TOPOLOGICALLY DISTINCT SETS OF NON-INTERSECTING CIRCLES IN THE PLANE

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Abstract. Nested parentheses are forms in an algebra which define orders of evaluations. A class of well-formed sets of associated opening and closing parentheses is well studied in conjunction with Dyck paths and Catalan numbers. Nested parentheses also represent cuts through circles on a line. These become topologies of non-intersecting circles in the plane if the underlying algebra is commutative.

This paper generalizes the concept and answers quantitatively—as recurrences and generating functions of matching rooted forests—the questions: how many different topologies of nested circles exist in the plane if (i) pairs of circles may intersect, or (ii) even triples of circles may intersect. That analysis is driven by examining the symmetry properties of the inner regions of the fundamental type(s) of the intersecting pairs and triples.

1. Paired Parentheses and Catalan Numbers

In a (non-commutative) algebra, opening and closing parentheses prescribe the order of grouping and evaluating expressions.

Definition 1. *A string of parentheses is* well-formed *if the total number of opening parentheses equals the number of closing parentheses, and if the subtotal count of opening parentheses is always larger than or equal to the subtotal count of closing parentheses while parsing the string left-to-right.*

Remark 1. *Equivalently one may demand that the subtotal of closing parentheses is always larger or equal to the subtotal of opening parentheses while parsing the string right-to-left.*

The well-formed nested parentheses form sets $\mathbb{P}_N$ of expressions with N pairs of parentheses.

Definition 2. $\mathbb{P}_N$ *is the set of all well-formed expressions with N pairs of parentheses.*

There are expressions that can be *factored*—in the algebra—by cutting the string at some places such that the left and right substrings are also well-formed. The number f of their factors puts the elements of $\mathbb{P}_N$ into disjoint subsets:

Definition 3. $\mathbb{P}_N^{(f)}$ *is the set of all well-formed expressions with N pairs of parentheses and $1 \leq f \leq N$ factors.*

N	$|\mathbb{P}_N|$	1	2	3	4	5	6	7	8	9
1	1	1								
2	2	1	1							
3	5	2	2	1						
4	14	5	5	3	1					
5	42	14	14	9	4	1				
6	132	42	42	28	14	5	1			
7	429	132	132	90	48	20	6	1		
8	1430	429	429	297	165	75	27	7	1	
9	4862	1430	1430	1001	572	275	110	35	8	1

TABLE 1. Catalan triangle: The number of nested expressions with N pairs of parentheses: the total count $|\mathbb{P}_N|$ and the number of nested expressions with $1 \le f \le N$ factors, $|\mathbb{P}_N^{(f)}|$.

$$\tag{1} \mathbb{P}_N = \bigcup_f \mathbb{P}_N^{(f)};$$

$$\tag{2} |\mathbb{P}_N| = \sum_{f=1}^{N} |\mathbb{P}_N^{(f)}|.$$

Example 1.

$$\tag{3} \mathbb{P}_1 = \mathbb{P}_1^{(1)} = \{()\};$$

$$\tag{4} \mathbb{P}_2^{(1)} = \{(())\};$$

$$\tag{5} \mathbb{P}_2^{(2)} = \{()()\};$$

$$\tag{6} \mathbb{P}_3^{(1)} = \{((())),(()())\};$$

$$\tag{7} \mathbb{P}_3^{(2)} = \{(())(),()(())\};$$

$$\tag{8} \mathbb{P}_3^{(3)} = \{()()()\};$$

$$\tag{9} \mathbb{P}_4^{(1)} = \{(((()))),((()())),((())()),(()(())),(()()())\};$$

Remark 2. *The opening and closing parentheses are the two letters in an alphabet of words, with a grammar that recursively admits words*

   (1) *that are the empty word,*
   (2) *that are concatenations of two words,*
   (3) *that are concatenations of the first letter, a word, and the second letter.*

If the opening parentheses are replaced by U and the closing parentheses replaced by D an equivalence with Dyck paths arises; the number of returns to the horizontal line in the paths is equivalent to the number of factors in the expression. This leads straight to the well-known Catalan triangle [11, A033184] of Table 1. The set sizes $|\mathbb{P}_N|$ are the Catalan numbers [3, §1.15][11, A000108][4].

Remark 3. *Because an expression distributes N left parenthesis at $2N$ places, the set size is limited by $|\mathbb{P}_N| < \binom{2N}{N}$, the central binomial coefficients. Actually one of them must be placed at the leftmost place and none can be placed at the rightmost place, which leaves $2N - 2$ places to distribute $N - 1$ of them: $|\mathbb{P}_N| < \binom{2N-2}{N-1}$.*

Remark 4. *A computer representation uses the two binary digits* 1 *and* 0 *to represent the opening and the closing parenthesis in the aforementioned alphabet of two letters* [11, A063171,A014486]. *(Then the most-significant bit is always 1. The swapped mapping is less useful because it needs to deal with the numerical representation of leading zeros in the binary number.) Because the rightmost part is absent for the unique case of missing parentheses, N = 0, or a closing parenthesis, all these representations are even numbers. This representation by non-negative integers induces a strict ordering in the set of nested parentheses.*

Example 2. () = 10$_2$; ()() = 1010$_2$; (()) = 1100$_2$; ()()() = 101010$_2$; ()(()) = 101100$_2$; (())() = 110010$_2$; (()()) = 110100$_2$; ((())) = 111000$_2$.

The expressions with one factor are given by embracing any expression with one pair less at the left and right end with a pair of matching parentheses:

$$(10) \qquad |\mathbb{P}_N^{(1)}| = |\mathbb{P}_{N-1}|.$$

The number of expressions with f factors is given by considering any concatenated "word" of factorizations [6],

$$(11) \qquad |\mathbb{P}_N^{(f)}| = \sum_{C(N):N=N_1+N_2+\cdots+N_f} |\mathbb{P}_{N_1}^{(1)}||\mathbb{P}_{N_2}^{(1)}|\cdots|\mathbb{P}_{N_f}^{(1)}|, \quad f \geq 2,$$

where the sum is over all compositions ("ordered" partitions) of N into positive parts N_j such that subexpressions do not factor any further.

A well-formed expression of parentheses represents a set of N nested circles if we join the upper and lower end of each associated pair of parentheses. The radii of the circles are growing functions of their spatial distance in the expressions; their mid points are on a straight line, and no perimeters of any pair of circles intersect. The string of opening and closing parentheses is a record of entering or leaving a circle while poking from the outside along the line through all circles. For each pair of circles (i) either the smaller one is entirely immersed in the larger one or (ii) they have no common points.

Example 3. (()())() $\mapsto$
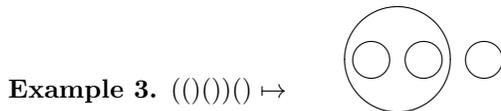
2. Nonintersecting Circle Sets in the Plane

2.1. **Nested Circle Sets.** If the algebra of Section 1 is a commutative algebra, some sets of nested parentheses are no longer considered distinct, because the order of the factors does no longer matter.

Definition 4. $\mathbb{C}_N$ *is the set of well-formed expressions of N pairs of parentheses where the order within factorizations does not matter.*

Definition 5. $\mathbb{C}_N^{(f)}$ *is the set of well-formed expressions of N pairs of parentheses with f factors where the order within factorizations does not matter.*

The number of factors still is a unique parameter of each well-formed set of expressions, so

$$(12) \qquad \mathbb{C}_N = \bigcup_f \mathbb{C}_N^{(f)};$$

$$(13) \qquad |\mathbb{C}_N| = \sum_{f=1}^N |\mathbb{C}_N^{(f)}|; \quad |\mathbb{C}_0| = 1.$$

We "lose" some of the sets of parenthesis relative to Section 1, because for example now the expressions ()(()) and (())() are considered the same: $|\mathbb{C}_N| \leq |\mathbb{P}_N|$. This reduction in the admitted expressions applies recursively to all sub-expressions that are obtained by "peeling" the surrounding pair of parentheses off expressions with a single factor.

