

SIMSUN PERMUTATIONS, SIMSUN SUCCESSIONS AND SIMSUN PATTERNS

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ABSTRACT. In this paper, we introduce the definitions of simsun succession, simsun cycle succession and simsun pattern. In particular, the ordinary simsun permutations are permutations avoiding simsun pattern 321. We study the descent and peak statistics on permutations avoiding simsun successions. We give a combinatorial interpretation of the q -Eulerian polynomials introduced by Brenti (J. Combin. Theory Ser. A 91 (2000), 137–170). We also present a bijection between permutations avoiding simsun pattern 132 and set partitions.

Keywords: Simsun patterns; Simsun successions; Simsun cycle successions; Descents

1. INTRODUCTION

Let \mathfrak{S}_n denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \dots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *descent* in π is an index i such that $\pi(i) > \pi(i+1)$. We say that π has no *double descents* if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$. Simsun permutation is defined by Sundaram and Simion [30]. A permutation $\pi \in \mathfrak{S}_n$ is called *simsun* if for all k , the subword of π restricted to $[k]$ (in the order they appear in π) contains no double descents. For example, 35142 is simsun, but 35241 is not. Let \mathcal{RS}_n be the set of simsun permutations of length n . Simion and Sundaram [30, p. 267] discovered that $\#\mathcal{RS}_n = E_{n+1}$, where E_n is the n th Euler number, which also is the number alternating permutations in \mathfrak{S}_n (see [27] for instance). Simsun permutation is a variant of André permutation, which was introduced by Foata and Schützenberge [10]. There has been much work related to simsun permutations (see [2, 5, 6, 8, 9, 11, 12, 18] for instance).

Let $\text{des}(\pi)$ be the number of descents of π . Let

$$S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\text{des}(\pi)} = \sum_{k=0}^{\lfloor n/2 \rfloor} S(n, k)x^k.$$

Following [30, p. 267], the numbers $S(n, k)$ satisfy the recurrence relation

$$S(n, k) = (k+1)S(n-1, k) + (n-2k+1)S(n-1, k-1),$$

with the initial conditions $S(0, 0) = 1$ and $S(0, k) = 0$ for $k \geq 1$, which is equivalent to

$$S_{n+1}(x) = (1+nx)S_n(x) + x(1-2x)S'_n(x),$$

with $S_0(x) = 1$. Let $D_n(x) = \sum_{k \geq 1} d(n, k)x^k$, where $d(n, k)$ denote the number of augmented André permutations of order n with $k-1$ left peaks (see [11]). It follows from [5, Proposition 4]

that $D_{n+1}(x) = xS_n(x)$ for $n \geq 0$. For each $\pi \in \mathfrak{S}_n$, we say that π has an *excedance* at i if $\pi(i) > i$. Let $\text{exc}(\pi)$ be the excedance number of π . The *Eulerian polynomials* are defined by

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)+1} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)+1} \quad \text{for } n \geq 1.$$

From [10, Proposition 2.7], we have

$$A_{n+1}(x) = x \sum_{k=0}^{\lfloor n/2 \rfloor} S(n, k)(2x)^k(1+x)^{n-2k}.$$

A *succession* in $\pi \in \mathfrak{S}_n$ is an index $i \in [n-1]$ such that $\pi(i+1) = \pi(i) + 1$. The study of successions began in 1940s (see [13, 22]), and there has been much recent activity. There are various variants of succession, including circular succession [31] and cycle succession [20]. The succession statistic has also been studied on various combinatorial structures, such as set partitions [19, 21], compositions and words [14]. For example, a succession in a partition of $[n]$ is an occurrence of two consecutive integers appear in the same block. Following [28, p. 137, Exercise 108], the number of partitions of $[n]$ with no successions is $B(n-1)$, where $B(n)$ is the n th *Bell number*, which also is the number of partitions of $[n]$.

This paper is organized as follows. In Section 2, we consider a subset of \mathcal{RS}_n with no successions. In Section 3, we introduce the definition of simsun succession. In Section 4, we give a combinatorial interpretation of the q -Eulerian polynomials introduced by Brenti [3]. In Section 5, we present a bijection between permutations avoiding simsun pattern 132 and set partitions.

2. A SUBSET OF \mathcal{RS}_n WITH NO SUCCESSIONS

Definition 1. Let \mathcal{BS}_n denote the set of permutations in \mathcal{RS}_n such that for all k , the subword of π restricted to $[k]$ (in the order they appear in π) does not contain successions.

For example, $\mathcal{BS}_5 = \{25143, 21435, 24135, 24153, 52413\}$. Now we present the first main result of this paper.

