# A bijection for two sequences in OEIS

David Callan

September 8, 2018

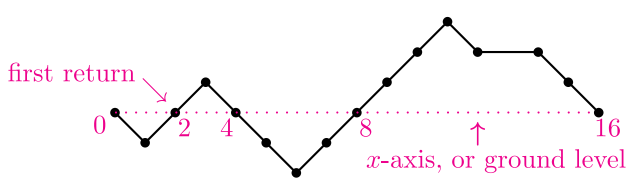

**Abstract**

We show that sequences A026737 and A111279 in The On-Line Encyclopedia of Integer Sequences are the same by giving a bijection between two classes of Grand Schröder paths.


## 1 Introduction

In a comment on sequence A026737 in OEIS [1], Andrew Plewe asks if it is the same as A111279. The answer is yes. As we will see, each of the sequences counts a class of *Grand Schröder paths*, that is, lattice paths of upsteps $U = (1, 1)$, flatsteps $F = (2, 0)$, and downsteps $D = (1, -1)$ starting at the origin (0,0) and ending on the $x$-axis, with size measured by # upsteps + # flatsteps. Thus a Grand Schröder path of size $n$ ends at $(2n, 0)$.

The $(n + 1)$-th term of A026737 counts Grand Schröder paths of size $n$ *all of whose flatsteps* (*if any*) *lie on the horizontal line* $y = 2$. This assertion follows immediately from the defining recurrence for the sequence. Let $\mathcal{A}_n$ denote this set of paths.

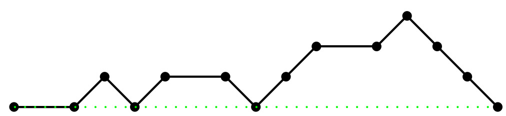



A path in $\mathcal{A}_8$

Figure 1

It is not too hard to find the generating function for $\mathcal{A}_n$ using the "first return" decomposition and a little recursion involving both nonnegative paths and the analogous generating functions when "$y = 2$" in the defining condition is replaced by "$y = 1$" and by "$y = 0$", and then observe that the generating function for $\mathcal{A}_n$ coincides with that for A111279. This answers Plewe's question in the affirmative. But it's nicer to give a bijective proof.

By definition, the $(n+1)$-th term of A111279 counts permutations of $[n+1]$ avoiding the three patterns {3241, 3421, 4321}. These permutations are in bijection [2] with the set $\mathcal{B}_n$ of Schröder paths of size $n$ *with at most one peak per component.* Recall that a *Schröder path* is a nonnegative Grand Schröder path, that is, one that never dips below the $x$-axis, and the interior vertices on the $x$-axis split a nonempty path that ends on the $x$-axis into its components.



A path in $\mathcal{B}_8$ with 4 components

Figure 2

## 2 The bijection

We give a simple bijection of the cut-and-paste type from $\mathcal{A}_n$ to $\mathcal{B}_n$ that preserves components. Thus it suffices to define our mapping on indecomposable (1-component) paths in $\mathcal{A}_n$. We refer to the horizontal line joining the terminal points of a path in $\mathcal{A}_n$ or $\mathcal{B}_n$ as *ground level*, GL for short, to eliminate the need for coordinate axes. Since a path in $\mathcal{A}_n$ contains no flatsteps at ground level, an indecomposable path in $\mathcal{A}_n$ cannot consist of a single flatstep and so lies entirely above or entirely below ground level.

If entirely below (and hence contains no flatsteps at all), flip it over the $x$-axis and replace all $UD$s (peaks) by $F$s to get an indecomposable Schröder $n$-path with *no* peak (Figure 3).

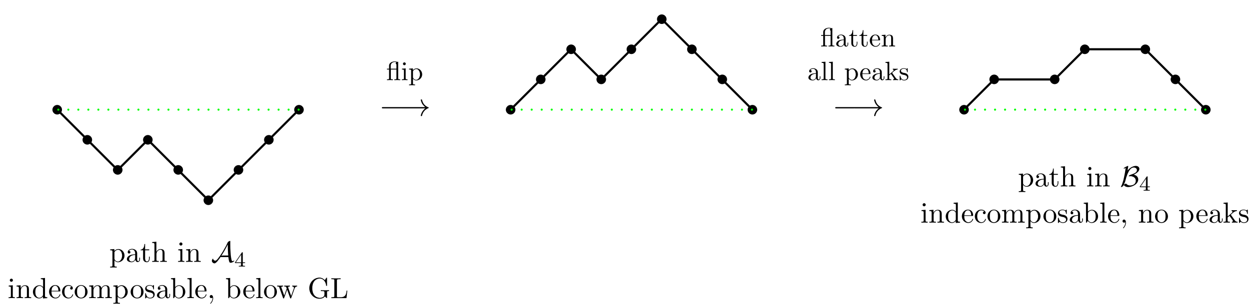



Figure 3

If entirely above, follow the sequence of operations illustrated in Figure 4 below to get an indecomposable Schröder $n$-path with *exactly one* peak.