Remark 5. *The reduction of equivalent expressions to a single representation requires some convention of which ordering of the factors is the admitted one. One convention is to map each factor to an integer with the representation of Remark 4, to put these factors into non-increasing or non-decreasing numerical order, and to concatenate their binary representations left-to-right to define the representative.*

Example 4.

$$(14) \qquad \mathbb{C}_1 = \mathbb{C}_1^{(1)} = \{()\};$$

$$(15) \qquad \mathbb{C}_2^{(1)} = \{(())\};$$

$$(16) \qquad \mathbb{C}_2^{(2)} = \{()()\};$$

$$(17) \qquad \mathbb{C}_3^{(1)} = \{((())), (()())\};$$

$$(18) \qquad \mathbb{C}_3^{(2)} = \{(())()\};$$

$$(19) \qquad \mathbb{C}_3^{(3)} = \{()()()\};$$

$$(20) \qquad \mathbb{C}_4^{(1)} = \{(((()))), ((()())), ((())()), (()()())\};$$

Table 2 shows $\mathbb{C}_N^{(f)}$ and their sums $|\mathbb{C}_N|$. The values are bootstrapped as follows: The consideration leading to Equation (10) leads also to

$$(21) \qquad |\mathbb{C}_N^{(1)}| = |\mathbb{C}_{N-1}|.$$

The decomposition of an expression with f factors needs to consider the number of ways of distributing the N pairs of parentheses over elements that do not factorize further. We partition N as $\pi(N) : N = \{N_1^{c_1}; N_2^{c_2}; \ldots N_f^{c_f}\} = \sum_{j=1}^f c_j N_j$ meaning that the expression contains c_1 factors with elements of $\mathbb{C}_1^{(1)}$, c_2 factors with elements of $\mathbb{C}_2^{(1)}$, and so on. For each part N_j with repetition c_j we compute the number of lists of c_j elements taken from a set of $|\mathbb{C}_{N_j}^{(1)}|$, possibly selecting some elements more than once or not at all. This is the number of weak compositions of c_j into $C_{N_j}^{(1)}$ parts of non-negative integers. This equals the number of compositions of $c_j + |\mathbb{C}_{N_j}^{(1)}|$ into $|\mathbb{C}_{N_j}^{(1)}|$ parts of positive integers, which is $\binom{c_j + |\mathbb{C}_{N_j}^{(1)}| - 1}{c_j}$ [12, §1.2].

$$(22)$$
$$|\mathbb{C}_N^{(f)}| = \sum_{\pi(N):N=\{N_1^{c_1};N_2^{c_2};\ldots N_f^{c_f}\}} \binom{|\mathbb{C}_{N_1}^{(1)}| + c_1 - 1}{c_1}\binom{|\mathbb{C}_{N_2}^{(1)}| + c_2 - 1}{c_2}\cdots\binom{|\mathbb{C}_{N_f}^{(1)}| + c_f - 1}{c_f}.$$

N	$|\mathbb{C}_N|$	1	2	3	4	5	6	7	8	9	10	11
1	1	1										
2	2	1	1									
3	4	2	1	1								
4	9	4	3	1	1							
5	20	9	6	3	1	1						
6	48	20	16	7	3	1	1					
7	115	48	37	18	7	3	1	1				
8	286	115	96	44	19	7	3	1	1			
9	719	286	239	117	46	19	7	3	1	1		
10	1842	719	622	299	124	47	19	7	3	1	1	
11	4766	1842	1607	793	320	126	47	19	7	3	1	1
12	12486	4766	4235	2095	858	327	127	47	19	7	3	1	1

TABLE 2. The counts $|\mathbb{C}_N|$ and $|\mathbb{C}_N^{(f)}|$ of nested nonintersecting circles [11, A033185,A000081].

Table 2 shows the phenomenon that at large N the values at large f converge to the sequence

(23) $\qquad |\mathbb{C}_N^{(N-i)}| = 1, 1, 3, 7, 19, 47, 127, 330, 889, 2378, \ldots, \quad i \geq 0, \quad N \to \infty,$

the *envelope* as Knopfmacher and Mays call it [7]. This is the Euler transform of $|\mathbb{C}_N|$ [1] and means that if the number of factors is large, most of the factors are the element $\mathbb{C}_1(1) = \{()\}$ and only few combinations remain to exhaust the others.

Returning to the interpretation of $\mathbb{P}_N$ as non-intersecting circles on a line, considering the order of factorizations unimportant means that $\mathbb{C}_N$ contains topologically distinct sets of non-intersecting circles that are free to move away from the line—as long as they stay within the boundaries of their surrounding circles. The two circles inside the bigger circle in Example 3 are allowed to bump around within the bigger circle, and the big and the outer small circle may also jointly move to other places.

Remark 6. *This is a planetary model of the circles in the sense that each circle can "rotate" inside its surrounding circle, and all these geometries are considered equivalent.*

Definition 6. *The generating function for the number of nested expressions in the commutative algebra is*

(24) $$C(z) = \sum_{N \geq 0} |\mathbb{C}_N| z^N.$$

It satisfies [5, I.5.2][10, 2]

(25) $$C(z) = \exp\left(\sum_{j \geq 1} z^j C(z^j)/j\right).$$

2.2. **Nested Circles Embedded in the Sphere Surface.** If the N circles are not embedded in the plane but embedded in the surface of a three-dimensional sphere, the topologies are counted by the unlabeled trees with $N + 1$ nodes as stated by Reshetnikov [11, A000055]:

(26) $\qquad\qquad 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, \ldots \quad N \geq 0.$

FIGURE 1. The 3 clusters of grouping the $|\mathbb{C}_4| = 9$ expressions with 4 pairs of parentheses into clusters of expressions equivalent under the flip transform.

Each unlabeled tree can be mapped to a circle set topology by constructing the line graph of the tree, associating circles with nodes of the line graph, and arranging the circles on the sphere such that they can touch (by moving them on the sphere and changing radii) iff they are connected in the line graph. The tree is a connectivity diagram of the regions on the sphere surface: an edge in the tree indicates one must cross a circle boundary to enter a different region.

Any expression of nested circles in the plane can be interpreted as a set of circles embedded in a sphere surface: draw a big circumscribing circle around the set of all circles and interpret it as an equator of the sphere. This defines a mapping of sets of circle topologies of the plane onto one circle topology of the sphere, because topologies that are related by flipping the interior and exterior region of a circle are no longer distinct on the sphere. In our notation of nested parentheses, such a flip starts with a nested expression `A ( B )`, where `A` and `B` are well-formed (potentially empty) subexpressions. After embedding in the sphere, the closing parenthesis can be torn across the back surface of the sphere to the opposite side, ending up with `( A ) B`. There are as many flip operations as there are factors in the expression because one can move any of them to the right before the flip—although some of their images may be the same because factors may be equal, and although in some cases the image may be the same as the original expression.

Example 5. *The flip operation on ()()() gives (()()).*

Example 6. *The flip operation on (())() gives ((())) if the () factor is flipped. It gives ()(()) when the (()) factor is flipped; in that case the image is the same as the original, because the order of the two factors does not matter here.*

Figures 1–3 illustrate for $N = 4$–6 how expressions transform under the flip-transform: edges in the graphs mean that the expression on one node is transformed to the expression of the other node by a flip-transformation.

So the $\mathbb{C}_N$ circle sets in the plane can be sorted into clusters which assemble all expressions that are mutually convertible by a chain of flip transformations. The number of clusters in $\mathbb{C}_N$ equals the number of topologically distinct circle sets embedded in the sphere surface.

FIGURE 2. The 6 clusters of grouping the $|\mathbb{C}_5| = 20$ expressions with 5 pairs of parentheses into clusters.