Theorem 2. For $n \geq 0$ and $0 \leq k \leq \lfloor n/2 \rfloor$, we have

$$S(n, k) = \#\{\pi \in \mathcal{BS}_{n+2} : \text{des}(\pi) = k + 1\}. \quad (1)$$

Equivalently, we have

$$S_n(x) = \sum_{\pi \in \mathcal{BS}_{n+2}} x^{\text{des}(\pi)-1}.$$

Proof. Let $r(n, k) = \#\{\pi \in \mathcal{BS}_{n+2} : \text{des}(\pi) = k+1\}$. There are two ways in which a permutation $\pi \in \mathcal{BS}_{n+2}$ with $k+1$ descents can be obtained from a permutation $\sigma \in \mathcal{BS}_{n+1}$:

- (a) If $\text{des}(\sigma) = k+1$, then we distinguish two cases:
 - (c₁) If $\sigma(n+1) = n+1$, then we can insert $n+2$ right after $\sigma(i)$, where i is a descent index;

(c₂) If $\sigma(n+1) < n+1$, then there exists an index $j \in [n]$ such that $\sigma(j) = n+1$. We can insert $n+2$ right after $\sigma(i)$, where i is a descent index and $i \neq j$. Moreover, we can also put $n+2$ at the end of σ .

In either case, we have $k+1$ choices for the positions of $n+2$. As we have $r(n-1, k)$ choices for σ . This gives $(k+1)r(n-1, k)$ possibilities.

(b) If $\text{des}(\sigma) = k$, then we can not insert $n+2$ immediately before or right after a decent index. Moreover, we can not put $n+2$ at the end of σ . Therefore, the entry $n+2$ can be inserted into any of the remaining $n-2k+1$ positions. As we have $r(n-1, k-1)$ choices for σ . This gives $(n-2k+1)r(n-1, k-1)$ possibilities.

Therefore, $r(n, k) = (k+1)r(n-1, k) + (n-2k+1)r(n-1, k-1)$. Note that $\mathcal{BS}_2 = \{21\}$. Thus $r(0, 0) = 1$ and $r(0, k) = 0$ for $k \geq 1$. Hence the numbers $r(n, k)$ satisfy the same recurrence and initial conditions as $S(n, k)$, so they agree. \square

Corollary 3. For $n \geq 1$, we have $\#\mathcal{BS}_n = E_{n-1}$.

3. THE DESCENT AND PEAK STATISTICS AND SIMSUN SUCCESSIONS

The number of peaks of permutations is certainly among the most important combinatorial statistics. See, e.g., [7, 15, 16, 24] and the references therein. We now recall some basic definitions. An *interior peak* in π is an index $i \in \{2, 3, \dots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. A *left peak* in $\pi \in \mathfrak{S}_n$ is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\text{pk}(\pi)$ (resp. $\text{lpk}(\pi)$) be the number of interior peaks (resp. left peaks) of π . Similarly, a *valley* in π is an index $i \in \{2, 3, \dots, n-1\}$ such that $\pi(i-1) > \pi(i) < \pi(i+1)$. Let $\text{val}(\pi)$ be the number of valleys of π . Clearly, interior peak and valley are equidistributed over \mathfrak{S}_n . Motivated by the study of longest increasing subsequences, Stanley [26] initiated a study of the longest alternating subsequences. An *alternating subsequence* of $\pi \in \mathfrak{S}_n$ is a subsequence $\pi(i_1), \pi(i_2), \dots, \pi(i_k)$ satisfying $\pi(i_1) > \pi(i_2) < \pi(i_3) > \dots > \pi(i_k)$, where $i_1 < i_2 < \dots < i_k$. Let $\text{as}(\pi)$ be the length (number of terms) of the longest alternating subsequence of a permutation $\pi \in \mathfrak{S}_n$.

Define

$$\begin{aligned} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle &= \#\{\pi \in \mathfrak{S}_n : \text{des}(\pi) = k-1\}; \\ P(n, k) &= \#\{\pi \in \mathfrak{S}_n : \text{pk}(\pi) = k\}; \\ \widehat{P}(n, k) &= \#\{\pi \in \mathfrak{S}_n : \text{lpk}(\pi) = k\}; \\ T(n, k) &= \#\{\pi \in \mathfrak{S}_n : \text{as}(\pi) = k\}. \end{aligned}$$

It is well known that these numbers satisfy the following recurrences (see [16, 26] for instance):

$$\begin{aligned} \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle &= k \left\langle \begin{matrix} n-1 \\ k \end{matrix} \right\rangle + (n-k+1) \left\langle \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rangle; \\ P(n, k) &= (2k+2)P(n-1, k) + (n-2k)P(n-1, k-1); \\ \widehat{P}(n, k) &= (2k+1)\widehat{P}(n-1, k) + (n-2k+1)\widehat{P}(n-1, k-1); \\ T(n, k) &= kT(n-1, k) + T(n-1, k-1) + (n-k+1)T(n-1, k-2). \end{aligned}$$

Definition 4. We say that π avoids simsun succession if for all k , the subword of π restricted to $[k]$ (in the order they appear in π) does not contain successions.