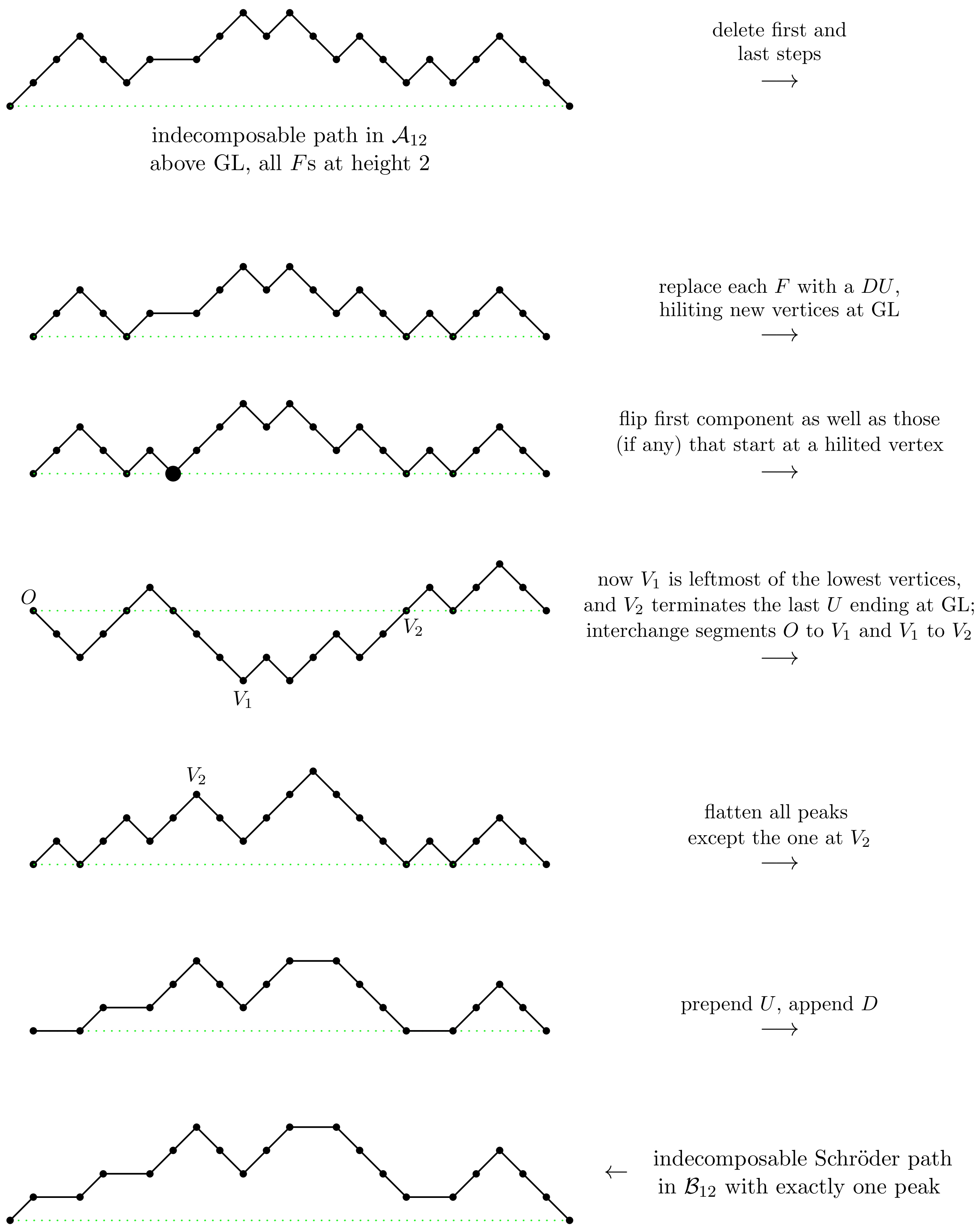

delete first and last steps ⟶
indecomposable path in $\mathcal{A}_{12}$ above GL, all $F$s at height 2
replace each $F$ with a $DU$, hiliting new vertices at GL ⟶
flip first component as well as those (if any) that start at a hilited vertex ⟶
$O$
$V_1$
$V_2$
now $V_1$ is leftmost of the lowest vertices, and $V_2$ terminates the last $U$ ending at GL; interchange segments $O$ to $V_1$ and $V_1$ to $V_2$ ⟶
$V_2$
flatten all peaks except the one at $V_2$ ⟶
prepend $U$, append $D$ ⟶
← indecomposable Schröder path in $\mathcal{B}_{12}$ with exactly one peak

Figure 4

All the steps are reversible, and we have the desired bijection.

Exercise. Check that the $\mathcal{A}_8$-path in Figure 1 corresponds to the $\mathcal{B}_8$-path in Figure 2 under this bijection.

# References


[1] The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2016.

[2] David Callan and Toufik Mansour, Five subsets of permutations enumerated as weak sorting permutations, submitted, http://front.math.ucdavis.edu/1602.05182



Department of Statistics, University of Wisconsin-Madison
1300 University Ave, Madison, WI 53706-1532
**callan@stat.wisc.edu**