin{aligned} &\texttt{A108626}\,(n+2) - 2\times\texttt{A108626}\,(n+1) - \texttt{A108626}\,(n) \\ &= 2\sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n-k+1}{i-1}\binom{n-k+1}{i}\binom{n-i+1}{k-i}. \end{aligned}$$

The previous relation actually happens to be the simplest case from a more general sequence of recurrence relations of order $2d$ given by:

$$\sum_{k=0}^{2d} c_k \texttt{A108626}\,(n+k) = (-1)^d \sum_{k=0}^{n}\sum_{i=0}^{k}\binom{n-k}{d+i}\binom{n-k}{i}\binom{n-i}{k-i},$$

where the numbers c_k are coefficients of order $2d-k$ in the characteristic polynomial

$$\frac{1}{2^d}\sum_{i=0}^{\lfloor \frac{d}{2}\rfloor}\binom{d}{2i}\left(x^4+2x^2-4x+1\right)^i\left(x^2+2x-1\right)^{d-2i}\,.$$

These recurrence relations, as well as similar ones related to the coefficients $a_{\nu,\mu}$, can be written as new generating functions, the diagonal of order d being made from the coefficients of terms with positive powers in

$$\begin{cases} \dfrac{\sum_{k=0}^{\lfloor \frac{d}{2}\rfloor}\binom{d}{2k}\left(x^4-4x^3+2x^2+1\right)^k\left(x^2+1\right)^{d-2k}}{\left(2x^3\right)^d\sqrt{x^4-4x^3+2x^2+1}} & \text{for } a_{n,n+d} \\[2ex] \dfrac{\sum_{k=0}^{\lfloor \frac{d}{2}\rfloor}\binom{d}{2k}\left(x^4+2x^2-4x+1\right)^k\left(x^2+2x-1\right)^{d-2k}}{\left(2x^3\right)^d\sqrt{x^4+2x^2-4x+1}} & \text{for } b_{n,n+d} \end{cases}.$$

References

[1] J. P. Allouche and T. Baruchel, Variations on an error sum function for the convergents of some powers of *e*, http://arxiv.org/abs/1408.2206

[2] A. Baker, *Transcendental Number Theory*, Cambridge University Press, 1975.

[3] F. Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* **11** (1979), 268-272.

[4] H. Cohen, Demonstration de l'irrationalite de $\zeta(3)$ (d'aprés R.Apéry), *Séminaire de Théorie des Nombres*, 5 octobre 1978, Grenoble, VI.1 - VI.9.

[5] H. Cohn, A short proof of the simple continued fraction expansion of e, *Amer. Math. Monthly* **113** (2006), 57-62.

[6] C. Elsner, Series of error terms for rational approximations of irrational numbers, *Journal of Integer Sequences* **14** (2011), Article 11.1.4;
http://www.cs.uwaterloo.ca/journals/JIS/VOL14/Elsner/elsner9.html

[7] C. Elsner and M. Stein, On error sum functions formed by convergents of real numbers, *Journal of Integer Sequences* **14** (2011), Article 11.8.6;
http://www.cs.uwaterloo.ca/journals/JIS/VOL14/Elsner2/elsner10.html

[8] C. Elsner and M. Stein, On the value distribution of Error Sums for approximations with rational numbers, *Integers* **12** (2012), A66, 1–28.

[9] C. Elsner, On error sums for square roots of positive integers with applications to Lucas and Pell numbers, *Journal of Integer Sequences*, **17** (2014), Article 14.4.4 .
https://cs.uwaterloo.ca/journals/JIS/VOL17/Elsner/elsner15.html

[10] C. Elsner and A. Klauke, Errorsums for the values of the exponential function, Forschungsberichte der FHDW Hannover, Bericht Nr. 02014/01, 1 - 19; RS 8153 (2014,1)

[11] C. Elsner and A. Klauke, Transcendence results and continued fraction expansions obtained from a combinatorial series, *Journal of Combinatorics and Number Theory* **5** (2013), 53 - 79.

[12] C. Elsner, On prime-detecting sequences from Apéry's recurrence formulae for $\zeta(3)$ and $\zeta(2)$, *Journal of Integer Sequences* **11** (2008), Article 08.5.1.

[13] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fifth ed., Clarendon Press, 1979.

[14] W. Koepf, *Hypergeometric Summation*, Vieweg, 1998.

[15] M. E. Larsen, *Summa Summarum*, CMS Treatises in Mathematics, A K Peters, Ltd., 2007.

[16] O. Perron, *Die Lehre von den Kettenbrüchen*, Bd. II, Wissenschaftliche Buchgesellschaft Darmstadt, 1977.

[17] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Bd. 1, 3rd ed., Springer, 1964.

[18] A. B. Shidlovskii, *Transcendental Numbers*, Walter de Gruyter, 1989.

[19] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.

List of OEIS sequence numbers.

A001850,
A003417,
A005258,
A005259,
A051451,
A108626,
A108626.