For example, the permutation $\pi = 321465$ contains a simsun succession, since π restricted to $[5]$ equals 32145 and it contains a succession. Let \mathcal{AS}_n denote the set of permutations in \mathfrak{S}_n that avoid simsun successions. In particular, $\mathcal{AS}_1 = \{1\}$, $\mathcal{AS}_2 = \{21\}$ and $\mathcal{AS}_3 = \{213, 321\}$.

Now we present the second main result of this paper.

Theorem 5. For $n \geq 2$, we have

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1} &= \sum_{\pi \in \mathcal{AS}_{n+1}} x^{\text{des}(\pi)}; \\ \sum_{\pi \in \mathfrak{S}_n} x^{\text{pk}(\pi)+1} &= \sum_{\pi \in \mathcal{AS}_{n+1}} x^{\text{lpk}(\pi)}; \\ \sum_{\pi \in \mathfrak{S}_n} x^{\text{lpk}(\pi)} &= \sum_{\pi \in \mathcal{AS}_{n+1}} x^{\text{val}(\pi)}; \\ \sum_{\pi \in \mathfrak{S}_n} x^{\text{as}(\pi)+1} &= \sum_{\pi \in \mathcal{AS}_{n+1}} x^{\text{as}(\pi)}. \end{aligned} \tag{2}$$

Proof. We only prove (2) and the others can be proved in a similar way. Let $a(n, k) = \#\{\pi \in \mathcal{AS}_{n+1} : \text{des}(\pi) = k\}$. There are two ways in which a permutation $\pi \in \mathcal{AS}_{n+2}$ with k descents can be obtained from a permutation $\sigma \in \mathcal{AS}_{n+1}$.

(a) If $\text{des}(\sigma) = k$, then we distinguish two cases:

- (c₁) If $\sigma(n+1) = n+1$, then we can insert $n+2$ right after $\sigma(i)$, where i is a descent index;
- (c₂) If $\sigma(n+1) < n+1$, then there exists an index $j \in [n]$ such that $\sigma(j) = n+1$. We can insert $n+2$ right after $\sigma(i)$, where i is a descent index and $i \neq j$. Moreover, we can also insert $n+2$ at the end of σ .

In either case, we have k choices for the positions of $n+2$. As we have $a(n, k)$ choices for σ . This gives $ka(n, k)$ possibilities.

(b) If $\text{des}(\sigma) = k-1$, we can not insert $n+2$ right after any decent index and we can not put $n+2$ at the end of σ . Hence the entry $n+2$ can be inserted into any of the remaining $n-k+2$ positions. As we have $a(n, k-1)$ choices for σ . This gives $(n-k+2)a(n, k-1)$ possibilities.

Therefore, $a(n+1, k) = ka(n, k) + (n-k+2)a(n, k-1)$. Note that $a(1, 1) = 1$ and $a(1, k) = 0$ for $k \geq 2$. Hence the numbers $a(n, k)$ satisfy the same recurrence and initial conditions as $\langle \binom{n}{k} \rangle$, so they agree. \square

4. q -EULERIAN POLYNOMIALS AND SIMSUN CYCLE SUCCESSIONS

Recall that $\pi \in \mathfrak{S}_n$ can be written in *standard cycle form*, where each cycle is written with its smallest entry first and the cycles are written in increasing order of their smallest entry. In this section, we always write π in standard cycle form. A *cycle succession* in π is an occurrence of two consecutive entries $i, i+1$ in that order within the same cycle for some $i \in [n-1]$ (see [20]). For example, the permutation $(1, 2, 6)(3, 5, 4)$ contains a cycle succession.

Definition 6. We say that π avoids simsun cycle succession if for all k , the subword of π restricted to $[k]$ (in the order they appear in π) does not contain cycle successions.