2.3. **Sets of Nested Circles and Squares.** If the geometric figures have k distinct hollow shapes— for example circles and squares with $k = 2$—the methods of circumscribing and placing side by side generalize the rules. There are k different ways of forming a single compound object from a set of objects with one element less because there are k options for the outermost shape. An upper-left index k specifies how many shapes are available. (21) and (22) turn into

$$(27) \qquad |^k\mathbb{C}_N^{(1)}| = k|^k\mathbb{C}_{N-1}|,$$

$$(28) \qquad |^k\mathbb{C}_N^{(f)}| = \sum_{\pi(N):N=\{N_1^{c_1};N_2^{c_2};...N_f^{c_f}\}} \prod_{j=1}^{f} \binom{|^k\mathbb{C}_{N_j}^{(1)}| + c_j - 1}{c_j}.$$

The implicit equation of the generating function for the total number of topologies in the plane is [8]

$$(29) \qquad {}^kC(z) = \exp\left(k\sum_{j\geq 1} z^j \, {}^kC(z^j)/j\right).$$

Example 7. *For $k = 2$, $|^2\mathbb{C}_2| = 7$ configurations exist: a circle inside a circle, a circle inside a square, a square inside a circle, a square inside a square, two disjoint circles, two disjoint squares, or a separated square and circle.*

The case with $k = 2$ shapes is further illustrated in Table 3, the case with $k = 3$ shapes in Table 4. The values on the diagonals are

$$(30) \qquad |^k\mathbb{C}_N^{(N-1)}| = \binom{N+k-1}{k-1}.$$

Row sums $|^k\mathbb{C}_N|$ are Euler transforms of the columns $|^k\mathbb{C}_N^{(1)}|$.
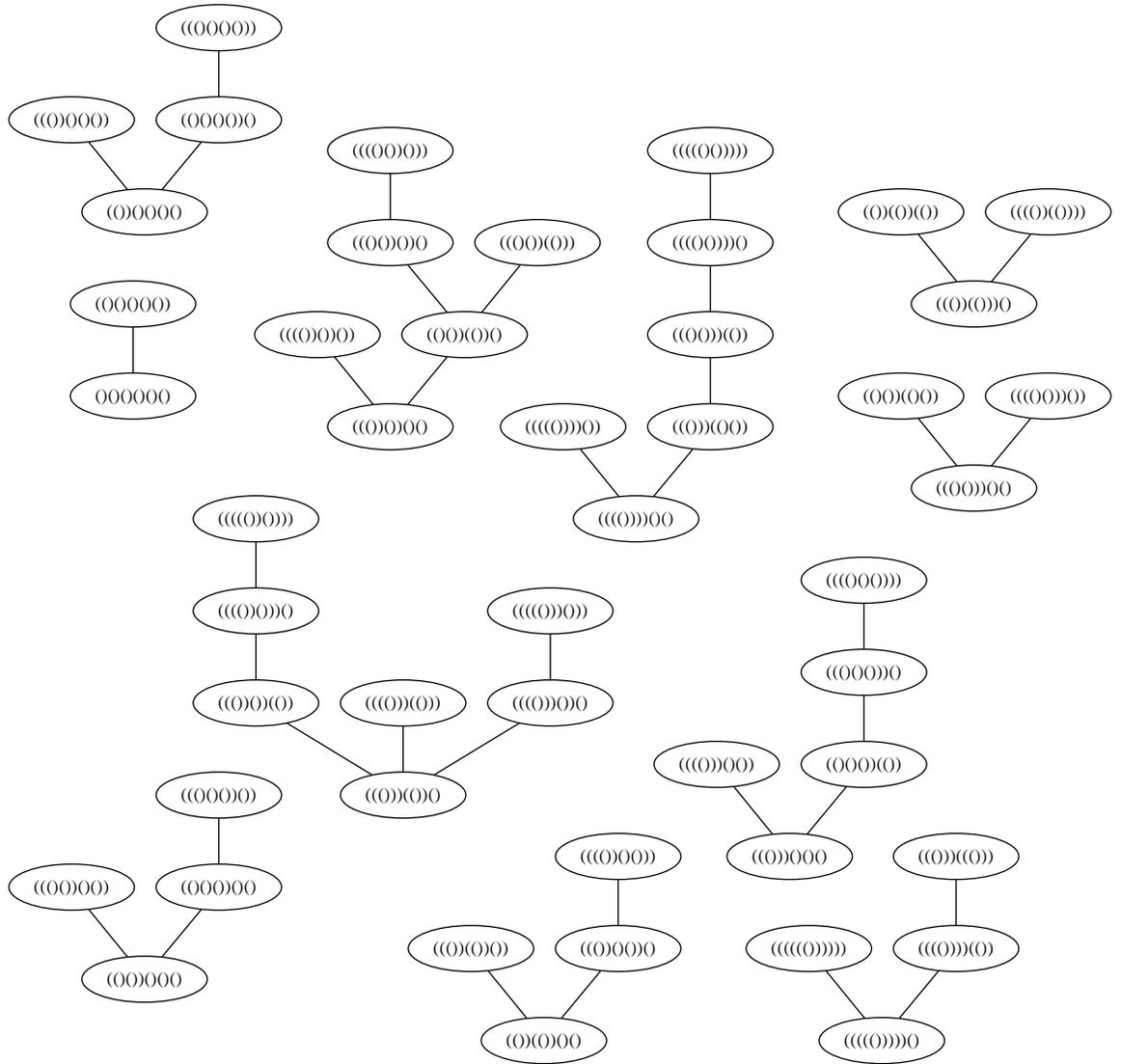
FIGURE 3. The 11 clusters of grouping the $|\mathbb{C}_6| = 48$ expressions with 6 pairs of parentheses into clusters.

3. TOPOLOGICALLY DISTINCT CIRCLE SETS, ONE CIRCLE MARKED

3.1. **Base-4 Notation.** In Section 2 the circles are moving without intersecting and qualitatively equal. We move on to the combinatorics of circle sets where one of them is marked (for example by a unique color, by morphing it into an ellipse or replacing it by a square). The equivalent modification in the commutative algebra is to introduce another symbol, a pair of brackets [], to locate the modified evaluation of a subexpression. Factorization is defined as before, and the marked circle is not intersecting with any of the other circles as before.

N	$|{}^2\mathbb{C}_N|$	1	2	3	4	5	6	7	8	9
1	2	2								
2	7	4	3							
3	26	14	8	4						
4	107	52	38	12	5					
5	458	214	160	62	16	6				
6	2058	916	741	288	86	20	7			
7	9498	4116	3416	1408	416	110	24	8		
8	44947	18996	16270	6856	2110	544	134	28	9	
9	216598	89894	78408	34036	10576	2812	672	158	32	10

TABLE 3. The counts $|{}^2\mathbb{C}_N|$ and $|{}^2\mathbb{C}_N^{(f)}|$ of nested nonintersecting circles and squares [11, A000151,A038055,A271878].

N	$|{}^3\mathbb{C}_N|$	1	2	3	4	5	6	7	8	9
1	3	3								
2	15	9	6							
3	82	45	27	10						
4	495	246	180	54	15					
5	3144	1485	1143	405	90	21				
6	20875	9432	7704	2856	720	135	28			
7	142773	62625	52731	20682	5385	1125	189	36		
8	1000131	428319	369969	150282	40914	8730	1620	252	45	
9	7136812	3000393	2638332	1104702	309510	68400	12891	2205	324	55

TABLE 4. The counts $|{}^3\mathbb{C}_N|$ and $|{}^3\mathbb{C}_N^{(f)}|$ of nested nonintersecting circles, squares and triangles [11, A006964,A038059,A271879].

