

Nonlocal conservation laws of the constant astigmatism equation

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Abstract. For the constant astigmatism equation, we construct a system of nonlocal conservation laws (an abelian covering) closed under the reciprocal transformations. We give functionally independent potentials modulo a Wronskian type relation.

1. Introduction

The constant astigmatism equation [2]

$$z_{yy} + \left(\frac{1}{z}\right)_{xx} + 2 = 0 \quad (1)$$

represents surfaces characterised by constant difference between the principal radii of curvature, with x, y being the curvature coordinates. The same equation represents orthogonal equiareal patterns on the unit sphere, closely related to sphere's plastic deformations within itself, see [11]. Both topics are classical, see, e.g., Bianchi [3, § 375], although equation (1) itself did not appear at that time.

It is clear from the geometry that the constant astigmatism equation is transformable to the sine–Gordon equation $\phi_{XY} = \sin \phi$ [11]. Then, of course, the constant astigmatism equation itself is integrable in the sense of soliton theory. Known are the zero curvature representation, see [2] or eq. (7) below, the bi-Hamiltonian structure and hierarchies of higher order symmetries and conservation laws [27], as well as multi-soliton solutions [10].

According to our previous paper [12], the constant astigmatism equation has six local

conservation laws with associated potentials $\chi, \xi, \eta, \zeta, \alpha, \beta$ satisfying

$$\begin{aligned} \chi_x &= z_y + y, & \chi_y &= \frac{z_x}{z^2} - x, \\ \eta_x &= xz_y, & \eta_y &= x\frac{z_x}{z^2} + \frac{1}{z} - x^2, \\ \xi_x &= -yz_y + z - y^2, & \xi_y &= -y\frac{z_x}{z^2}, \\ \zeta_x &= xy z_y - xz + \frac{1}{2}xy^2, & \zeta_y &= xy\frac{z_x}{z^2} + \frac{y}{z} - \frac{1}{2}x^2y, \\ \alpha_x &= \frac{\sqrt{(z_x + zz_y)^2 + 4z^3}}{z}, & \alpha_y &= \frac{\sqrt{(z_x + zz_y)^2 + 4z^3}}{z^2}, \\ \beta_x &= \frac{\sqrt{(z_x - zz_y)^2 + 4z^3}}{z}, & \beta_y &= -\frac{\sqrt{(z_x - zz_y)^2 + 4z^3}}{z^2}. \end{aligned}$$

Potentials α, β correspond to the independent variables X, Y of the related sine–Gordon equation [2].

Potentials ξ, η are images of x, y under the reciprocal transformations [12] \mathcal{Y}, \mathcal{X} , respectively; see formulas (15) below. Applying \mathcal{X} to ξ and \mathcal{Y} to η , we obtain new nonlocal potentials and the process can be continued indefinitely. It is then natural to ask what is the minimal set of potentials closed under the action of \mathcal{X} and \mathcal{Y} . They are also nonlocal conservation laws of the sine–Gordon equation, but available descriptions [30, 34] are not of much help.

In this paper we first generate hierarchies of nonlocal conservation laws from the zero curvature representation and then look how they are acted upon by \mathcal{X} and \mathcal{Y} . This allows us to find a set of potentials closed under the reciprocal transformations rather easily.

It has been known from the very beginning of the soliton theory that hierarchies of conservation laws arise through expansion in terms of the spectral parameter [24]. Literature on the subject is vast any many ways to connect integrability and hierarchies of conservation laws have been proposed (see, for example, [29, 5] or [8, Prop. 1.5] or [25, Sect. 5d]).

However, the constant astigmatism equation belongs to the rare cases when nonlocal conservation laws can be obtained in a very straightforward way, almost effortlessly. It is also easily seen how they are acted upon by the transformations.

2. Preliminaries

Let \mathcal{E} be a system of partial differential equations in two independent variables x, y . A conservation law is a 1-form $f dx + g dy$ such that $f_y - g_x = 0$ as a consequence of the system \mathcal{E} . A *potential*, say ϕ , corresponding to this conservation law is a variable which formally satisfies the compatible system $\phi_x = f, \phi_y = g$.

Let \mathfrak{g} be a matrix Lie algebra. A \mathfrak{g} -valued *zero curvature representation* [35] of the system \mathcal{E} is a 1-parametric family of \mathfrak{g} -valued forms $\alpha(\lambda) = A(\lambda) dx + B(\lambda) dy$ such that

$A_y - B_x + [A, B] = 0$ as a consequence of the system \mathcal{E} .

Let Q be an arbitrary matrix (called a *gauge matrix*) belonging to the associated Lie group \mathcal{G} . The *gauge transformation* [35] with respect to Q sends $\alpha = A dx + B dy$ to ${}^Q\alpha = {}^Q A dx + {}^Q B dy$, where

$${}^Q A = Q_x Q^{-1} + Q A Q^{-1}, \quad {}^Q B = Q_y Q^{-1} + Q B Q^{-1}. \quad (2)$$

We also say that ${}^Q A dx + {}^Q B dy$ is *gauge equivalent* to $A dx + B dy$.

The shortest way to conservation laws is from a zero curvature representation that vanishes at some value λ_0 of λ . Without loss of generality we assume that $\lambda_0 = 0$, i.e., $A(0) = B(0) = 0$. Consider the associated compatible linear system [35] (or a differential covering [16, 4])

$$\Phi_x = A\Phi, \quad \Phi_y = B\Phi, \quad (3)$$

where Φ is a column vector. Expanding Φ into the formal power series

$$\Phi = \sum_{i=0}^{\infty} \Phi_i \lambda^i$$

around zero and inserting into (3), we obtain compatible equations

$$\Phi_{n,x} = \sum_{i=1}^n A_i \Phi_{n-i}, \quad \Phi_{n,y} = \sum_{i=1}^n B_i \Phi_{n-i}, \quad n \geq 0, \quad (4)$$

where A_i, B_i are the coefficients of the Taylor expansion of A, B around $\lambda = 0$. Here we start from $i = 1$ since $A_0 = B_0 = 0$. By formulas (4), each of the derivatives $\Phi_{n,x}, \Phi_{n,y}$ is explicitly expressed in terms of $\Phi_0, \dots, \Phi_{n-1}$. Moreover, $\Phi_{0,x} = \Phi_{0,y} = 0$, meaning that Φ_0 is a constant vector. Choosing Φ_0 suitably, we thus obtain what may be called a hierarchy of vectorial potentials

$$\begin{aligned} \Phi_{1,x} &= A_1 \Phi_0, & \Phi_{1,y} &= B_1 \Phi_0, \\ \Phi_{2,x} &= A_1 \Phi_1 + A_2 \Phi_0, & \Phi_{2,y} &= B_1 \Phi_1 + B_2 \Phi_0, \\ \Phi_{3,x} &= A_1 \Phi_2 + A_2 \Phi_1 + A_3 \Phi_0, & \Phi_{3,y} &= B_1 \Phi_2 + B_2 \Phi_1 + B_3 \Phi_0, \\ & \vdots & & \end{aligned}$$

The 1-forms

$$\sum_{i=1}^n A_i \Phi_{n-i} dx + \sum_{i=1}^n B_i \Phi_{n-i} dy$$

then constitute a hierarchy of vectorial conservation laws, linear in the potentials Φ_i . Their components are the scalar conservation laws sought. They are termed ‘nonlocal’ since they depend on the potentials. The whole hierarchy of potentials is also a special abelian covering [4].