For example, $\pi = (1543)(2)$ avoids simsun cycle succession. Let \mathcal{CS}_n be the set of permutations in \mathfrak{S}_n that avoid simsun cycle successions. In particular, $\mathcal{CS}_1 = \{(1)\}$, $\mathcal{CS}_2 = \{(1)(2)\}$ and $\mathcal{CS}_3 = \{(1)(2)(3), (13)(2)\}$.

In [3], Brenti considered a q -analog of the classical Eulerian polynomials defined by

$$A_0(x; q) = 1, \quad A_n(x; q) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)} \quad \text{for } n \geq 1,$$

where $\text{cyc}(\pi)$ is the number of cycles in π . The first few of the q -Eulerian polynomials are

$$A_0(x; q) = 1, A_1(x; q) = q, A_2(x; q) = q(x + q), A_3(x; q) = q[x^2 + (3q + 1)x + q^2].$$

Clearly, $A_n(x) = xA_n(x; 1)$ for $n \geq 1$. The real-rootedness of $A_n(x; q)$ has been studied in [1, 17].

The third main result of this paper is the following.

Theorem 7. For $n \geq 1$, we have

$$\sum_{\pi \in \mathfrak{S}_n} x^{\text{exc}(\pi)} q^{\text{cyc}(\pi)+1} = \sum_{\sigma \in \mathcal{CS}_{n+1}} x^{\text{exc}(\sigma)} q^{\text{cyc}(\sigma)}. \quad (3)$$

Let $\mathcal{S}_{n,k,\ell} = \{\pi \in \mathfrak{S}_n : \text{exc}(\pi) = k, \text{cyc}(\pi) = \ell\}$ and $\mathcal{CS}_{n,k,\ell} = \{\pi \in \mathcal{CS}_n : \text{exc}(\pi) = k, \text{cyc}(\pi) = \ell\}$. In the rest of this section, we give a constructive proof of (3). We now introduce two definitions of labeled permutations.

Definition 8. Suppose $\pi \in \mathcal{S}_{n,k,\ell}$ and $i_1 < i_2 < \dots < i_k$ are the excedances of π . Then we put superscript label u_r right after i_r , where $1 \leq r \leq k$. In the remaining positions except the first position of each cycle, we put superscript labels v_1, v_2, \dots, v_{n-k} from left to right.

Definition 9. Suppose $\pi \in \mathcal{CS}_{n,k,\ell}$ and $i_1 < i_2 < \dots < i_k$ are the excedances of π . Then we put superscript label p_r right after i_r , where $1 \leq r \leq k$. In the remaining positions except the first position of each cycle and the position right after the entry n , we put superscript labels $q_1, q_2, \dots, q_{n-k-1}$ from left to right.

In the following discussion, we always add labels to permutations in $\mathcal{S}_{n,k,\ell}$ and $\mathcal{CS}_{n,k,\ell}$. As an example, for $\pi = (135)(26)(4)$, if we say that $\pi \in \mathcal{S}_{6,3,3}$, then the labels of π is given by $(1^{u_1} 3^{u_2} 5^{v_1})(2^{u_3} 6^{v_2})(4^{v_3})$; if we say that $\pi \in \mathcal{CS}_{6,3,3}$, then the labels of π is given by $(1^{p_1} 3^{p_2} 5^{q_1})(2^{p_3} 6)(4^{q_2})$.

Now we start to construct a bijection, denoted by Φ , between $\mathcal{S}_{n,k,\ell}$ and $\mathcal{CS}_{n+1,k,\ell+1}$. When $n = 1$, we have $\mathcal{S}_{1,0,1} = \{(1)\}$ and $\mathcal{CS}_{2,0,2} = \{(1)(2)\}$. Set $\Phi((1)) = (1)(2)$. This gives a bijection between $\mathcal{S}_{1,0,1}$ and $\mathcal{CS}_{2,0,2}$. Let $n = m$. Suppose Φ is a bijection between $\mathcal{S}_{m,k,\ell}$ and $\mathcal{CS}_{m+1,k,\ell+1}$ for all k and ℓ . Consider the case $n = m + 1$. Given $\pi \in \mathcal{S}_{m,k,\ell}$. Assume $\Phi(\pi) = \sigma$. Then $\sigma \in \mathcal{CS}_{m+1,k,\ell+1}$. Consider the following three cases:

- (i) If $\widehat{\pi}$ is obtained from π by inserting the entry $m + 1$ to the position of π with label u_r , then we insert $m + 2$ to the position of σ with label p_r . In this case, $\text{exc}(\widehat{\pi}) = \text{exc}(\Phi(\widehat{\pi})) = k$ and $\text{cyc}(\widehat{\pi}) + 1 = \text{cyc}(\Phi(\widehat{\pi})) = \ell + 1$.