Definition 7. $\mathbb{M}_N^{(f)}$ *is the set of N circles with f factors, one of these circles marked.*

$$
(31) \qquad \mathbb{M}_N = \bigcup_{f=1}^{N} \mathbb{M}_N^{(f)};
$$

$$
(32) \qquad |\mathbb{M}_N| = \sum_{f=1}^{N} |\mathbb{M}_N^{(f)}|; \quad |\mathbb{M}_0| = 1.
$$

N	$\|\mathbb{M}_N\|$	1	2	3	4	5	6	7	8	9	10	11
1	1	1										
2	3	2	1									
3	9	5	3	1								
4	26	13	9	3	1							
5	75	35	26	10	3	1						
6	214	95	75	30	10	3	1					
7	612	262	214	91	31	10	3	1				
8	1747	727	612	268	95	31	10	3	1			
9	4995	2033	1747	790	284	96	31	10	3	1		
10	14294	5714	4995	2308	848	288	96	31	10	3	1	
11	40967	16136	14294	6737	2506	864	289	96	31	10	3	1

TABLE 5. The number of nonintersecting circles with one of them marked. $\|\mathbb{M}_N\|$ are the row sums and $\|\mathbb{M}_N^{(f)}\|$ the entries with f factors [11, A000243,A000107].

Example 8.

$$(33) \qquad \mathbb{M}_1 \;=\; \mathbb{M}_1^{(1)} = \{[]\};$$

$$(34) \qquad \mathbb{M}_2^{(1)} \;=\; \{[()], ([])\};$$

$$(35) \qquad \mathbb{M}_2^{(2)} \;=\; \{[]()\};$$

$$(36) \qquad \mathbb{M}_3^{(1)} \;=\; \{(([])), ([()]), ([]()), [()()], [(())]\};$$

$$(37) \qquad \mathbb{M}_3^{(2)} \;=\; \{([])(), [()](), [](())\};$$

$$(38) \qquad \mathbb{M}_3^{(3)} \;=\; \{[]()()\};$$

$$\mathbb{M}_4^{(1)} \;=\; \{((())[]), ((([]))), (([()])), (([])()), (([]())), (([]0)]),$$
$$(39) \qquad\qquad ([()]()), ([(())]), ([]()()), [()()()], [(())()], [(()())], [((()))]\};$$

$$\mathbb{M}_4^{(2)} \;=\; \{(())()[], ((()))[], (([]))(), ([()])(), ([])(()),$$
$$(40) \qquad\qquad ([]())(), [()()](), [()](()), [(())]()\};$$

3.2. **Recurrences.** An overview of how many distinct arrangements exist is given in Table 5. The set of $\mathbb{M}_N^{(1)}$ is created by either (i) wrapping an expression without a marked sphere into a bracket, or by (ii) wrapping an expression that already contains a marked sphere into a pair of parentheses:

$$(41) \qquad\qquad \|\mathbb{M}_N^{(1)}\| = \|\mathbb{C}_{N-1}\| + \|\mathbb{M}_{N-1}\|.$$

For expressions with $f \geq 2$ factors we may always move the factor with the bracket to some pivotal (say the leftmost) factor because the order of factors does not matter in Sections 2 and 3. That pivotal factor needs, say, N' pairs of parentheses (including the marked), and all the other factors may be varied as a set of the $\mathbb{C}$ type:

$$(42) \qquad\qquad \|\mathbb{M}_N^{(f)}\| = \sum_{N'=1}^{N-1} \|\mathbb{M}_{N'}^{(1)}\|\,\|\mathbb{C}_{N-N'}^{(f-1)}\|, \quad f \geq 2.$$

Definition 8. *The generating function of the topologies of non-intersecting circles with one marked is*

(43) $$M(z) = \sum_{N \geq 0} |\mathbb{M}_N| z^N.$$

If we sum on both sides of (42) over f, insert (13) for the sum over the $\mathbb{C}$ and (41) to eliminate the $\mathbb{M}_{N'}^{(1)}$ on the right hand side, Jovovic's relation shows up [11, A000243]

(44) $$M(z) = 1 + \frac{zC^2(z)}{1 - zC(z)}.$$

For sufficiently large N the count with $N-1$ factors is $|\mathbb{M}_N^{(N-1)}| = 3$ because the set contains the expressions of the form $[()]()() \cdots$, $([])()() \cdots$, and $[](())()() \cdots$.

3.3. Inner Void Circle. There is a subset of expressions $\mathbb{M}_N^{(v)} \subseteq \mathbb{M}_N$—indicated with an upper v like *void*—where the bracket does not contain any subexpression with parentheses, i.e., where the marked circle does not circumscribe any other circle.

Definition 9. $\mathbb{M}_N^{(f,v)}$ *is the set of N circles with f factors, where one of these circles is marked and does not contain other circles.*

(45) $$\mathbb{M}_N^{(v)} = \bigcup_{f=1}^{N} \mathbb{M}_N^{(f,v)} \subseteq \mathbb{M}_N^{(f)};$$

(46) $$|\mathbb{M}_N^{(v)}| = \sum_{f=1}^{N} |\mathbb{M}_N^{(f,v)}|.$$

Example 9. *Removing in Example 8 the expressions where the bracket pair embraces other parentheses yields:*

(47) $$\mathbb{M}_1^{(v)} = \mathbb{M}_1^{(1,v)} = \{[]\};$$

(48) $$\mathbb{M}_2^{(1,v)} = \{([])\};$$

(49) $$\mathbb{M}_2^{(2,v)} = \{[]()\};$$

(50) $$\mathbb{M}_3^{(1,v)} = \{(([])), ([]()), \};$$

(51) $$\mathbb{M}_3^{(2,v)} = \{([])(), [](())\};$$

(52) $$\mathbb{M}_3^{(3,v)} = \{[]()()\};$$

(53) $$\mathbb{M}_4^{(1,v)} = \{((())[]), ((([]))), (([])()), (([]())), ([]()()), \};$$

(54) $$\mathbb{M}_4^{(2,v)} = \{(()())[], ((()))[], (([]))(), ([])(()), ([]())()\}.$$

The topologies with that scenario are counted in Table 6. With the same argument as in Equation (42), scenarios with an empty bracket need to locate the bracket at some fixed factor, and let the other factors generate all possible diagrams with the remaining parentheses:

(55) $$|\mathbb{M}_N^{(f,v)}| = \sum_{N'=1}^{N-1} |\mathbb{M}_{N'}^{(1,v)}| |\mathbb{C}_{N-N'}^{(f-1)}|.$$

N	$\lvert\mathbb{M}_N^{(v)}\rvert$	1	2	3	4	5	6	7	8	9	10	11
1	1	1										
2	2	1	1									
3	5	2	2	1								
4	13	5	5	2	1							
5	35	13	13	6	2	1						
6	95	35	35	16	6	2	1					
7	262	95	95	46	17	6	2	1				
8	727	262	262	128	49	17	6	2	1			
9	2033	727	727	364	139	50	17	6	2	1		
10	5714	2033	2033	1029	401	142	50	17	6	2	1	
11	16136	5714	5714	2930	1147	412	143	50	17	6	2	1
12	45733	16136	16136	8344	3299	1184	415	143	50	17	6	2	1

TABLE 6. The number of nonintersecting circles, one marked. $\lvert\mathbb{M}_N^{(v)}\rvert$ and $\lvert\mathbb{M}_N^{(f,v)}\rvert$ for $N, f \geq 1$ [11, A000107].

On the diagonals of Tables 5 and 6 we find

$$(56) \qquad\qquad \lvert\mathbb{M}_N^{(N)}\rvert = \lvert\mathbb{M}_N^{(N,v)}\rvert = 1,$$

because the only expressions with as many factors as circles is the product of singletons, $\mathbb{M}_N^{(N)} = \mathbb{M}_N^{(N,v)} = \{[]()()\cdots()\}$.

The number in column $\mathbb{M}_N^{(1)}$ in Table 6 duplicates the total of the previous row:

$$(57) \qquad\qquad \lvert\mathbb{M}_N^{(1,v)}\rvert = \lvert\mathbb{M}_{N-1}^{(v)}\rvert.$$

This is easily understood because each element of the set $\mathbb{M}_N^{(1)v}$ is created by surrounding the expression of an element of the set $\mathbb{M}_{N-1}^{(v)}$ by a pair of (non-marked) parentheses, so the "void" within the bracket is conserved.