Example 1. The integrable Harry Dym equation $u_t = u^3 u_{xxx}$ (see, e.g., [1, 7, 9, 13, 19, 26, 32]) has the zero curvature representation

$$A = \begin{pmatrix} 0 & 2\lambda \\ -\frac{1}{2}\lambda/u^2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2\lambda^2 u_x & -8\lambda^3 u \\ \lambda u_{xx} + 2\lambda^3/u & -2\lambda^2 u_x \end{pmatrix}$$

polynomial in λ and vanishing at $\lambda = 0$. Using equations (4), we find an explicit infinite hierarchy of nonlocal conservation laws

$$\begin{aligned} q_{i,x} &= -\frac{1}{2} p_{i-1}/u^2, & q_{i,t} &= -u_x q_{i-2} + u_{xx} p_{i-1} + p_{i-3}/u, & i &\geq 1, \\ p_{i,x} &= q_{i-1}, & p_{i,t} &= p_{i-2} u_x - 2q_{i-3} u, & i &\geq 2, \end{aligned} \tag{5}$$

undoubtedly known to experts in the field. Here $p_i = q_i = 0$ for $i < 0$ and $p_0 = q_0 = 1$, $p_1 = x$.

It can also happen that A_0, B_0 vanish modulo a gauge transformation. The condition can be easily revealed (see below), since zero curvature representations gauge equivalent to zero have zero characteristic element [20].

If A_0, B_0 , instead of vanishing, belong to a solvable subalgebra of \mathfrak{g} , then the procedure remains essentially the same; Φ_0 itself corresponds to a finite hierarchy of potentials [22] (for instance, this is so for the famous mKdV equation). Cases when a gauge equivalent zero curvature representation fits into a solvable subalgebra are easy to recognise, too. Such a zero curvature representation admits a local solution of the associated Riccati equation [22].

As is well known, linearly independent conservation laws can have functionally dependent potentials; cf. the discussion of local or potential dependence in [28].

Example 2. Continuing the Harry Dym example, the relations

$$\sum_{i=0}^{2n} (-1)^i q_i p_{2n-i} = c_n, \quad n \geq 0,$$

among potentials are compatible with the system (5). Here c_n denote arbitrary constants.

Below we shall be interested in how conservation laws change under various local and nonlocal symmetries of the equation. It is typical for an integrable equation that its symmetries either preserve the zero curvature representation or send it to a gauge-equivalent one (see [6] for an infinitesimal criterion). If the zero curvature representation is preserved, then so are the hierarchies. If a gauge transformation (2) corresponds to the symmetry, then Φ transforms to $Q\Phi$, since (3) is equivalent to

$$(Q\Phi)_x = {}^Q A Q\Phi, \quad (Q\Phi)_y = {}^Q B Q\Phi. \tag{6}$$

3. The zero curvature representation

From now on, we deal with the constant astigmatism equation (1). The zero curvature representation we shall start with is $\alpha = A^\circ dx + B^\circ dy$, where

$$\begin{aligned} A^\circ(\lambda) &= \begin{pmatrix} \frac{(1+\lambda^2)z_x}{8\lambda z} + \frac{(1-\lambda^2)z_y}{8\lambda} & \frac{(\lambda+1)^2\sqrt{z}}{4\lambda} \\ \frac{(1-\lambda)^2\sqrt{z}}{4\lambda} & -\frac{(1+\lambda^2)z_x}{8\lambda z} - \frac{(1-\lambda^2)z_y}{8\lambda} \end{pmatrix}, \\ B^\circ(\lambda) &= \begin{pmatrix} \frac{(1-\lambda^2)z_x}{8\lambda z^2} + \frac{(1+\lambda^2)z_y}{8\lambda z} & \frac{1-\lambda^2}{4\lambda\sqrt{z}} \\ \frac{1-\lambda^2}{4\lambda\sqrt{z}} & -\frac{(1-\lambda^2)z_x}{8\lambda z^2} - \frac{(1+\lambda^2)z_y}{8\lambda z} \end{pmatrix} \end{aligned} \quad (7)$$

are two $\mathfrak{sl}(2)$ matrices rationally depending on λ . The characteristic matrix C° [21], as determined from the condition

$$A_y^\circ - B_x^\circ + [A^\circ, B^\circ] = [z_{yy} + (1/z)_{xx} + 2] \cdot C^\circ,$$

is

$$C^\circ(\lambda) = \begin{pmatrix} \frac{1-\lambda^2}{8\lambda} & 0 \\ 0 & -\frac{1-\lambda^2}{8\lambda} \end{pmatrix}.$$

We see that $C^\circ(\lambda)$ vanishes when $\lambda = \pm 1$. Consequently, both $\alpha(1)$ and $\alpha(-1)$ are gauge equivalent to zero, which makes them suitable for immediate generation of conservation laws. However, $A(\lambda)dx + B(\lambda)dy$ and $A(-\lambda)dx + B(-\lambda)dy$ turn out to be gauge equivalent and, therefore, $\lambda = \pm 1$ lead to equivalent results. Choosing $\lambda = -1$, the trivialising gauge matrix is

$$Q = \begin{pmatrix} z^{1/4} & 0 \\ xz^{1/4} & z^{-1/4} \end{pmatrix},$$

since we have

$$A^\circ(-1) = \begin{pmatrix} -\frac{z_x}{4z} & 0 \\ -\sqrt{z} & \frac{z_x}{4z} \end{pmatrix} = -Q^{-1}Q_x, \quad B^\circ(-1) = \begin{pmatrix} -\frac{z_y}{4z} & 0 \\ 0 & \frac{z_y}{4z} \end{pmatrix} = -Q^{-1}Q_y.$$

Otherwise said, matrices

$${}^Q A^\circ(\lambda) = Q_x Q^{-1} + Q A^\circ(\lambda) Q^{-1}, \quad {}^Q B^\circ(\lambda) = Q_y Q^{-1} + Q B^\circ(\lambda) Q^{-1}$$

vanish at $\lambda = -1$. Then

$$A(\lambda) = {}^Q A^\circ(\lambda - 1), \quad B(\lambda) = {}^Q B^\circ(\lambda - 1)$$

vanish at $\lambda = 0$, i.e., fit the assumption $A(0) = B(0) = 0$ of the previous section. Explicitly,

$$\begin{aligned}
 A(\lambda) &= \begin{pmatrix} \frac{\lambda(\lambda-2)K_1}{2(\lambda-1)} - \frac{\lambda^2 z L_1}{2(\lambda-1)} & \frac{\lambda^2 z}{4(\lambda-1)} \\ \frac{\lambda(\lambda-2)K_2}{2(\lambda-1)} - \frac{\lambda^2 z L_2}{2(\lambda-1)} & -\frac{\lambda(\lambda-2)K_1}{2(\lambda-1)} + \frac{\lambda^2 z L_1}{2(\lambda-1)} \end{pmatrix}, \\
 B(\lambda) &= \begin{pmatrix} \frac{\lambda(\lambda-2)L_1}{2(\lambda-1)} - \frac{\lambda^2 K_1}{2(\lambda-1)z} & -\frac{\lambda(\lambda-2)}{4(\lambda-1)} \\ \frac{\lambda(\lambda-2)L_2}{2(\lambda-1)} - \frac{\lambda^2 K_2}{2(\lambda-1)z} & -\frac{\lambda(\lambda-2)L_1}{2(\lambda-1)} + \frac{\lambda^2 K_1}{2(\lambda-1)z} \end{pmatrix},
 \end{aligned} \tag{8}$$

where

$$\begin{aligned}
 K_1 &= -\frac{z_y}{4}, & L_1 &= -\frac{z_x}{4z^2} + \frac{x}{2}, \\
 K_2 &= -\frac{xz_y}{2}, & L_2 &= -\frac{xz_x}{2z^2} - \frac{1}{2z} + \frac{x^2}{2}.
 \end{aligned} \tag{9}$$

Thus, we can derive a double hierarchy of nonlocal conservation laws by expansion of a 2-component vector Φ satisfying system (3), i.e.,

$$\Phi_x = A\Phi, \quad \Phi_y = B\Phi. \tag{10}$$

This will be done in the next section. Our primary interest in this section are the transformation properties of these hierarchies under local and nonlocal symmetries. These will be derived from the transformation properties of the zero curvature representation.