- (ii) If $\hat{\pi}$ is obtained from π by inserting the entry $m+1$ to the position of π with label v_j , then we insert $m+2$ to the position of σ with label q_j . In this case, $\text{exc}(\hat{\pi}) = \text{exc}(\Phi(\hat{\pi})) = k+1$ and $\text{cyc}(\hat{\pi}) + 1 = \text{cyc}(\Phi(\hat{\pi})) = \ell + 1$.
- (iii) If $\hat{\pi}$ is obtained from π by appending $(m+1)$ to π as a new cycle, then we append $(m+2)$ to σ as a new cycle. In this case, $\text{exc}(\hat{\pi}) = \text{exc}(\Phi(\hat{\pi})) = k$ and $\text{cyc}(\hat{\pi}) + 1 = \text{cyc}(\Phi(\hat{\pi})) = \ell + 2$.

By induction, we see that Φ is the desired bijection between $\mathcal{S}_{n,k,\ell}$ and $\mathcal{CS}_{n+1,k,\ell+1}$ for all k and ℓ , which also gives a constructive proof of (3).

Example 10. Given $\pi = (135)(2)(4) \in \mathcal{S}_{5,2,3}$. The correspondence between π and $\Phi(\pi)$ is built up as follows:

$$\begin{aligned} (1^{v_1}) &\Leftrightarrow (1^{q_1})(2); \\ (1^{v_1})(2^{v_2}) &\Leftrightarrow (1^{q_1})(2^{q_2})(3); \\ (1^{u_1}3^{v_1})(2^{v_2}) &\Leftrightarrow (1^{p_1}4)(2^{q_1})(3^{q_2}); \\ (1^{u_1}3^{v_1})(2^{v_2})(4^{v_3}) &\Leftrightarrow (1^{p_1}4^{q_1})(2^{q_2})(3^{q_3})(5); \\ (1^{u_1}3^{u_2}5^{v_1})(2^{v_2})(4^{v_3}) &\Leftrightarrow (1^{p_1}4^{p_2}6)(2^{q_1})(3^{q_2})(5^{q_3}). \end{aligned}$$

5. PERMUTATIONS AVOIDING THE SIMSUN PATTERN 132 AND SET PARTITIONS

In this section, containment and avoidance will always refer to consecutive patterns. Let m, n be two positive integers with $m \leq n$, and let $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$. We say that π contains τ as a *consecutive pattern* if it has a subsequence of adjacent entries order-isomorphic to τ . A permutation π avoids a pattern τ if π does not contain τ .

Definition 11. Let $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$. We say that π avoids simsun pattern τ if for all k , the subword of π restricted to $[k]$ (in the order they appear in π) does not contain the consecutive pattern τ .

Let $\mathcal{SP}_n(\tau)$ denote the set of permutations in \mathfrak{S}_n that avoid simsun pattern τ . In particular, $\mathcal{SP}_n(321) = \mathcal{RS}_n$. Using the reverse map, we get $\#\mathcal{SP}_n(321) = \#\mathcal{SP}_n(123) = E_{n+1}$. In the following, we shall establish a connection between $\mathcal{SP}_n(132)$ and set partitions of $[n]$.

A *partition* p of $[n]$ is a collection of nonempty disjoint subsets, called blocks, whose union is $[n]$. As usual, we always write $p \vdash [n]$ and put $p = B_1/B_2/\cdots/B_k$ in standard form with $\min B_1 < \min B_2 < \cdots < \min B_k$. It is well known that the number of partitions of $[n]$ with exactly k blocks is the *Stirling number of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. Let $\prod_n = \{p : p \vdash [n]\}$ and let $\text{block}(p)$ be the number of blocks of p . Let $p = B_1/B_2/\cdots/B_k$ be a partition of $[n]$. For $c \in B_s$ and $d \in B_t$, we say that the pair (c, d) is a *free rise* of p if $c < d$, where $1 \leq s < t \leq k$. Let $\text{fr}(p)$ be the number of free rises of p . For example, $\text{fr}(\{1, 2, 3\}/\{4\}/\{5\}) = 7$. A *singleton* of a partition is a block with exactly one element (see [29] for instance). Let $\text{singleton}(p)$ be the number of singletons of p . We say that a block of p is *non-singleton* if it contains at least two elements. Let $\text{nsingleton}(p)$ be the number of non-singletons of p . For example, $\text{singleton}(\{1, 2, 3\}/\{4\}/\{5\}) = 2$ and $\text{nsingleton}(\{1, 2, 3\}/\{4\}/\{5\}) = 1$.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *right peak* of π is an entry $\pi(i)$ with $i \in \{2, 3, \dots, n\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(n+1) = 0$. Let $\text{rpk}(\pi)$ be the number of right peaks of π . An *inversion* of π is a pair $(\pi(i), \pi(j))$ such that $i < j$ and $\pi(i) > \pi(j)$. Let $\text{inv}(\pi)$ be the number of inversions of π . We say that the entry $\pi(i)$ is an *exterior double descent* of π if $\pi(i-1) > \pi(i) > \pi(i+1)$, where $i \in [n]$ and we take $\pi(0) = +\infty$ and $\pi(n+1) = 0$. Let $\text{exddes}(\pi)$ be the number of exterior double descents of π . For example, $\text{rpk}(42315) = 2$, $\text{inv}(42315) = 5$ and $\text{exddes}(42315) = 1$.