In a similar manner $\lvert\mathbb{M}_N^{(1,v)}\rvert = \lvert\mathbb{M}_N^{(2,v)}\rvert$ is understood by "peeling off" the outermost pair of parentheses of the element of $\mathbb{M}_N^{(1,v)}$ and placing it as an extra factor () aside from the peeled expression. This association works because the outermost pair of parentheses is never the bracket.

In summary, all entries of Table 5 and 6 can be recursively generated from Table 2 with the aforementioned 4 formulas.

Remark 7. *The serialized representation of the circle sets with two types of parentheses on a computer is possible by moving from the binary digit representation of Sections 1 and 2 to a base-4 representation* $) \mapsto 0$, $(\mapsto 1$, $] \mapsto 2$, $[\mapsto 3$. *The mapping is* $[[]] \mapsto 3322_4$, $([[()]]()) \mapsto 1331022100_4$, *for example.*

4. CIRCLE SETS WITH ONE PAIR INTERSECTING

4.1. **Serialized Notation.** Another derivative of the non-intersecting circle sets of Section 2 are circle sets where exactly one pair of circles intersects at two points of their rims.

These two intersecting circles are a natural reference frame for the other $N-2$. In the serialized notation we introduce the expression $[[]]$ with two bracket pairs to indicate crossing of the rims of the first, then of the second circle, then leaving the first

and finally leaving the second. The notation provides 5 regions that host the $N-2$ remaining circles. The well-formed general expression is $reg4\,[reg3\,[reg2]\,reg1]\,reg0$ if the regions are enumerated 0–4.
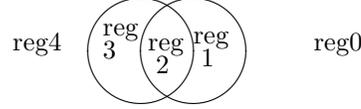
$$reg4 \qquad \qquad reg0$$

$reg4\,[reg3\,[reg2]\,reg1]\,reg0 \mapsto$

The serialized notation is well-suited for computerized managing, but again has the drawback that the freedom of moving circle sets around as long as no new intersections are induced is not strictly enforced. We add the following constraints to the serialized notation to avoid over-counting those circle sets with two intersections:

(1) The regions *reg1*, *reg2* and *reg3* host members of the $\mathbb{C}_N$ collection. This basically ensures that their circle sets do not introduce intersections by peeking beyond the enclosures defined by the bracket pair. Note that no such rule is enforced on *reg0* and *reg4* because we allow the crossing circles to be inside other circles; so an expression like ([[]]) is well-formed, although the isolated left and right parentheses are not individually members of $\mathbb{P}$.

(2) If the entire core region of the crossing circles is removed—leaving the concatenated expression *reg4 reg0*—this must be a well-formed $\mathbb{P}$ expression. This ensures that circles that rotate in the space outside the crossing circles are considered equivalent; eventually expressions like (()[[]]) and ([[]]()) for example are counted only once.

(3) From the two expressions obtained by swapping *reg1* and *reg3* only one is admitted. These are the regions inside one of the intersecting circles but not in the intersection. The rule ensures that a sort of mirror operation at the center of the intersection—which does not change the topology—is admitted only once in the circle sets.

Definition 10. $\mathbb{X}_N^{(f)}$ *is the set of N circles with f factors, two circle rims intersecting in two points.*

$$(58) \qquad \mathbb{X}_N \;=\; \bigcup_{f=1}^{N} \mathbb{X}_N^{(f)};$$

$$(59) \qquad |\mathbb{X}_N| \;=\; \sum_{f=1}^{N} |\mathbb{X}_N^{(f)}|; \quad |\mathbb{X}_1| = 0.$$

Example 10. $([[]])() \mapsto$

$([[]()]) \mapsto$

$[[]](()) \mapsto$

$[[()]()]() \mapsto$

N	$\|\mathbb{X}_N\|$	1	2	3	4	5	6	7	8	9	10
2	1	1	0								
3	4	3	1	0							
4	15	10	4	1	0						
5	50	30	15	4	1	0					
6	162	91	50	16	4	1	0				
7	506	268	162	55	16	4	1	0			
8	1558	790	506	185	56	16	4	1	0		
9	4727	2308	1558	594	190	56	16	4	1	0	
10	14227	6737	4727	1878	617	191	56	16	4	1	0
11	42521	19609	14227	5825	1970	622	191	56	16	4	1	0

TABLE 7. Topologically distinct sets of N circles with one pair intersecting, total (row sums) $\|\mathbb{X}_N\|$ and $\|\mathbb{X}_N^{(f)}\|$ classified according to number of factors $1 \le f \le N$.

Example 11.

$$(60) \qquad \mathbb{X}_2 = \mathbb{X}_2^{(1)} = \{[[]]\};$$

$$(61) \qquad \mathbb{X}_2^{(2)} = \{\};$$

$$(62) \qquad \mathbb{X}_3^{(1)} = \{([[]]), [[()]], [[()]]\};$$

$$(63) \qquad \mathbb{X}_3^{(2)} = \{[[]]()\};$$

$$(64) \qquad \mathbb{X}_3^{(3)} = \{\};$$

$$\mathbb{X}_4^{(1)} = \{((([[]])), ([[()]]), ([[()]]), ([[]]()),$$
$$(65) \qquad\qquad [()[]()], [[()()]], [[()]()], [[(())]], [[]()()], [[](())]\};$$

$$(66) \qquad \mathbb{X}_4^{(2)} = \{([[]])(), [[()]](), [[()]](), [[]](())\};$$

$$(67) \qquad \mathbb{X}_4^{(3)} = \{[[]]()()\};$$

$$(68) \qquad \mathbb{X}_4^{(4)} = \{\};$$

4.2. **Recurrences.** Table 7 shows how many expressions are in the sets $\mathbb{X}_N$ and $\mathbb{X}_N^{(f)}$. The first three values of $\|\mathbb{X}_N\|$ are mentioned in the Encyclopedia of Integer Sequences [11, A261070].

Obviously $\|\mathbb{X}_N^{(N)}\| = 0$ and $\|\mathbb{X}_N^{(N-1)}\| = 1$ because we always spend two circles in the bracket—which does not factorize—and the expression $[[]]()()() \cdots$ is the only member of $\mathbb{X}_N^{(N-1)}$.

For sufficiently large N there are $\|\mathbb{X}_N^{(N-2)}\| = 4$ expressions, namely $[[()]]()() \cdots$, $[[()]()() \cdots$, $[[]](())() \cdots$, and $([[]])()() \cdots$ with $N - 3$ trailing isolated circles.

The argument of isolating the factor that contains the bracket pair that led to Equation (42) remains valid, so

$$(69) \qquad\qquad \|\mathbb{X}_N^{(f)}\| = \sum_{N'=1}^{N-1} \|\mathbb{X}_{N'}^{(1)}\| \, \|\mathbb{C}_{N-N'}^{(f-1)}\|, \quad f \ge 2.$$

The dismantling of the sole factor of an expression of $\mathbb{X}_N^{(1)}$ that contains the two brackets shows two variants: if the outer parentheses are the round parenthesis, the

expression has been formed by embracing any expression with $N-1$ circles, which contributes $|\mathbb{X}_{N-1}|$. If alternatively the expression is of the form stripped down to where $reg4$ and $reg0$ are empty, we count the number of ways of construction $reg3$, $reg2$ and $reg1$ with a total of $N-2$ circles by a function D_{n-2}:

$$|\mathbb{X}_N^{(1)}| = |\mathbb{X}_{N-1}| + D_{N-2}. \tag{70}$$

$$D_N = 1, 2, 6, 15, 41, 106, 284, 750, 2010, 5382, 14523, 39290\ldots; \quad N \geq 0. \tag{71}$$

Example 12. *The 6 expressions that contribute to* $D_2 = 6$ *are* $[[]()()]$, $[[](())]$, $[[()]()]$, $[()[]()]$, $[[(())]]$, *and* $[[()()]]$.