To start with, the constant astigmatism equation is invariant under the involution

$$\bar{x} = y, \quad \bar{y} = x, \quad \bar{z} = \frac{1}{z}, \tag{11}$$

henceforth called *duality*. By applying involution to the zero curvature representation $A dx + B dy$, we obtain $\bar{A} d\bar{x} + \bar{B} d\bar{y} = \bar{B} dx + \bar{A} dy$, where \bar{A}, \bar{B} result from A, B by replacing K_i, L_i with

$$\begin{aligned}
 \bar{K}_1 &= \frac{z_x}{4z^2}, & \bar{L}_1 &= \frac{z_y}{4} + \frac{y}{2}, \\
 \bar{K}_2 &= \frac{yz_x}{2z^2}, & \bar{L}_2 &= \frac{yz_y}{2} - \frac{z}{2} + \frac{y^2}{2}.
 \end{aligned} \tag{12}$$

Thus, the *dual conservation laws* will be derived by expansion of a vector $\bar{\Phi}$ satisfying the system

$$\bar{\Phi}_x = \bar{B}\bar{\Phi}, \quad \bar{\Phi}_y = \bar{A}\bar{\Phi}. \tag{13}$$

An easy computation reveals that the zero curvature representation $\bar{A}(\lambda) d\bar{x} + \bar{B}(\lambda) d\bar{y} = \bar{B}(\lambda) dx + \bar{A}(\lambda) dy$ is gauge equivalent to $A(\lambda) dx + B(\lambda) dy$. The gauge matrix and its inverse are

$$H = \frac{1}{\lambda - 2} \begin{pmatrix} -x & 1 \\ -xy - 1 + \frac{2}{\lambda} & y \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} \lambda y & -\lambda \\ \lambda xy + \lambda - 2 & -\lambda x \end{pmatrix} \quad (14)$$

unless $\lambda = 0, 2$. Consequently, $H^{-1}\bar{\Phi}$ satisfies system (10).

Furthermore, the constant astigmatism equation is invariant under the reciprocal transformations [12] $\mathcal{X}(x, y, z) = (x', y', z')$ and $\mathcal{Y}(x, y, z) = (x^*, y^*, z^*)$, where

$$\begin{aligned} x' &= \frac{xz}{x^2z + 1}, & y' &= \eta, & z' &= \frac{(x^2z + 1)^2}{z}, \\ x^* &= \xi, & y^* &= \frac{y}{y^2 + z}, & z^* &= \frac{z}{(y^2 + z)^2}, \end{aligned} \quad (15)$$

see the Introduction for ξ, η . Neglecting the integration constants, we have $\mathcal{X} \circ \mathcal{X} = \text{Id} = \mathcal{Y} \circ \mathcal{Y}$. Since \mathcal{X} is related to \mathcal{Y} by the involution (11), we shall focus on \mathcal{X} only.

The image of the zero curvature representation $A dx + B dy$ under \mathcal{X} is $A' dx' + B' dy' = \tilde{A} dx + \tilde{B} dy$, where

$$\begin{aligned} \tilde{A}(\lambda) &= \begin{pmatrix} -\frac{\lambda(\lambda - 2)K_1}{2(\lambda - 1)} + \frac{\lambda^2 z L_1}{2(\lambda - 1)} & \frac{\lambda(\lambda - 2)K_2}{2(\lambda - 1)} - \frac{\lambda^2 z L_2}{2(\lambda - 1)} \\ \frac{\lambda^2 z}{4(\lambda - 1)} & \frac{\lambda(\lambda - 2)K_1}{2(\lambda - 1)} - \frac{\lambda^2 z L_1}{2(\lambda - 1)} \end{pmatrix}, \\ \tilde{B}(\lambda) &= \begin{pmatrix} -\frac{\lambda(\lambda - 2)L_1}{2(\lambda - 1)} + \frac{\lambda^2 K_1}{2(\lambda - 1)z} & \frac{\lambda(\lambda - 2)L_2}{2(\lambda - 1)} - \frac{\lambda^2 K_2}{2(\lambda - 1)z} \\ -\frac{\lambda(\lambda - 2)}{4(\lambda - 1)} & \frac{\lambda(\lambda - 2)L_1}{2(\lambda - 1)} - \frac{\lambda^2 K_1}{2(\lambda - 1)z} \end{pmatrix}, \end{aligned}$$

with K_i, L_i being given by formulas (9). Thus, the *reciprocal conservation laws* will be derived by expansion of a vector Φ' satisfying the system

$$\Phi'_x = \tilde{A}\Phi', \quad \Phi'_y = \tilde{B}\Phi'. \quad (16)$$

It is easy to check that $\tilde{A} dx + \tilde{B} dy$ is gauge equivalent to $A dx + B dy$ through the gauge matrix

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = S^{-1}. \quad (17)$$

Consequently, $S^{-1}\Phi' = S\Phi'$ satisfies system (10).

Finally, *reciprocal dual conservation laws* are derived from $\bar{A}' d\bar{x}' + \bar{B}' d\bar{y}'$. Omitting details, $P^{-1}\bar{\Phi}'$ satisfies system (10), where

$$P^{-1} = \begin{pmatrix} \frac{\lambda x z \eta}{x^2 z + 1} + \lambda - 2 & -\frac{\lambda x z}{x^2 z + 1} \\ \lambda \eta & -\lambda \end{pmatrix}.$$

4. The hierarchies

Denote

$$\Phi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \bar{\Phi} = \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}, \quad \Phi' = \begin{pmatrix} u' \\ v' \end{pmatrix}, \quad \bar{\Phi}' = \begin{pmatrix} \bar{u}' \\ \bar{v}' \end{pmatrix}$$

the vectors generating ordinary, dual, reciprocal, and reciprocal dual hierarchy of nonlocal conservation laws. Expanding $A(\lambda), B(\lambda)$ into series $\sum_i A_i \lambda^i, \sum_i B_i \lambda^i$ around $\lambda = 0$, we find that $A_0 = B_0 = 0$, as expected, and

$$A_1 = \begin{pmatrix} K_1 & 0 \\ K_2 & -K_1 \end{pmatrix}, \quad A_2 = A_3 = \dots = \begin{pmatrix} \frac{1}{2} K_1 + \frac{1}{2} z L_1 & -\frac{1}{4} z \\ \frac{1}{2} K_2 + \frac{1}{2} z L_2 & -\frac{1}{2} K_1 - \frac{1}{2} z L_1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} L_1 & -\frac{1}{2} \\ L_2 & -L_1 \end{pmatrix}, \quad B_2 = B_3 = \dots = \begin{pmatrix} \frac{1}{2} L_1 + \frac{1}{2} K_1/z & -\frac{1}{4} \\ \frac{1}{2} L_2 + \frac{1}{2} K_2/z & -\frac{1}{2} L_1 - \frac{1}{2} K_1/z \end{pmatrix}.$$

It follows that formulas (4) simplify to

$$\Phi_{n,x} = A_1 \Phi_{n-1} + A_2 \sum_{i=0}^{n-2} \Phi_i, \quad \Phi_{n,y} = B_1 \Phi_{n-1} + B_2 \sum_{i=0}^{n-2} \Phi_i. \quad (18)$$

Substituting

$$\Phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Phi_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \quad i > 1,$$

into formulas (18), we immediately obtain the following construction of potentials u_i, v_i .