Now we present the fourth main result of this paper.

Theorem 12. *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathcal{SP}_n(132)} x^{\text{des}(\pi)+1} y^{\text{rpk}(\pi)} z^{\text{exddes}(\pi)} q^{\text{inv}(\pi)} = \sum_{p \in \Pi_n} x^{\text{block}(p)} y^{\text{nsingleton}(p)} z^{\text{singleton}(p)} q^{\text{fr}(p)}. \quad (4)$$

In the following, we shall present a constructive proof of (4).

Recently, Chen et al. [4] present a grammatical labeling of partitions of $[n]$: For $p \in \Pi_n$, we label a block of p by letter b and label the partition itself by letter a , and the weight of a partition is defined to be the product of its labels. Hence $w(p) = ab^k$ if $\text{block}(p) = k$. They deduced that $\sum_{p \in \Pi_n} w(p) = a \sum_{k=0}^n \binom{n}{k} b^k$. As a variant of the grammatical labeling, we introduce the following definition.

Definition 13. *Suppose $p = B_1/B_2/\cdots/B_k$ is a partition of $[n]$. Then we label B_i by letter b_{k+1-i} , where $1 \leq i \leq k$. Moreover, we put label a at the end of p .*

Note that in order to get a permutation $\hat{\pi} \in \mathcal{SP}_{n+1}(132)$ from $\pi \in \mathcal{SP}_n(132)$, we can't insert the entry $n+1$ right after any ascent entry of π , where an ascent entry is a value $\pi(i)$ such that $\pi(i) < \pi(i+1)$. We now introduce a definition of labeled permutations.

Definition 14. *Suppose $\pi \in \mathcal{SP}_n(132)$ with $k-1$ descents, where $1 \leq k \leq n$. Let $i_1 < i_2 < \cdots < i_{k-1}$ be the descent indices of π . We put superscript labels s_r right after $\pi(i_r)$, where $1 \leq r \leq k-1$. Moreover, we put superscript label s_k at the end of π and superscript label t at the front of π .*

For example, the partition $\{1, 3\}/\{2, 4, 5\}$ and the permutation 42315 are labeled as follows:

$$\{1, 3\}_{b_2}/\{2, 4, 5\}_{b_1} a, \quad {}^t 4^{s_1} 23^{s_2} 15^{s_3}.$$

For $1 \leq k \leq n$ and $0 \leq \ell \leq \binom{n}{2}$, we define $\Pi_{n,k,\ell} = \{p \in \Pi_n : \text{block}(p) = k, \text{fr}(p) = \ell\}$ and

$$\mathcal{SP}_{n,k,\ell}(132) = \{\pi \in \mathcal{SP}_n(132) : \text{des}(\pi) = k-1, \text{inv}(\pi) = \ell\}.$$

In the following discussion, we always add labels to partitions and permutations.

Now we construct a bijection, denoted Ψ , between $\mathcal{SP}_{n,k,\ell}(132)$ and $\Pi_{n,k,\ell}$. When $n = 1$, we have $\mathcal{SP}_{1,1,0}(132) = \{1\}$ and $\Pi_{1,1,0} = \{\{1\}\}$. The bijection between $\mathcal{SP}_{1,1,0}(132)$ and $\Pi_{1,1,0}$ is given by

$${}^t 1^{s_1} \Leftrightarrow \{1\}_{b_1} a.$$

When $n = 2$, if the entry 2 is inserted to the position of ${}^t1^{s_1}$ with label t , then we append the block $\{2\}$ to $\{1\}_{b_1}a$; if the entry 2 is inserted to the position of ${}^t1^{s_1}$ with label s_1 , then we insert the element 2 into the block $\{1\}$. In other words, the bijection Ψ is given by

$$\begin{aligned} {}^t2^{s_1}1^{s_2} &\Leftrightarrow \{1\}_{b_2}/\{2\}_{b_1}a; \\ {}^t12^{s_1} &\Leftrightarrow \{12\}_{b_1}a. \end{aligned}$$

It should be noted that the block with label b_1 consists of the elements lie before the label s_1 , and the block with label b_2 (if exists) consists of the elements lie between the labels s_1 and s_2 .