The distribution of the N circles over $reg3$, $reg2$ and $reg1$ has no further restrictions to place any member of $\mathbb{C}$ into $reg2$, which reduces D by composition to another function $\hat{D}$ of the form

$$D_N = \sum_{N'=0}^{N} |\mathbb{C}_{N'}| \hat{D}_{N-N'}. \tag{72}$$

$\hat{D}_N$ counts the number of ways of placing N circles in total into $reg3$ and $reg1$ such that each expression is a member of $\mathbb{C}$ and such that the third rule of Section 4.1 of counting only the "ordered" pairs is obeyed. If N is odd, the expressions in two regions necessarily differ because they must have a different number of circles, so the rule may for example be implemented by putting always the expression with the lower number into one region:

$$\hat{D}_{N,odd} = \sum_{N'=0}^{\lfloor N/2 \rfloor} |\mathbb{C}_{N'}| \, |\mathbb{C}_{N-N'}|. \tag{73}$$

If N is even, an additional format appears where the expressions in $reg3$ and $reg1$ have the same number of circles. Because these elements of $|\mathbb{C}_{N/2}|$ may be put into a strict order, the triangular number with that argument counts the "non-ordered" pairs of these:

$$\hat{D}_{N,even} = \sum_{N'=0}^{N/2} |\mathbb{C}_{N'}| \, |\mathbb{C}_{N-N'}| + \frac{|\mathbb{C}_{N/2}|(|\mathbb{C}_{N/2}|+1)}{2}. \tag{74}$$

In terms of the generating functions (24), (43) and

$$\hat{D}(z) = \sum_{N \geq 0} \hat{D}_N z^N, \quad D(z) = \sum_{N \geq 0} D_N z^N, \tag{75}$$

this type of half convolution in the previous two equations may be summarized as [11, A027852]

$$\hat{D}(z) = \frac{1}{2}[C(z)^2 + C(z^2)]. \tag{76}$$

Remark 8. *The symmetry enforced to the contents of reg1 and reg3 is the symmetry of the cyclic group of order 2. The cycle index for this group is* $(t_1^2 + t_2)/2$ *[5, I60]. Substitution of* $t_j \mapsto C(z^j)$ *gives the same result [3, p. 252].*

$$\hat{D}_N = 1, 1, 3, 6, 16, 37, 96, 239, 622, 1607, 4235, 11185, 29862\ldots; \quad N \geq 0. \tag{77}$$

The convolution (72) turns into a product of the generating functions:

$$(78) \qquad\qquad\qquad D(z) = C(z)\hat{D}(z).$$

Example 13. $\hat{D}_1 = 1$ *is the size of the set* $\{[\,[\,]()]\}$.

Example 14. $\hat{D}_2 = 3$ *is the size of the set* $\{[()[\,]()], [\,[\,]()()], [\,[(())]\}$.

Example 15. $\hat{D}_3 = 6$ *is the size of the set* $\{[\,[()()()], [\,[(())()], [\,[((()))], [\,[(()())],$ $[()[\,]()()], [()[\,]((())]\}$.

Summing (69) over f and using (70) leads to

$$(79) \qquad\qquad\qquad X(z) = 1 + \frac{z^2 D(z) C(z)}{1 - zC(z)}.$$

4.3. **Pair of Touching Circles.** If there are no further circles in the area of the intersection, the two intersecting circles may be moved apart until they touch in a single point. These borderline cases are destilled from the previous analysis by counting expressions only where the two inner brackets appear side by side, i.e, where *reg2* is empty. We call these sets of configurations $\mathbb{X}_N^{(f,t)}$ where the label t indicates touching.

Definition 11. $\mathbb{X}_N^{(f,t)}$ *is the set of* N *circles with* f *factors, two circle rims touching at one point.*

$$(80) \qquad\qquad \mathbb{X}_N^{(t)} \quad = \quad \bigcup_{f=1}^{N} \mathbb{X}_N^{(f,t)} \subseteq \mathbb{X}_N;$$

$$(81) \qquad\qquad |\mathbb{X}_N^{(t)}| \quad = \quad \sum_{f=1}^{N} |\mathbb{X}_N^{(f,t)}| \le |\mathbb{X}_N|;$$

Example 16. *If we remove the expressions from Example 11 where other circles appear within the innermost of the two square brackets, the following list emerges:*

$$(82) \qquad \mathbb{X}_2^{(t)} \quad = \quad \mathbb{X}_2^{(1,t)} = \{[\,[\,]]\};$$

$$(83) \qquad \mathbb{X}_2^{(2,t)} \quad = \quad \{\};$$

$$(84) \qquad \mathbb{X}_3^{(1,t)} \quad = \quad \{([\,[\,]]), [\,[\,]()]\};$$

$$(85) \qquad \mathbb{X}_3^{(2,t)} \quad = \quad \{[\,[\,]()\};$$

$$(86) \qquad \mathbb{X}_3^{(3,t)} \quad = \quad \{\};$$

$$(87) \qquad \mathbb{X}_4^{(1,t)} \quad = \quad \{(([\,[\,]])), ([\,[\,]()]), ([\,[\,]()), \{[()[\,]()], [\,[\,]()()], [\,[(())]\}\};$$

$$(88) \qquad \mathbb{X}_4^{(2,t)} \quad = \quad \{([\,[\,]])(), [\,[\,]()](), [\,[\,]((())\};$$

$$(89) \qquad \mathbb{X}_4^{(3,t)} \quad = \quad \{[\,[\,]()()\};$$

$$(90) \qquad \mathbb{X}_4^{(4)} \quad = \quad \{\};$$

Table 8 shows how many expressions are in the sets $\mathbb{X}_N^{(t)}$ and $\mathbb{X}_N^{(f)t}$.

As before

$$(91) \qquad\qquad\qquad |\mathbb{X}_N^{(N,t)}| = 0; \quad |\mathbb{X}_N^{(N-1,t)}| = 1.$$

N	$\lvert\mathbb{X}_N^{(t)}\rvert$	1	2	3	4	5	6	7	8	9	10
1	0	0									
2	1	1	0								
3	3	2	1	0							
4	10	6	3	1	0						
5	30	16	10	3	1	0					
6	91	46	30	11	3	1	0				
7	268	128	91	34	11	3	1	0			
8	790	364	268	108	35	11	3	1	0		
9	2308	1029	790	327	112	35	11	3	1	0	
10	6737	2930	2308	992	344	113	35	11	3	1	0
11	19609	8344	6737	2962	1055	348	113	35	11	3	1	0

TABLE 8. Topologically distinct sets of N circles with one pair touching, total and classified according to number of factors $1 \leq f \leq N$ [11, A269800].

The strategy of isolating the factor with the brackets that lead to (70) remains valid:

$$(92) \qquad \lvert\mathbb{X}_N^{(f,t)}\rvert = \sum_{N'=1}^{N-1} \lvert\mathbb{X}_{N'}^{(1,t)}\rvert\,\lvert\mathbb{C}_{N-N'}^{(f-1)}\rvert, \quad f \geq 2.$$

The formula that distributed the $N-2$ circles within the three regions in the intersecting circles now needs to skip the cases where some of them are in *reg2*. And instead of (70) we immediately skip to

$$(93) \qquad \lvert\mathbb{X}_N^{(1,t)}\rvert = \lvert\mathbb{X}_{N-1}^{(t)}\rvert + \hat{D}_{N-2}$$

and replace (79) by the generating function

$$(94) \qquad X^{(t)}(z) = \sum_{N \geq 0} X_N^{(t)} z^N = 1 + \frac{z^2 \hat{D}(z) C(z)}{1 - z C(z)}.$$

4.4. **One or More Intersecting Pairs.** The topologies of the members of the sets $\mathbb{X}_N$ are mapped onto rooted trees representing the dependence of "being a circle inside another" as "being a branch of a node that represents the enclosing circle." The plane is the root of the tree. There is no limit of how many branches a node can have. Moving around circle clusters freely means that the nodes are counted without a notion of order. The sole exception—which distinguishes $\mathbb{C}_N$ from $\mathbb{X}_N$—is that the tree must have a single node representing the intersecting circle pair which has up to three nodes (the three regions) that partially respect order because the circle clusters in *reg2* are topologically considered different from circle clusters in *reg1* or *reg3*.