Construction 1. *Denote*

$$u_0 = 1, \quad v_0 = 0,$$

and define potentials u_n, v_n by induction

$$u_{n,x} = K_1 u_{n-1} + \frac{1}{2} (K_1 + z L_1) \sum_{i=0}^{n-2} u_i - \frac{1}{4} z \sum_{i=0}^{n-2} v_i,$$

$$u_{n,y} = L_1 u_{n-1} - \frac{1}{2} v_{n-1} + \frac{1}{2} \left(L_1 + \frac{K_1}{z} \right) \sum_{i=0}^{n-2} u_i - \frac{1}{4} \sum_{i=0}^{n-2} v_i, \quad (19)$$

$$v_{n,x} = K_2 u_{n-1} - K_1 v_{n-1} + \frac{1}{2} (K_2 + z L_2) \sum_{i=0}^{n-2} u_i - \frac{1}{2} (K_1 + z L_1) \sum_{i=0}^{n-2} v_i,$$

$$v_{n,y} = L_2 u_{n-1} - L_1 v_{n-1} + \frac{1}{2} \left(L_2 + \frac{K_2}{z} \right) \sum_{i=0}^{n-2} u_i - \frac{1}{2} \left(L_1 + \frac{K_1}{z} \right) \sum_{i=0}^{n-2} v_i,$$

for all $n > 0$, with K_1, K_2, L_1, L_2 being as introduced by formulas (9) above.

By construction, u_i, v_i are potentials of nonlocal conservation laws of the constant astigmatism equation. Observe that $u_{1,x} = K_1$, $u_{1,y} = L_1$, $v_{1,x} = K_2$, $v_{1,y} = L_2$ are local. As we shall see later, the potentials u_n, v_n are mutually independent. Choosing a different initial vector $\Phi_0 \neq 0$, we obtain another set of potentials, linearly dependent on the potentials just constructed.

The constant astigmatism equation is invariant under the involution (11), while Construction 1 is not. Mutatis mutandis, we obtain the dual construction.

Construction 2. Denote

$$\bar{u}_0 = 1, \quad \bar{v}_0 = 0,$$

and define potentials \bar{u}_n, \bar{v}_n by induction

$$\begin{aligned} \bar{u}_{n,x} &= \bar{L}_1 \bar{u}_{n-1} - \frac{1}{2} \bar{v}_{n-1} + \frac{1}{2} (\bar{L}_1 + z \bar{K}_1) \sum_{i=0}^{n-2} \bar{u}_i - \frac{1}{4} \sum_{i=0}^{n-2} \bar{v}_i, \\ \bar{u}_{n,y} &= \bar{K}_1 \bar{u}_{n-1} + \frac{1}{2} \left(\bar{K}_1 + \frac{\bar{L}_1}{z} \right) \sum_{i=0}^{n-2} \bar{u}_i - \frac{1}{4z} \sum_{i=0}^{n-2} \bar{v}_i, \\ \bar{v}_{n,x} &= \bar{L}_2 \bar{u}_{n-1} - \bar{L}_1 \bar{v}_{n-1} + \frac{1}{2} (\bar{L}_2 + z \bar{K}_2) \sum_{i=0}^{n-2} \bar{u}_i - \frac{1}{2} (\bar{L}_1 + z \bar{K}_1) \sum_{i=0}^{n-2} \bar{v}_i, \\ \bar{v}_{n,y} &= \bar{K}_2 \bar{u}_{n-1} - \bar{K}_1 \bar{v}_{n-1} + \frac{1}{2} \left(\bar{K}_2 + \frac{\bar{L}_2}{z} \right) \sum_{i=0}^{n-2} \bar{u}_i - \frac{1}{2} \left(\bar{K}_1 + \frac{\bar{L}_1}{z} \right) \sum_{i=0}^{n-2} \bar{v}_i, \end{aligned} \tag{20}$$

for all $n > 0$, with $\bar{K}_1, \bar{K}_2, \bar{L}_1, \bar{L}_2$ being given by formulas (12) above.

By construction, \bar{u}_i, \bar{v}_i are nonlocal potentials of the constant astigmatism equation. They are said to be *dual* to u_i, v_i . However, $u_i, v_i, \bar{u}_i, \bar{v}_i$ are not functionally independent, as we shall see below.

We shall not present any construction of potentials $u'_i, v'_i, \bar{u}'_i, \bar{v}'_i$. Instead, we shall show how they depend on the potentials $u_i, v_i, \bar{u}_i, \bar{v}_i$ already constructed; see Proposition 2 below.

Remark 1. The potentials χ, ξ, η, ζ (see the Introduction) are functions of x, y, z, u_1, v_1, w_1 , namely

$$\begin{aligned} \eta &= -2v_1, & \chi &= -4u_1 + xy, \\ \xi &= -2w_1, & \zeta &= 2u_1^2 + 2u_1 - 4u_2 - 2yv_1 - \frac{1}{2} \ln z + \frac{1}{4} x^2 y^2. \end{aligned}$$

5. Linear and Wronskian relations

As we know from Section 3, vectors $\Phi, H^{-1}\Phi, S^{-1}\Phi, P^{-1}\Phi$ satisfy system (10). More generally, we can write

$$\Phi_x^{(k)} = A\Phi^{(k)}, \quad \Phi_y^{(k)} = B\Phi^{(k)}, \quad k = 1, \dots, m. \tag{21}$$

Let h be the dimension of the vectors $\Phi^{(k)}$, i.e., let A, B belong to $\mathfrak{gl}(h)$. Let $W_{k_1 \dots k_h}$ denote the Wronskian determinant composed of h columns $\Phi^{(k_1)}, \dots, \Phi^{(k_h)}$. Denoting $c_{k_1 \dots k_h}$ arbitrary constants, possibly depending on λ , relations

$$W_{k_1 \dots k_h} = c_{k_1 \dots k_h}$$

are compatible with system (21). Moreover, if $m > n$, then columns $\Phi^{(h+1)}, \dots, \Phi^{(m)}$ are linear combinations of $\Phi^{(1)}, \dots, \Phi^{(h)}$ with constant coefficients:

$$\Phi^{(l)} = c_{k_1}^l \Phi^{(k_1)} + \dots + c_{k_h}^l \Phi^{(k_h)}, \quad l = h+1, \dots, m. \quad (22)$$

Example 3. Let $\mathfrak{g} = \mathfrak{sl}(2)$, let $\Phi^{(k)} = (\phi_1^{(k)}, \phi_2^{(k)})^\top$ satisfy (21). For every pair of indices $1 \leq k_1 < k_2 \leq m$, let $W_{k_1 k_2} = \phi_1^{(k_1)} \phi_2^{(k_2)} - \phi_1^{(k_2)} \phi_2^{(k_1)}$ denote the corresponding Wronskian determinant. If $m = 2$, then we have one relation $W_{12} = c_{12}$, and we can solve it, say, for $\phi_2^{(2)}$ to get

$$\phi_2^{(2)} = \frac{\phi_2^{(1)} \phi_1^{(2)} + c_{12}}{\phi_1^{(1)}}. \quad (23)$$

If $m \geq 3$, then, additionally to (23), we have linear relations

$$c_{k_1 k_2} \phi_1^{(k_3)} - c_{k_1 k_3} \phi_1^{(k_2)} + c_{k_2 k_3} \phi_1^{(k_1)} = 0, \quad c_{k_1 k_2} \phi_2^{(k_3)} - c_{k_1 k_3} \phi_2^{(k_2)} + c_{k_2 k_3} \phi_2^{(k_1)} = 0$$

and we can express $\phi_i^{(k)}$, $k > 2$, in terms of $\phi_i^{(1)}, \phi_i^{(2)}$. It is perhaps worth noticing that the Wronskian determinants $W_{k_1 k_2}$ themselves are not independent if $m \geq 4$. As a consequence, relations $c_{k_1 k_2} c_{k_3 k_4} - c_{k_1 k_3} c_{k_2 k_4} + c_{k_1 k_4} c_{k_2 k_3} = 0$ are imposed, one for each quadruple of mutually distinct indices k_1, k_2, k_3, k_4 .