Let $n = m$, where $m \geq 2$. Suppose Ψ is a bijection between $\mathcal{SP}_{m,k,\ell}(132)$ and $\prod_{m,k,\ell}$. Consider the case $n = m + 1$. Suppose $\pi \in \mathcal{SP}_{m,k,\ell}(132)$ and $\hat{\pi}$ is obtained from π by inserting the entry $m + 1$ into π . Set $\Psi(\pi) = p$. Suppose further that the block of p with label b_1 consists of the elements of π lie before the label s_1 , and for $1 < i \leq k$, the block of p with label b_i consists of the elements of π lie between the labels s_{i-1} and s_i . Consider the following two cases:

- (i) If the entry $m + 1$ is inserted to the position of π with label t , then we append the block $\{m + 1\}$ to p . In this case, $\text{des}(\hat{\pi}) = \text{des}(\pi) + 1 = k$ and $\text{inv}(\hat{\pi}) = \text{inv}(\pi) + m = \ell + m$. Moreover, $\text{block}(\Psi(\hat{\pi})) = \text{block}(p) + 1 = k + 1$ and $\text{fr}(\Psi(\hat{\pi})) = \text{fr}(p) + m = \ell + m$.
- (ii) If the entry $m + 1$ is inserted to the position of π with label s_i , then we insert the element $m + 1$ into the block of p with label b_i . In this case, $\text{des}(\hat{\pi}) + 1 = \text{block}(\Psi(\hat{\pi}))$ and $\text{inv}(\hat{\pi}) = \text{fr}(\Psi(\hat{\pi}))$. More precisely, we distinguish two cases:
 - (c₁) if $i = k$, then $\text{des}(\hat{\pi}) = \text{des}(\pi) = k - 1$, $\text{inv}(\hat{\pi}) = \text{inv}(p) = \ell$, $\text{block}(\hat{\pi}) = \text{block}(p) = k$ and $\text{fr}(\hat{\pi}) = \text{fr}(p) = \ell$.
 - (c₂) if $1 \leq i < k$ and the label s_i lies right after $\pi(j)$, then $\pi(j) > \pi(j + 1)$. By the induction hypothesis, there are $m - j$ elements in the union of the blocks of p with labels $b_{i+1}, b_{i+2}, \dots, b_k$. Therefore, $\text{des}(\hat{\pi}) = \text{des}(\pi) = k - 1$, $\text{inv}(\hat{\pi}) = \ell + m - j$, $\text{block}(\hat{\pi}) = \text{block}(p) = k$. and $\text{fr}(\hat{\pi}) = \text{fr}(p) + m - j = \ell + m - j$.

After the above step, we label the obtained permutations and partitions. It is clear that the block of $\Psi(\hat{\pi})$ with label b_1 consists of the elements of $\hat{\pi}$ lie before the label s_1 , and for $1 < i \leq k$, the block of $\Psi(\hat{\pi})$ with label b_i consists of the elements of $\hat{\pi}$ lie between the labels s_{i-1} and s_i , and the block of $\Psi(\hat{\pi})$ with label b_{k+1} (if exists) consists of the elements of $\hat{\pi}$ lie between the labels s_k and s_{k+1} . By induction, we see that Ψ is the desired bijection between $\mathcal{SP}_{n,k,\ell}(132)$ and $\prod_{n,k,\ell}$ for all k and ℓ . Using Ψ , we see that if $\pi(i)$ is a right peak of π , then $\pi(i - 1)$ and $\pi(i)$ are in the same block and $\pi(i)$ is the largest element of that block. Moreover, if $\pi(i)$ is an exterior double descent of π , then $\{\pi(i)\}$ is a singleton of $\Psi(\pi)$.

