# ON THE NUMBER OF COMMUTATION CLASSES OF THE LONGEST ELEMENT IN THE SYMMETRIC GROUP 

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#### Abstract

Using the standard Coxeter presentation for the symmetric group $S_{n}$, two reduced expressions for the same group element are said to be commutation equivalent if we can obtain one expression from the other by applying a finite sequence of commutations. The resulting equivalence classes of reduced expressions are called commutation classes. How many commutation classes are there for the longest element in $S_{n}$ ?


Original proposer of the open problem: Donald E. Knuth
The year when the open problem was proposed: 1992 [11, §9]

A Coxeter system is a pair $(W, S)$ consisting of a distinguished (finite) set $S$ of generating involutions and a group

$$
\left.W=\langle S|(s t)^{m(s, t)}=e \text { for } m(s, t)<\infty\right\rangle
$$

called a Coxeter group, where $e$ is the identity, $m(s, t)=1$ if and only if $s=t$, and $m(s, t)=$ $m(t, s)$. It turns out that the elements of $S$ are distinct as group elements and that $m(s, t)$ is the order of $s t$. Since the elements of $S$ have order two, the relation $(s t)^{m(s, t)}=e$ can be written to allow the replacement

$$
\underbrace{s t s \cdots}_{m(s, t)} \mapsto \underbrace{t s t \cdots}_{m(s, t)}
$$

which is called a commutation if $m(s, t)=2$ and a braid move if $m(s, t) \geq 3$.
Given a Coxeter system $(W, S)$, a word $\mathbf{w}=s_{x_{1}} s_{x_{2}} \cdots s_{x_{m}}$ in the free monoid $S^{*}$ is called an expression for $w \in W$ if it is equal to $w$ when considered as a group element. If $m$ is minimal among all expressions for $w$, the corresponding word is called a reduced expression for $w$. In this case, we define the length of $w$ to be $\ell(w)=m$. According to [8], every finite Coxeter group contains a unique element of maximal length, which we refer to as the longest element and denote by $w_{0}$.

Let $(W, S)$ be a Coxeter system and let $w \in W$. Then $w$ may have several different reduced expressions that represent it. However, Matsumoto's Theorem [7, Theorem 1.2.2]

[^0]| 312312 |  |  | 231231 |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 132312 | 321232 | 232123 | 123121 | 213231 | 123212 | 212321 |  |
| 312132 |  |  |  | 21321 | 231213 |  |  |
| 132132 |  |  | 213213 |  |  |  |  |

Figure 1. Reduced expressions and the corresponding commutation classes for the longest element in $S_{4}$.
states that every reduced expression for $w$ can be obtained from any other by applying a finite sequence of commutations and braid moves.

Following [13, we define a relation $\sim$ on the set of reduced expressions for $w$. Let w and $\mathrm{w}^{\prime}$ be two reduced expressions for $w$ and define $\mathrm{w} \sim \mathrm{w}^{\prime}$ if we can obtain $\mathrm{w}^{\prime}$ from w by applying a single commutation. Now, define the equivalence relation $\approx$ by taking the reflexive transitive closure of $\sim$. Each equivalence class under $\approx$ is called a commutation class.

The Coxeter system of type $A_{n-1}$ is generated by $S\left(A_{n-1}\right)=\left\{s_{1}, s_{2}, \ldots, s_{n-1}\right\}$ and has defining relations (i) $s_{i} s_{i}=e$ for all $i$; (ii) $s_{i} s_{j}=s_{j} s_{i}$ when $|i-j|>1$; and (iii) $s_{i} s_{j} s_{i}=s_{j} s_{i} s_{j}$ when $|i-j|=1$. The corresponding Coxeter group $W\left(A_{n-1}\right)$ is isomorphic to the symmetric group $S_{n}$ under the correspondence $s_{i} \mapsto(i, i+1)$. It is well known that the longest element in $S_{n}$ is given in 1-line notation by

$$
w_{0}=[n, n-1, \ldots, 2,1]
$$

and that $\ell\left(w_{0}\right)=\binom{n}{2}$.
Let $c_{n}$ denote the number of commutation classes of the longest element in $S_{n}$. The longest element $w_{0}$ in $S_{4}$ has length 6 and is given by the permutation $(1,4)(2,3)$. There are 16 distinct reduced expressions for $w_{0}$ while $c_{4}=8$. The 8 commutation classes for $w_{0}$ are given in Figure 1, where we have listed the reduced expressions that each class contains. Note that for brevity, we have written $i$ in place of $s_{i}$.

In [12], Stanley provides a formula for the number of reduced expressions of the longest element $w_{0}$ in $S_{n}$. However, the following question is currently unanswered.
Open Problem. What is the number of commutation classes of the longest element in $S_{n}$ ?
To our knowledge, this problem was first introduced in 1992 by Knuth in Section 9 of [11], but not using our current terminology. A more general version of the problem appears in Section 5.2 of [9]. In the paragraph following the proof of Proposition 4.4 of [14], Tenner explicitly states the open problem in terms of commutation classes.

According to sequence A006245 of The On-Line Encyclopedia of Integer Sequences [1], the first 10 values for $c_{n}$ are $1,1,2,8,62,908,24698,1232944,112018190,18410581880$. To


Figure 2. Heaps for the longest element in $S_{4}$.


Figure 3. Minimal ladder lotteries corresponding to the primitive sorting networks on 4 elements.
date, only the first 15 terms are known. The current best upper-bound for $c_{n}$ was obtained by Felsner and Valtr. They prove that for sufficiently large $n, c_{n} \leq 2^{0.6571 n^{2}}$ [5, Theorem 2], although their result is stated in terms of arrangements of pseudolines.

The commutation classes of the longest element of the symmetric group are in bijection with a number of interesting objects. It turns out that $c_{n}$ is equal to the number of

- heaps for the longest element in $S_{n}$ [13, Proposition 2.2];
- primitive sorting networks on $n$ elements [2, 10, 11, 15, 16];
- rhombic tilings of a regular $2 n$-gon (where all side lengths of the rhombi and the $2 n$-gon are the same) [3, 14];
- oriented matroids of rank 3 on $n$ elements [6, 9];
- arrangements of $n$ pseudolines [4, 5, 11].

In Figure 2, we have drawn lattice point representations of the 8 heaps that correspond to the commutation classes for the longest element in $S_{4}$. Note that our heaps are sideways versions of the heaps that usually appear in the literature. The minimum ladder lotteries (or ghost legs) corresponding to the 8 primitive sorting networks on 4 elements are provided in Figure 3, The 8 distinct rhombic tilings of a regular octagon are depicted in Figure 4 ,


Figure 4. Rhombic tilings of a regular octagon.
Very little is known about the number of commutation classes of the longest element in other finite Coxeter groups.

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