The linear relations are to be employed first. We have

$$\begin{aligned} \Phi^{(1)} = \Phi &= \begin{pmatrix} u \\ v \end{pmatrix}, & \Phi^{(2)} = H^{-1} \bar{\Phi} &= \begin{pmatrix} \lambda y u' - \lambda v' \\ (\lambda x y + \lambda - 2) u' - \lambda x v' \end{pmatrix}, \\ \Phi^{(3)} = S \bar{\Phi} &= \begin{pmatrix} \bar{v} \\ \bar{u} \end{pmatrix}, & \Phi^{(4)} = P^{-1} \bar{\Phi}' &= \begin{pmatrix} \left(\frac{\lambda x z \eta}{x^2 z + 1} + \lambda - 2 \right) \bar{u}' - \frac{\lambda x z}{x^2 z + 1} \bar{v}' \\ \lambda \eta \bar{u}' - \lambda \bar{v}' \end{pmatrix}. \end{aligned} \quad (24)$$

We obtain two linear relations (22), namely

$$\begin{aligned} \begin{pmatrix} \bar{v} \\ \bar{u} \end{pmatrix} &= c_1^3 \begin{pmatrix} u \\ v \end{pmatrix} + c_2^3 \begin{pmatrix} \lambda y u' - \lambda v' \\ (\lambda x y + \lambda - 2) u' - \lambda x v' \end{pmatrix}, \\ \begin{pmatrix} \left(\frac{\lambda x z \eta}{x^2 z + 1} + \lambda - 2 \right) \bar{u}' - \frac{\lambda x z}{x^2 z + 1} \bar{v}' \\ \lambda \eta \bar{u}' - \lambda \bar{v}' \end{pmatrix} &= c_1^4 \begin{pmatrix} u \\ v \end{pmatrix} + c_2^4 \begin{pmatrix} \lambda y u' - \lambda v' \\ (\lambda x y + \lambda - 2) u' - \lambda x v' \end{pmatrix}. \end{aligned}$$

The choice of constants $c_1^3, c_2^3, c_1^4, c_2^4$ is more or less arbitrary and influences the extension of \mathcal{X} to higher potentials. We would like to preserve the relation $\mathcal{X} \circ \mathcal{X} = \text{Id}$. In particular, we want $v'_1 = -\frac{1}{2}y$, in view of $y' = \eta = -2v_1$ from Remark 1. For this reason, we have to

withstand the temptation to set $v'_i = u_i$, $u'_i = v_i$ by choosing $c_1^3 = 1$, $c_2^3 = 0$. We would also like \mathcal{X} to preserve constants, i.e., we want $u'_0 = 1$, $v'_0 = 0$, $\bar{u}'_0 = 1$, $\bar{v}'_0 = 0$. Led by these considerations, we set

$$c_1^3 = 0, \quad c_2^3 = -\frac{1}{2}, \quad c_1^4 = -2, \quad c_2^4 = 0.$$

Under this choice, the above linear relations yield

$$\begin{aligned} u' &= \left(1 - \frac{\lambda}{2}(xy + 1)\right)\bar{u} + \frac{\lambda}{2}x\bar{v}, & \bar{u}' &= \frac{2}{2-\lambda}\left(u - \frac{xz}{x^2z+1}v\right), \\ v' &= \frac{\lambda}{2}y\bar{u} + \frac{\lambda}{2}\bar{v}, & \bar{v}' &= \frac{2}{\lambda}v + \frac{4v_1}{\lambda-2}\left(u - \frac{xz}{x^2z+1}v\right). \end{aligned} \tag{25}$$

Moreover, $p'' = p$ for each potential $p = u, v, \bar{u}, \bar{v}$.

Proposition 1. *Potentials $u_i, v_i, \bar{u}_i, \bar{v}_i$ transform under \mathcal{X} as follows:*

$$\begin{aligned} u'_i &= \bar{u}_i - \frac{1}{2}(1 + xy)\bar{u}_{i-1} + \frac{1}{2}x\bar{v}_{i-1}, \\ v'_i &= -\frac{1}{2}y\bar{u}_{i-1} + \frac{1}{2}\bar{v}_{i-1}, \\ \bar{u}'_i &= \sum_{j=0}^i \frac{1}{2^{i-j}} \left(u_j - \frac{xz}{x^2z+1}v_j\right), \\ \bar{v}'_i &= 2v_{i+1} - 2v_1 \sum_{j=0}^i \frac{1}{2^{i-j}} \left(u_j - \frac{xz}{x^2z+1}v_j\right). \end{aligned}$$

Moreover, $p''_i = p$ for each potential $p_i = u_i, v_i, \bar{u}_i, \bar{v}_i$.

Proof. By expanding formulas (25) in powers of λ . □

Proposition 2. *Potentials $u_i, v_i, \bar{u}_i, \bar{v}_i$ transform under \mathcal{Y} as follows:*

$$\begin{aligned} u_i^* &= \sum_{j=0}^i \frac{1}{2^{i-j}} \left(\bar{u}_j - \frac{y}{y^2+z}\bar{v}_j\right), \\ v_i^* &= 2v_{i+1} - 2\bar{v}_1 \sum_{j=0}^i \frac{1}{2^{i-j}} \left(\bar{u}_j - \frac{y}{y^2+z}\bar{v}_j\right), \\ \bar{u}_i^* &= u_i - \frac{1}{2}(1 + xy)u_{i-1} + \frac{1}{2}yv_{i-1}, \\ \bar{v}_i^* &= -\frac{1}{2}xu_{i-1} + \frac{1}{2}v_{i-1}. \end{aligned}$$

Moreover, $p_i^{**} = p$ for each potential $p_i = u_i, v_i, \bar{u}_i, \bar{v}_i$.

Proof. By $\mathcal{Y} = \mathcal{I} \circ \mathcal{X} \circ \mathcal{I}$, where $\mathcal{I} : x \leftrightarrow y, z \leftrightarrow 1/z, u_i \leftrightarrow \bar{u}_i, v_i \leftrightarrow \bar{v}_i$ is the duality transformation. □

We see that equalities $\mathcal{X} \circ \mathcal{X} = \text{Id} = \mathcal{Y} \circ \mathcal{Y}$ still hold after extension of \mathcal{X}, \mathcal{Y} to the higher potentials.

Turning back to ordinary and dual conservation laws, we recall that $H^{-1}\bar{\Phi}$ and Φ satisfy one and the same linear system (10). This implies constancy of the Wronskian determinant of $\Phi = (u, v)^\top$ and $H^{-1}\bar{\Phi} = H^{-1}(\bar{u}, \bar{v})^\top$, which is equivalent to

$$R(u, v, \bar{u}, \bar{v}, \lambda) \equiv (\lambda - 2)u\bar{u} + \lambda(xu - v)(y\bar{u} - \bar{v}) = c(\lambda). \quad (26)$$

It follows that hierarchies $u_i, v_i, \bar{u}_i, \bar{v}_i$ are not independent.

Proposition 3. *For all integers $n \geq -1$ and constants c_i , we have the relations*

$$\sum_{i=0}^n u_i \bar{u}_{n-i} + \sum_{i=0}^n (xu_i - v_i)(y\bar{u}_{n-i} - \bar{v}_{n-i}) - 2 \sum_{i=0}^{n+1} u_i \bar{u}_{n-i+1} - c_{n+1}. \quad (27)$$

Proof. The statement results from formula (26) by expanding around $\lambda = 0$ and setting $c(\lambda) = \sum c_n \lambda^n$. \square

We have the freedom to choose $c(\lambda) \neq 0$ (if $c = 0$ then the three hierarchies are functionally dependent again). To have $\bar{u}_0 = 1$, we choose $c = -2$, i.e., $c_0 = -2$, $c_i = 0$ for $i > 0$. Solving (27) with respect to \bar{u}_{k+1} and renaming \bar{v}_i to w_i , we get the recursion formulas

$$\begin{aligned} \bar{u}_0 &= 1, \\ \bar{u}_{k+1} &= \frac{1+xy}{2} \sum_{i=0}^k u_i \bar{u}_{k-i} - \frac{y}{2} \sum_{i=0}^k \bar{u}_i v_{k-i} + \frac{1}{2} \sum_{i=0}^k (v_i - xu_i) w_{k-i} - \sum_{i=1}^{k+1} u_i \bar{u}_{k-i+1}. \end{aligned} \quad (28)$$

E.g., $\bar{u}_1 = -u_1 + \frac{1}{2}(1+xy)$, etc.