Furthermore, we define a map $\varphi : \prod_n \rightarrow \mathcal{SP}_n(132)$ as follows: For $p = B_1/B_2/\dots/B_k \in \prod_n$, let $p^r = B_k/B_{k-1}/\dots/B_1$. Let $\varphi(p)$ be a permutation obtained from p^r by erasing all of the braces of blocks and bars of p^r . For example, if $p = \{1\}/\{2,4\}/\{3,5,7\}/\{6\}$, then $\varphi(p) = 6357241$. Combining Ψ , we see that φ is also a bijection and $\mathcal{SP}_{n,k,\ell}(132) = \{\varphi(p) : p \in \prod_{n,k,\ell}\}$. It is clear that if B_i is a non-singleton of p with the largest element m , then m is a right peak of $\varphi(p)$. Moreover, if $\{c\}$ is a singleton of p , then c is an exterior double descent of $\varphi(p)$. In conclusion, using the bijections Ψ and φ , we get a constructive proof of (4).

Example 15. Given $\pi = 42351 \in \mathcal{SP}_{5,3,6}(132)$. The correspondence between π and $\Psi(\pi)$ can be done if you proceed as follows:

$$\begin{aligned} {}^t 1^{s_1} &\Leftrightarrow \{1\}_{b_1} a; \\ {}^t 2^{s_1} 1^{s_2} &\Leftrightarrow \{1\}_{b_2} / \{2\}_{b_1} a; \\ {}^t 23^{s_1} 1^{s_2} &\Leftrightarrow \{1\}_{b_2} / \{2, 3\}_{b_1} a; \\ {}^t 4^{s_1} 23^{s_2} 1^{s_3} &\Leftrightarrow \{1\}_{b_3} / \{2, 3\}_{b_2} / \{4\}_{b_1} a; \\ {}^t 4^{s_1} 235^{s_2} 1^{s_3} &\Leftrightarrow \{1\}_{b_3} / \{2, 3, 5\}_{b_2} / \{4\}_{b_1} a. \end{aligned}$$

Let $B_n(x) = \sum_{k=0}^n \{n\}_k x^k$ be the Stirling polynomials. Taking $y = z = q = 1$ in (4) leads to the following.

Corollary 16. For $n \geq 1$, we have

$$B_n(x) = \sum_{\pi \in \mathcal{SP}_n(132)} x^{\text{des}(\pi)+1}.$$

By using the reverse and complement maps, it is clear that

$$B_n(x) = \sum_{\pi \in \mathcal{SP}_n(231)} x^{\text{asc}(\pi)+1} = \sum_{\pi \in \mathcal{SP}_n(312)} x^{\text{asc}(\pi)+1} = \sum_{\pi \in \mathcal{SP}_n(213)} x^{\text{des}(\pi)+1},$$

where $\text{asc}(\pi)$ is the number of ascents of π , i.e., the number of indices i such that $\pi(i) < \pi(i+1)$.

Using the bijection Ψ and [28, p. 137, Exercise 108], we get the following result.

Proposition 17. The number of permutations in $\mathcal{SP}_n(132)$ with no successions is $B(n-1)$.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. We say that an element $\pi(i)$ is a *left-to-right maximum* of π if $\pi(i) > \pi(j)$ for every $j < i$. Let $\text{lrn}(\pi)$ be the number of left-to-right maxima of π . For example, $\text{lrn}(2314) = 3$. Let $p = B_1/B_2/\cdots/B_k$. Following [23], we define a_i to be the number of $c \in B_i$ with $c > \min B_{i-1}$, where $2 \leq i \leq k$. Let

$$\widehat{Des}(p) = \{2^{a_2}, 3^{a_3}, \dots, k^{a_k}\}$$

be the *dual descent multiset* of p , where i^d means that i is repeated d times. For example, $\widehat{Des}(\{1, 3, 5\}/\{2\}/\{4, 6, 7\}) = \{2^1, 3^3\}$. Let $\text{dudes}(p) = \#\widehat{Des}(p)$. Using the bijections Ψ and φ , it is easy to verify the following result.

Proposition 18. For $n \geq 1$, we have

$$\sum_{\pi \in \mathcal{SP}_n(132)} x^{\text{lrn}(\pi)} = \sum_{p \in \prod_n} x^{n - \text{dudes}(p)}.$$

Let $D(\pi) = \{i : \pi(i) > \pi(i+1)\}$ be the descent set of π . The *major index* of π is the sum of the descents: $\text{maj}(\pi) = \sum_{i \in D(\pi)} i$. Along the same lines as the proof of [28, Eq. (1.41)], it is routine to check that

$$\sum_{\pi \in \mathcal{SP}_n(132)} x^{\text{inv}(\pi)} = \sum_{\pi \in \mathcal{SP}_n(132)} x^{\binom{n}{2} - \text{maj}(\pi)}.$$

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