If one chops the node representing the plane off the tree, it becomes a rooted forest, where the number of rooted trees is the factor f of the interpretation as nested parentheses.

The natural extension of these rules is to symmetrize the rules for the branches in that rooted forest, i.e., to allow circles and the three regions in the intersecting circle pair to host any number of intersecting circle clusters or intersecting circle

pairs. The restriction that remains is that intersection of more than two circles are still not considered.

Definition 12. $^2\mathbb{X}_N$ *is the set of topologies of N nested circles in the plane where each circle intersects with at most one other circle.* $^2\mathbb{X}_N^{(f)}$ *is the set of topologies of N nested circles with f factors, i.e., with f of these objects that are not inside any other of these objects.*

Definition 13. *The generating function is*

$$(95) \qquad ^2X(z) = \sum_{N \geq 0} |^2\mathbb{X}_N| z^N.$$
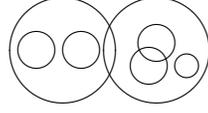
Example 17. *This is a circle bundle in $^2\mathbb{X}_7^{(1)}$ which is not in $\mathbb{X}_7$:*
The outer pair of 2 intersecting circles contains another pair of 2 intersecting circles (amongst others) in one of its three regions.

The grand book-keeping of placing these objects side by side works as before, and the empty plane is the unique way of not having any circles:

$$(96) \qquad ^2\mathbb{X}_N = \bigcup_{f=1}^{N} {}^2\mathbb{X}_N^{(f)}; \quad |^2\mathbb{X}_N| = \sum_{f=1}^{N} |^2\mathbb{X}_N^{(f)}|; \quad |^2\mathbb{X}_0| = 1.$$

$$(97) \qquad |^2\mathbb{X}_N^{(f)}| = \sum_{\pi(N):N=\{N_1^{c_1};N_2^{c_2};\dots N_f^{c_f}\}} \prod_{j=1}^{f} \binom{|^2\mathbb{X}_{N_j}^{(1)}| + c_j - 1}{c_j}; \quad f \geq 2.$$

The difference starts where the objects at $f = 1$ are dismantled. These are not the two types considered in (41), (70) or (93) nor the k types as in (27). The compound object is either a circle that hosts the same type of objects with one circle less, or a pair of intersecting circles with other objects of the same type in their regions:

$$(98) \qquad |^2\mathbb{X}_N^{(1)}| = |^2\mathbb{X}_{N-1}| + \bar{D}_{N-2}.$$

$\bar{D}_{N-2}$ is the number of ways of distributing objects of the $^2\mathbb{X}$ type with a total of $N - 2$ circles into three regions with the symmetry rules of Section 4.2.

With the splitting rule of Section 4.2 the overlapping *reg2* may contain any number of the elements of $^2\mathbb{X}$ and the other two regions share the remaining number of circles as if the set was ordered. Copying from (72) and (76),

$$(99) \qquad \bar{D}_N = \sum_{N'=0}^{N} |^2\mathbb{X}_{N'}| \tilde{D}_{N-N'};$$

$$(100) \qquad \tilde{D}(z) = \sum_{N \geq 0} \tilde{D}_N z^N = \frac{1}{2}[^2X(z)^2 + {}^2X(z^2)].$$

(101)
$$\bar{D}_N = 1, 2, 8, 26, 99, 364, 1417, 5541, 22193, 89799, 368160, 1523020, \dots, N \geq 0.$$

N	$\lvert{}^2\mathbb{X}_N\rvert$	1	2	3	4	5	6	7	8	9
1	1	1								
2	3	2	1							
3	8	5	2	1						
4	27	16	8	2	1					
5	90	53	26	8	2	1				
6	330	189	100	30	8	2	1			
7	1225	694	375	115	30	8	2	1		
8	4729	2642	1473	453	120	30	8	2	1	
9	18554	10270	5823	1827	473	120	30	8	2	1
10	74234	40747	23479	7432	1936	479	120	30	8	2	1
11	300828	164033	95618	30622	7954	1961	479	120	30	8	2	1

TABLE 9. The number of topologies of nested N circles intersecting at most as binaries, $\lvert{}^2\mathbb{X}_N\rvert$, and the subcounts with f factors, $\lvert{}^2\mathbb{X}_N^{(f)}\rvert$, $1 \le f \le N$. The row sums are the Euler transform of the column $f = 1$.

The number of ways of distributing N circles over *reg1* and *reg3* of two intersecting circles is

(102)
$$\tilde{D}_N = 1, 1, 4, 11, 41, 141, 537, 2041, 8042, 32028, 129780, 531331, 2198502, \ldots N \ge 0.$$

Example 18. $\tilde{D}_2 = 4$ *counts the three ways of Example 14 plus the one way of putting two intersecting circles in one of the two regions.*

Example 19. $\tilde{D}_3 = 11$ *counts the six ways of Example 15 plus the following five ways that are new in* ${}^2\mathbb{X}_N$:

(1) *putting* [[]]() *in one region,*
(2) *putting* [[]] *in one region and* () *in the other,*
(3) *putting* ([[]]) *in one region,*
(4) *putting* [[]()] *in one region,*
(5) *putting* [[()]] *in one region.*

5. OUTLOOK: 3-CIRCLE INTERSECTIONS

5.1. **6 New Topologies.** Adding 3-circle intersections introduces 6 new topologies beyond those of ${}^2\mathbb{X}_N$ [11, A250001]:
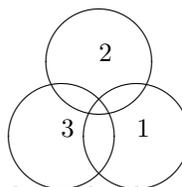
(1) The "RGB spot diagram" ${}^{3,1}\mathbb{X}_3$: The 7 regions inside the circles may be labeled by the circles that cover them: *1*, *12*, *2*, *23*, *3*, *13*, *123*. The symmetry of the diagram is established by three mirror lines that pass through *12*, *23* and *13*, and a symmetry for rotations by multiples of 120° around the center. The symmetry group is the noncyclic group of order 6. A permutation representation is (1)(23) for the first generator and (123) for the second [9]. The elements are

- the unit element $(1)(2)(3)$ which contributes t_1^3 to the cycle polynomial,
- the first generator which contributes $t_1 t_2$,
- the second generator which contributes t_3,
- the square of the second generator, (132), which contributes t_3,
- the element (12) which contributes $t_1 t_2$, and
- the element (13) which contributes $t_1 t_2$.

The cycle index is $(t_1^3 + 3t_1 t_2 + 2t_3)/6$.
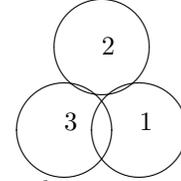
(2) The torn version of this with an uncovered central area ${}^{3,2}\mathbb{X}_3$:
The 7 regions inside the circles may be labeled by the circles that cover
them: *1, 12, 2, 23, 3, 13, $\notin$ 123*. The symmetry is the same as for ${}^{3,1}\mathbb{X}_3$
above.
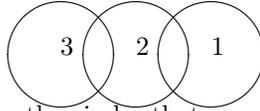
(3) A linear chain ${}^{3,3}\mathbb{X}_3$:   The 5 regions inside the
circles may be labeled by the circles that cover them: *1, 12, 2, 23, 3*. The
symmetry is the same left-right mirror symmetry as in Remark 8; the cycle
index is $(t_1^2 + t_2)/2$.
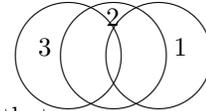
(4) The left-right compressed version of the previous diagram, ${}^{3,4}\mathbb{X}_3$:
The 7 regions inside the circles may be labeled by the circles that cover
them: *1, 12, 123, 23, 3, $\overline{2}$, $\underline{2}$*, using overline and underline to register the
upper and lower regions of the pieces of circle 2. The appearance of the
regions $\overline{2}$ and $\underline{2}$ introduces an additional up-down mirror symmetry. The
symmetry group is the noncyclic group of order 4, which has the generators
(34) and (12) [9]. The elements are
  - the unit element $(1)(2)(3)(4)$ which contributes t_1^4 to the cycle poly-
    nomial,
  - the first generator which contributes $t_1^2 t_2$,
  - the second generator which contributes $t_1^2 t_2$,
  - the element $(12)(34)$ which contributes t_2^2.