Assuming assignments (28) and renaming \bar{v}_i to w_i , systems (19) and (20) reduce to covering (29) below.

Construction 3. *Let*

$$u_0 = 1, \quad v_0 = w_0 = 0,$$

and define potentials u_n, v_n, w_n by induction

$$\begin{aligned}
 u_{n,x} &= K_1 u_{n-1} + \frac{1}{2} (K_1 + zL_1) \sum_{i=0}^{n-2} u_i - \frac{1}{4} z \sum_{i=0}^{n-2} v_i, \\
 u_{n,y} &= L_1 u_{n-1} - \frac{1}{2} v_{n-1} + \frac{1}{2} \left(L_1 + \frac{K_1}{z} \right) \sum_{i=0}^{n-2} u_i - \frac{1}{4} \sum_{i=0}^{n-2} v_i, \\
 v_{n,x} &= K_2 u_{n-1} - K_1 v_{n-1} + \frac{1}{2} (K_2 + zL_2) \sum_{i=0}^{n-2} u_i - \frac{1}{2} (K_1 + zL_1) \sum_{i=0}^{n-2} v_i, \\
 v_{n,y} &= L_2 u_{n-1} - L_1 v_{n-1} + \frac{1}{2} \left(L_2 + \frac{K_2}{z} \right) \sum_{i=0}^{n-2} u_i - \frac{1}{2} \left(L_1 + \frac{K_1}{z} \right) \sum_{i=0}^{n-2} v_i, \\
 w_{n,x} &= \bar{L}_2 \bar{u}_{n-1} - \bar{L}_1 w_{n-1} + \frac{1}{2} (\bar{L}_2 + z\bar{K}_2) \sum_{i=0}^{n-2} \bar{u}_i - \frac{1}{2} (\bar{L}_1 + z\bar{K}_1) \sum_{i=0}^{n-2} w_i, \\
 w_{n,y} &= \bar{K}_2 \bar{u}_{n-1} - \bar{K}_1 w_{n-1} + \frac{1}{2} \left(\bar{K}_2 + \frac{\bar{L}_2}{z} \right) \sum_{i=0}^{n-2} \bar{u}_i - \frac{1}{2} \left(\bar{K}_1 + \frac{\bar{L}_1}{z} \right) \sum_{i=0}^{n-2} w_i,
 \end{aligned} \tag{29}$$

with \bar{u}_i being given by formulas (28).

By construction, equations (29) are compatible and yield a triple hierarchy of conservation laws of the constant astigmatism equation.

Remark 2. The constant astigmatism equation possesses a scaling symmetry. The variables have weights according to Table 1.

| Variable | x | y | z | u_i | v_i | w_i |
|----------|-----|-----|-----|-------|-------|-------|
| Weight | -1 | 1 | 2 | 0 | -1 | 1 |

Table 1. Weights of variables under the scaling symmetry

6. Independence

The last part of this paper is devoted to the proof that there are no relations among $u_i, v_i, \bar{u}_i, \bar{v}_i$ other than (27). Actually we prove that there are no relations among u_i, v_i, w_i determined by Construction 3. Otherwise the method is that of recent works [17, 18].

Consider the base coordinates x, y and the jet coordinates z_μ . The *total derivatives* are the vector fields

$$D_x = \frac{\partial}{\partial x} + \sum_{\mu} z_{\mu x} \frac{\partial}{\partial z_{\mu}}, \quad D_y = \frac{\partial}{\partial y} + \sum_{\mu} z_{\mu y} \frac{\partial}{\partial z_{\mu}}, \tag{30}$$

where μ runs over all monomials in x, y (according to a convenient, but abandoned tradition, μ can be thought of as a monomial $x^i y^j$).

To impose the constant astigmatism equation \mathcal{E} , we assign

$$z_{yy} = -\left(\frac{1}{z}\right)_{xx} - 2, \quad z_{\underbrace{x \cdots x}_n yy} = -D_x^{n+2}(1/z), \quad n > 0. \quad (31)$$

The first equation is the constant astigmatism equation solved for z_{yy} , the others are the differential consequences of the first equation. The total derivatives can be restricted to \mathcal{E} (since they are tangent to \mathcal{E}). Obviously, $D_x|_{\mathcal{E}}, D_y|_{\mathcal{E}}$ commute. They are given by the same formulas (30) under assignments (31) and with z_μ running only over the unassigned jet coordinates (with μ running over all monomials not divisible by y^2).

Joint systems (19) and (20) give a covering $\tilde{\mathcal{E}}$ equipped with total derivatives given by

$$\begin{aligned} \tilde{D}_x^{(4)} &= D_x + \sum_{n=1}^{\infty} \left(u_{n,x} \frac{\partial}{\partial u_n} + v_{n,x} \frac{\partial}{\partial v_n} + \bar{u}_{n,x} \frac{\partial}{\partial \bar{u}_n} + \bar{v}_{n,x} \frac{\partial}{\partial \bar{v}_n} \right), \\ \tilde{D}_y^{(4)} &= D_y + \sum_{n=1}^{\infty} \left(u_{n,y} \frac{\partial}{\partial u_n} + v_{n,y} \frac{\partial}{\partial v_n} + \bar{u}_{n,y} \frac{\partial}{\partial \bar{u}_n} + \bar{v}_{n,y} \frac{\partial}{\partial \bar{v}_n} \right). \end{aligned}$$

Reducing by relations (27), we obtain covering (29) with total derivatives given by

$$\begin{aligned} \tilde{D}_x^{(3)} &= D_x + \sum_{n=1}^{\infty} \left(u_{n,x} \frac{\partial}{\partial u_n} + v_{n,x} \frac{\partial}{\partial v_n} + w_{n,x} \frac{\partial}{\partial w_n} \right), \\ \tilde{D}_y^{(3)} &= D_y + \sum_{n=1}^{\infty} \left(u_{n,y} \frac{\partial}{\partial u_n} + v_{n,y} \frac{\partial}{\partial v_n} + w_{n,y} \frac{\partial}{\partial w_n} \right). \end{aligned}$$

We are going to prove that there are no relations among $x, y, z_\mu, u_n, v_n, w_n$. A covering $\tilde{\mathcal{E}}$ is said to be *differentially connected* in the sense of [14, 15] if every function $F : \tilde{\mathcal{E}} \rightarrow \mathbb{R}$ such that $\tilde{D}_x F = \tilde{D}_y F = 0$ is a constant (possibly depending on the spectral parameter).

If $\tilde{D}_x F = \tilde{D}_y F = 0$ and F is not a constant, then condition $F = c(\lambda)$ determines a relation compatible with the covering $\tilde{\mathcal{E}}$. Needless to say, the constant astigmatism equation itself is differentially connected. Moreover, being differentially connected is a typical property of the covering (3) associated with a non-degenerate zero curvature representation.

In what follows, δ_{ij} is the Kronecker delta. The binomial coefficients

$$\binom{n}{k} = \prod_{i=1}^k \frac{n-i+1}{i} \quad (32)$$

are defined for all integer values of n, k . For nonnegative n they are nonzero if and only if $0 \leq k \leq n$. For negative n they are zero if and only if $n < k < 0$.

Proposition 4. *Let F be a smooth function of a finite number of variables $x, y, z_\mu, u_n, v_n, w_n, n = 1, \dots, N$. If $\tilde{D}_x^{(3)} F = \tilde{D}_y^{(3)} F = 0$, then F is a constant.*

Proof. Let $\tilde{\mathcal{E}}^{(N)}$ be the covering corresponding to the potentials $u_n, v_n, w_n, n = 1, \dots, N$, so that $\tilde{\mathcal{E}}^{(N)}$ is the domain of F . Since F depends on a finite number of variables, there is a maximal degree monomial $\nu = x^i$ such that F depends on $z_{\nu x}$ or $z_{\nu y}$. Then we have

$$\frac{\partial F}{\partial z_{\nu x}} = \frac{\partial}{\partial z_{\nu x x}} \tilde{D}_x F = 0, \quad \frac{\partial F}{\partial z_{\nu y}} = \frac{\partial}{\partial z_{\nu x y}} \tilde{D}_y F = 0$$

for all $i \geq 0$. It follows that F cannot actually depend on $z_{\nu x}$ or $z_{\nu y}$ for any ν . Therefore, F can only depend on the $3N + 3$ variables x, y, z and u_n, v_n, w_n , where $n = 1, \dots, N$ for some $N \in \mathbb{N}$. Since $u_{n,x}, u_{n,y}, v_{n,x}, v_{n,y}, w_{n,x}, w_{n,y}$ are linear in z_x, z_y , so are $\tilde{D}_x F$ and $\tilde{D}_y F$, and we can decompose

$$\tilde{D}_x F = F_{11} z_x + F_{12} z_y + F_{10}, \quad \tilde{D}_y F = F_{21} z_x + F_{22} z_y + F_{20},$$

where F_{ij} are independent of z_x, z_y . It is easy to check that $F_{22} = F_{11}$ and $F_{12} = z^2 F_{21}$, while the others are independent. Define vector fields X_1, X_0, Y_1, Y_0 by

$$X_1 F = F_{11}, \quad X_0 F = F_{10}, \quad Y_1 F = F_{21}, \quad Y_0 F = F_{20}.$$

In coordinates,

$$\begin{aligned} X_1 &= \frac{\partial}{\partial z} - \frac{1}{8z} \sum_{i=0}^N \sum_{j=0}^{i-2} u_j \frac{\partial}{\partial u_i} - \frac{1}{8z} \sum_{i=0}^N \sum_{j=0}^{i-2} (2xu_j - v_j) \frac{\partial}{\partial v_i} \\ &\quad + \frac{1}{8z} \sum_{i=0}^N \sum_{j=0}^{i-2} (2y\bar{u}_j - w_j) \frac{\partial}{\partial w_i}, \\ Y_1 &= -\frac{1}{8z^2} \sum_{i=0}^N \sum_{j=0}^{i-1} (1 + \delta_{j,i-1}) u_j \frac{\partial}{\partial u_i} - \frac{1}{8z^2} \sum_{i=0}^N \sum_{j=0}^{i-1} (1 + \delta_{j,i-1}) (2xu_j - v_j) \frac{\partial}{\partial v_i} \\ &\quad + \frac{1}{8z^2} \sum_{i=0}^N \sum_{j=0}^{i-1} (1 + \delta_{j,i-1}) (2y\bar{u}_j - w_j) \frac{\partial}{\partial w_i}, \\ X_0 &= \frac{\partial}{\partial x} + \frac{z}{4} \sum_{i=0}^N \sum_{j=0}^{i-2} (xu_j - v_j) \frac{\partial}{\partial u_i} + \frac{1}{4} \sum_{i=0}^N \sum_{j=0}^{i-2} [(x^2 z + 1)u_j - xv_j] \frac{\partial}{\partial v_i} \\ &\quad + \frac{1}{4} \sum_{i=0}^N \sum_{j=0}^{i-1} (1 + \delta_{j,i-1}) [(y^2 - z)\bar{u}_j - yw_j] \frac{\partial}{\partial w_i}, \\ Y_0 &= \frac{\partial}{\partial y} + \frac{1}{4} \sum_{i=0}^N \sum_{j=0}^{i-1} (1 + \delta_{j,i-1}) (xu_j - v_j) \frac{\partial}{\partial u_i} \\ &\quad + \frac{1}{4z} \sum_{i=0}^N \sum_{j=0}^{i-1} (1 + \delta_{j,i-1}) [(x^2 z - 1)u_j - xv_j] \frac{\partial}{\partial v_i} \\ &\quad + \frac{1}{4z} \sum_{i=0}^N \sum_{j=0}^{i-2} [(y^2 - z)\bar{u}_j - yw_j] \frac{\partial}{\partial w_i}. \end{aligned}$$

To prove that F is a constant, it suffices to show that it is invariant under $3N + 3$ linearly independent fields. Since F_{ij} are zero when $\tilde{D}_x F = \tilde{D}_y F = 0$, the function F is invariant under the fields X_1, X_0, Y_1, Y_0 . Then F is also invariant under all nested Lie brackets of arbitrary depth. Thus, to prove that F is a constant, it suffices to construct $3N + 3$ linearly independent nested Lie brackets.

To compute the first sequence Z_n° of nested brackets, we introduce the linear combinations

$$\begin{aligned} Z_1 &= zX_1 - z^2Y_1 = z\frac{\partial}{\partial z} + \frac{1}{4}\sum_{i=1}^N u_{i-1}\frac{\partial}{\partial u_i} + \frac{1}{4}\sum_{i=1}^N (2xu_{i-1} - v_{i-1})\frac{\partial}{\partial v_i} \\ &\quad - \frac{1}{4}\sum_{i=1}^N (2y\bar{u}_{i-1} - w_{i-1})\frac{\partial}{\partial w_i}, \\ Z_0 &= X_0 - zY_0 = \frac{\partial}{\partial x} - z\frac{\partial}{\partial y} - \frac{z}{2}\sum_{i=1}^N (xu_{i-1} - v_{i-1})\frac{\partial}{\partial u_i} \\ &\quad + \frac{1}{2}\sum_{i=1}^N [(1 - x^2z)u_{i-1} + xzv_{i-1}]\frac{\partial}{\partial v_i} + \frac{1}{2}\sum_{i=1}^N [(y^2 - z)\bar{u}_{i-1} - yw_{i-1}]\frac{\partial}{\partial w_i}. \end{aligned}$$

Starting from $Z_0^\circ = Z_0 + 2xzZ_1$, we compute recursively

$$\begin{aligned} Z_n^\circ &= [Z_{n-1}^\circ, Z_1] + Z_{n-1}^\circ \\ &= \frac{\partial}{\partial x} + \frac{z}{2^{n+1}}\sum_{i=n+1}^N (v_{i-n-1} - xu_{i-n-1})\frac{\partial}{\partial u_i} + \frac{xz}{2^{n+1}}\sum_{i=n+1}^N (v_{i-n-1} - xu_{i-n-1})\frac{\partial}{\partial v_i} \\ &\quad - \sum_{i=1}^N \sum_{j=0}^{i-1} \frac{1}{(-2)^{i-j}} \binom{n+1}{i-j} u_j \frac{\partial}{\partial v_i} - \sum_{i=1}^N \sum_{j=0}^{i-1} \frac{1}{(-2)^{i-j}} \binom{n}{i-j-1} (y^2\bar{u}_j - yw_j) \frac{\partial}{\partial w_i} \\ &\quad + \frac{z}{2^{n+1}}\sum_{i=n}^N (2\bar{u}_{i-n} - \bar{u}_{i-n-1})\frac{\partial}{\partial w_i}, \quad n > 0. \end{aligned}$$