The cycle index is $(t_1^4 + 2t_1^2 t_2 + t_2^2)/4$. This is replaced by the direct product
$(t_1^2 + t_2)/2 \times (t_1'^2 + t_2')/2$ as we wish to represent the combined regions *1 $\cup$ 12*
and *3 $\cup$ 23* that are related by one of the C_2 symmetries differently from
the regions $\overline{2}$ and $\underline{2}$ by the other C_2 symmetry.
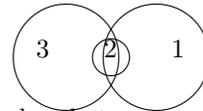
(5) The previous diagram with a shrunk center circle, ${}^{3,5}\mathbb{X}_3$:
The 7 regions inside the circles may be labeled by the circles that cover
them: *1, 12, 123, 23, 3, $\overline{13}$, $\underline{13}$*, using overline and underline to register the

upper and lower regions. The symmetry is the same as in the preceding diagram $^{3,4}\mathbb{X}_3$.

(6) The asymmetric bundle $^{3,6}\mathbb{X}_3$: The 5 regions inside the circles may be labeled by the circles that cover them: *1, 12, 123, 23, 2*. The cycle index is t_1.
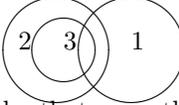
Let $^3\mathbb{X}_N \supseteq {}^2\mathbb{X}_N$ denote the arrangements of N nested circles which admit the topologies of simple circles, the one topology of two-circle intersections and the six topologies of three-circle intersections in the subregions.

Example 20. *This is an element of* $^3\mathbb{X}_{10}^{(2)}$ *which is not in* $^2\mathbb{X}_N$:
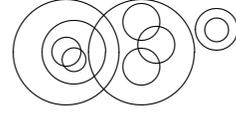
5.2. **Recurrences.**

Definition 14. *Generating function of the topologies with up-to-three intersections:*

$$(103) \qquad \sum_N |^3\mathbb{X}_N| z^N = {}^3X(z).$$

The multiset interpretation as a forest of rooted trees with non-factoring elements $^3\mathbb{X}_N^{(1)}$ in the roots holds again:

$$(104) \qquad |^3\mathbb{X}_N^{(f)}| = \sum_{\pi(N): N = \{N_1^{c_1}; N_2^{c_2}; \dots N_f^{c_f}\}} \prod_{j=1}^{f} \binom{|^3\mathbb{X}_{N_j}^{(1)}| + c_j - 1}{c_j}.$$

There is one type of compound objects constructed by wrapping a circle around others, one type of covering them with two intersecting circles, and six types of covering them with three intersecting circles. Because the two types $^{3,1}\mathbb{X}$ and $^{3,2}\mathbb{X}$ of the 3-circles have the same number of regions and the same symmetry, we count the first type twice and drop the second; because types $^{3,4}\mathbb{X}$ and and $^{3,5}\mathbb{X}$ have the same number of regions and the same symmetry, we also count $^{3,4}\mathbb{X}$ twice and drop $^{3,5}\mathbb{X}$. The upgrade of (98) is

$$(105) \quad |^3\mathbb{X}_N^{(1)}| = |^3\mathbb{X}_{N-1}| + {}^2D_{N-2} + 2\,{}^{3,1}D_{N-3} + {}^{3,3}D_{N-3} + 2\,{}^{3,4}D_{N-3} + {}^{3,6}D_{N-3}.$$

The generating functions are defined in the obvious way preserving the upper left type indices: $\sum_{N\geq 0} {}^{\cdots}D_N z^N = {}^{\cdots}D(z)$. They are all anchored at $^{\cdots}D_0 = 1$ and zero for negative N.

(1) The three regions in $^2\mathbb{X}_N$ are populated as before, but now also accepting elements of $^3\mathbb{X}$ in their subregions such that their values differ from the values of $\bar{D}_N$ of Equation (99):

$$(106) \qquad {}^2D(z) = \frac{1}{2}\,{}^3X(z)[^3X^2(z) + {}^3X(z^2)].$$

(2) In $^{3,1}\mathbb{X}_N$ region *123* is populated without restriction. The remaining 6 regions associated via symmetry are then incorporated with $t_j \mapsto {}^3X^2(z^j)$, $j \geq 1$, in the cycle index, so

$$(107) \qquad {}^{3,1}D(z) = \frac{1}{6}\,{}^3X(z)[^3X^6(z) + 3\,{}^3X^2(z)\,{}^3X^2(z^2) + 2\,{}^3X^2(z^3)].$$

N	$\lvert^3\mathbb{X}_N\rvert$	1	2	3	4	5	6	7	8	9
1	1	1								
2	3	2	1							
3	14	11	2	1						
4	61	44	14	2	1					
5	252	169	66	14	2	1				
6	1019	609	323	70	14	2	1			
7	4127	2253	1431	356	70	14	2	1		
8	17242	8779	6320	1695	361	70	14	2	1	
9	74007	36319	27420	8081	1739	361	70	14	2	1
10	325615	157297	119821	37849	8455	1745	361	70	14	2	1
11	1458604	701901	528557	176894	40549	8510	1745	361	70	14	2	1

TABLE 10. The number of topologies of nested N circles intersecting at most as triples, $\lvert^3\mathbb{X}_N\rvert$, and the subcounts with f factors, $\lvert^3\mathbb{X}_N^{(f)}\rvert$, $1 \le f \le N$. The row sums are the Euler transform of the column $f = 1$.

(3) Region 2 in $^{3,3}\mathbb{X}_N$ is populated without restrictions, which contributes a factor $^3X(z)$. The pair regions 1 and 12 are populated without restrictions with is represented by $f(z) = {}^3X^2(z)$. The regions 3 and 23 associated to them via symmetry are then incorporated with $t_j \mapsto f(z^j)$ in the cycle index, so

$$(108) \qquad {}^{3,3}D(z) = \frac{1}{2}\,{}^3X(z)[{}^3X^4(z) + {}^3X^2(z^2)].$$

(4) In $^{3,4}\mathbb{X}_N$ region 123 is populated without restriction which contributes a factor $^3X(z)$. Regions 1 and 12 are represented by $f(z)$ and $\overline{2}$ is represented by $^3X(z)$. Substituting $t_j = f(z^j)$, $t'_j = {}^3X(z)$ in the cycle index yields

$$(109) \qquad {}^{3,4}D(z) = \frac{1}{4}\,{}^3X(z)[{}^3X^4(z) + {}^3X^2(z^2)][{}^3X^2(z) + {}^3X(z^2)].$$

(5) The five regions in $^{3,6}\mathbb{X}_N$ are populated without restrictions:

$$(110) \qquad {}^{3,6}D(z) = {}^3X^5(z).$$

The numerical evaluation of the recurrences leads to Table 10. The first difference in comparison to Table 9 is where the 6 additional topologies offer new branches as soon as at least 3 circles are involved: $\lvert^3\mathbb{X}_3^{(1)}\rvert = \lvert^2\mathbb{X}_3^{(1)}\rvert + 6$.

In a wider context one would like to construct and count all circle sets of N circles with an arbitrary number of intersections [11, A250001]. That is out of reach of this paper; in Table 10 that analysis is only complete up to $N = 3$.

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