Observe that $n + 1$ is the lowest i to occur in $\partial/\partial u_i$. To compute another sequence X_n^+

of nested brackets, we first introduce

$$\begin{aligned}
 Z_n^+ &= \sum_{k=0}^n (-1)^k \binom{n}{k} Z_{n+k}^\circ \\
 &= \sum_{k=0}^n \frac{(-1)^k z}{2^{n+k+1}} \binom{n}{k} \sum_{i=n+k+1}^N (v_{i-n-k-1} - xu_{i-n-k-1}) \frac{\partial}{\partial u_i} \\
 &\quad + \sum_{k=0}^n \frac{(-1)^k xz}{2^{n+k+1}} \binom{n}{k} \sum_{i=n+k+1}^N (v_{i-n-k-1} - xu_{i-n-k-1}) \frac{\partial}{\partial v_i} \\
 &\quad + \sum_{k=0}^n \frac{(-1)^k z}{2^{n+k+1}} \binom{n}{k} \sum_{i=n+k}^N (2\bar{u}_{i-n-k} - \bar{u}_{i-n-k-1}) \frac{\partial}{\partial w_i} \\
 &\quad - \sum_{k=0}^n \frac{(-1)^k}{2^{n+k}} \binom{n+1}{k} \sum_{i=n+k}^N u_{i-n-k} \frac{\partial}{\partial v_i} \\
 &\quad - \sum_{k=0}^n \frac{(-1)^k}{2^{n+k}} \binom{n}{k-1} \sum_{i=n+k}^N (y^2 \bar{u}_{i-n-k} - yw_{i-n-k}) \frac{\partial}{\partial v_i}, \quad n > 0
 \end{aligned}$$

and then

$$\begin{aligned}
 X_n^+ &= [X_0, Z_n^+] = \sum_{k=0}^N \frac{(-1)^{k+1}}{2^{n+k}} B(n-1, k-1) \sum_{i=n+k}^N u_{i-n-k} \frac{\partial}{\partial u_i} \\
 &\quad + \sum_{k=0}^N \frac{(-1)^{k+1}}{2^{n+k-1}} B(n-1, k-1) \sum_{i=n+k}^N (xu_{i-n-k} + v_{i-n-k}) \frac{\partial}{\partial v_i} \\
 &\quad + \sum_{k=0}^N \frac{(-1)^k}{2^{n+k-1}} B(n-1, k-1) \sum_{i=n+k}^N (y\bar{u}_{i-n-k} + w_{i-n-k}) \frac{\partial}{\partial w_i},
 \end{aligned}$$

where

$$B(n, k) = \sum_{i=0}^k (-1)^{i-k} \binom{n-k+i+1}{i} = \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{n-2i}{k-2i}. \quad (33)$$

Recall that binomial coefficients are defined by formula (32). For us, the important fact is that $B(n, k) = 0$ for $k < 0$ and arbitrary n . For $k \leq n$, the values $B(n, k)$ are nonzero since they constitute the last Mendelson triangle [23], see sequence A035317 in the The On-Line Encyclopedia of Integer Sequences [31]. For $k > n + 1$, we have $B(n, k) = (-1)^i 2^{i-n-2}$, nonzero as well. Finally, $B(n, n+1) = 1$ if $n \geq -1$ is odd and $B(n, n+1) = 0$ otherwise.

To finish the proof, we consider the $3N + 3$ rd order determinant Δ composed of the coefficients of the fields

$$Y_1, X_1^+, \dots, X_{N-1}^+, Z_1^\circ, \dots, Z_{2N+1}^\circ, Z_1, Y_0$$

at p with respect to the basis

$$\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_N}, \frac{\partial}{\partial w_1}, \dots, \frac{\partial}{\partial w_N}, -\frac{\partial}{\partial x}, \frac{\partial}{\partial v_1}, \dots, \frac{\partial}{\partial v_N}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}.$$

Since the coefficients are rational functions in the variables x, y, z, u_i, v_i, w_i (polynomial except in z), so is Δ . It follows that if Δ is nonzero at one point $p \in \tilde{\mathcal{E}}^{(N)}$, then it is nonzero in a dense subset of $\tilde{\mathcal{E}}^{(N)}$ (the zeroes of Δ constitute a proper algebraic submanifold in $\tilde{\mathcal{E}}^{(N)}$).

Recall that $u_0 = 1, v_0 = w_0 = 0$. Let p be the point $x = y = u_1 = \dots = u_N = v_1 = \dots = v_N = w_1 = \dots = w_N = 0, z = 1$, then $\bar{u}_i = 1/2^i$ for all $i \geq 0$. It is easy to check that $\Delta(p)$ has enough zero entries to decompose into a product of four simple determinants. As a matter of fact, a thorough inspection reveals that $\Delta(p)$ decomposes into the blocks

$$\Delta(p) = \begin{vmatrix} M_{11} & 0 & 0 & M_{14} \\ 0 & M_{22} & 0 & M_{24} \\ 0 & M_{32} & M_{33} & M_{34} \\ 0 & 0 & 0 & M_{44} \end{vmatrix}$$

with a division after N th, $2N$ th and $3N + 1$ st row and column. Therefore, $\Delta(p) = |M_{11}| \cdot |M_{22}| \cdot |M_{33}| \cdot |M_{44}|$. The simplest 2×2 block M_{44} corresponds to the fields Z_1, Y_0 and the basis vectors $\partial/\partial y, \partial/\partial z$. It follows that M_{44} is a unit matrix, hence $|M_{44}| = 1$. The second simplest $N \times N$ block M_{22} corresponds to the fields $Z_1^\circ, \dots, Z_N^\circ$ and the basis vectors $\partial/\partial w_1, \dots, \partial/\partial w_N$. As such, M_{22} is a diagonal matrix with the entries $2^{-1}, 2^{-2}, \dots, 2^{-N}$ and, therefore, $|M_{22}| = 2^{-(N+1)/2} \neq 0$. Only slightly more involved is M_{11} , which corresponds to the fields $Y_1, X_1^+, \dots, X_{N-1}^+$ and the basis vectors $\partial/\partial u_1, \dots, \partial/\partial u_N$. The first column $-\frac{1}{4}, -\frac{1}{8}, \dots, -\frac{1}{8}$ corresponds to Y_1 , whereas the rest corresponds to X_n^+ at p , which is

$$\sum_{k=0}^N \frac{(-1)^{k+1}}{2^{n+k}} B(n-1, k-1) \sum_{i=n+k}^N u_{i-n-k} \frac{\partial}{\partial u_i}.$$

Here the only nonzero terms are those with $i - n - k = 0$ (since $u_i = 0$ except for $i = 0$). Hence, the element at the crossing of the $n + 1$ st column and i th row is

$$\frac{(-1)^{k+1}}{2^i} B(n-1, i-n-1).$$

By the description of B following equation (33), we see that M_{11} is lower triangular with $-\frac{1}{4}, 2^{-2}, 2^{-3}, \dots, 2^{-N}$ on the diagonal and, again, $|M_{11}| \neq 0$.

Finally, the remaining block M_{33} corresponds to the fields $Z_{N+1}^\circ, \dots, Z_{2N+1}^\circ$ and the basis vectors $-\partial/\partial x, \partial/\partial v_1, \dots, \partial/\partial v_N$. It follows that the ij th element of M_{33} is

$$\frac{(-1)^i}{2^{i-1}} \binom{N+j+1}{i-1}$$

and, therefore,

$$|M_{33}| = \prod_{i=1}^{N+1} (-1)^i / 2^{i-1} \neq 0.$$

Summing up, the fields $Y_1, X_1^+, \dots, X_{N-1}^+, Z_1^\circ, \dots, Z_{2N+1}^\circ, Z_1, Y_0$ are linearly independent in a dense subset of the $3N + 3$ -manifold $\tilde{\mathcal{E}}^{(N)}$ and, therefore, $F = \text{const}$, which finishes the proof. \square

Corollary 1. *There is no possible functional dependence among the potentials u_i, v_i, w_i .*

Proof. A relation $F(u_1, \dots, u_N, v_1, \dots, v_N, w_1, \dots, w_N) = 0$ would imply that $\tilde{D}_x^{(3)} F = \tilde{D}_y^{(3)} F = 0$. \square

Cf. [17, Corollary 1].

7. Conclusions

We found four infinite series of linearly independent nonlocal conservation laws of the constant astigmatism equation. The hierarchies are closed with respect to duality \mathcal{I} and reciprocal transformations \mathcal{X}, \mathcal{Y} . The corresponding potentials $u_i, v_i, \bar{u}_i, \bar{v}_i$ exhibit functional dependence of Wronskian type.

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