PLURIASSOCIATIVE ALGEBRAS I: THE PLURIASSOCIATIVE OPERAD

SAMUELE GIRAUDO

Abstract. Diassociative algebras form a categoy of algebras recently introduced by Loday. A diassociative algebra is a vector space endowed with two associative binary operations satisfying some very natural relations. Any diassociative algebra is an algebra over the diassociative operad, and, among its most notable properties, this operad is the Koszul dual of the dendriform operad. We introduce here, by adopting the point of view and the tools offered by the theory of operads, a generalization on a nonnegative integer parameter γ of diassociative algebras, called γ-pluriassociative algebras, so that 1-pluriassociative algebras are diassociative algebras. Pluriassociative algebras are vector spaces endowed with 2γ associative binary operations satisfying some relations. We provide a complete study of the γ-pluriassociative operads, the underlying operads of the category of γ-pluriassociative algebras. We exhibit a realization of these operads, establish several presentations by generators and relations, compute their Hilbert series, show that they are Koszul, and construct the free objects in the corresponding categories. We also study several notions of units in γ-pluriassociative algebras and propose a general way to construct such algebras. This paper ends with the introduction of an analogous generalization of the triassociative operad of Loday and Ronco.

Contents

Date: March 7, 2016.

2010 *Mathematics Subject Classification.* 05E99, 05C05, 18D50.

Key words and phrases. Tree; Rewrite rule; Operad; Koszul operad; Diassociative operad; Triassociative operad; Poincaré-Birkhoff-Witt basis.

Introduction

In the recent years, several algebraic structures on vector spaces based on various sets of combinatorial objects and endowed with more or less complicated operations on these have been considered by algebraic combinatorists. As most famous examples, we can cite free pre-Lie algebras [CL01], which are vector spaces of rooted trees endowed with a grafting product, and free dendriform algebras [Lod01], which are vector spaces of binary trees endowed with two products operating by shuffling binary trees. Other well-known examples include free Zinbiel algebras [Lod95, Lod01] endowing the space of all permutations with a shuffle product, nonassociative permutative algebras [MY91, Liv06] endowing the space of all rooted trees with a grafting product at the root, and duplicial algebras [Lod08] endowing the space of all binary with two grafting operations.

Instead of studying all these algebraic structures separately, it is possible to ask and treat some general questions about these under a uniform point of view. The theory of operads is an efficient tool to regard different categories of algebraic structures in a unified manner. This theory (see [LV12] for a complete exposition and also [Cha08] for an exposition highlighting the combinatorial aspects of the theory) has been introduced in the context of algebraic topology [May72, BV73]. Roughly speaking, an operad is a space of abstract operators consisting in several inputs and one output that can be composed to form bigger ones. The point is that any operad encodes a category of algebras and working with an operad amounts to work with the algebras all together of this category. Moreover, the use of the theory of operads leads to the discovery of connections between differents sorts of algebras by terms of morphisms of operads. As a simple example, the well-known fact that any associative algebra gives rise to a Lie algebra by considering its associator as a Lie bracket comes from the fact that there is a morphism from the underlying operad of the category of Lie algebras to the underlying operad of the category of associative algebras.

The present work is concerned with the definition of a coherent generalization of dialgebras, algebraic structures introduced by Loday in [Lod01]. A dialgebra is a vector space endowed with two associative binary operations $\dashv$ and $\vdash$ satisfying some relations. From a combinatorial point of view, the bases of the free dialgebra over one generator are indexed by ordered pairs (n, k) of integers, denoted by $\mathfrak{e}_{n,k}$, and satisfying $1 \leqslant k \leqslant n$. The operations $\dashv$ and $\vdash$ admit simple set-theoretic descriptions over this basis [Cha05]. In a previous work [Gir12, Gir15], we introduced a new construction for the operad Dias, the underlying operad of the category of diassociative algebras, and we raised the question whether this construction can be extended to obtain operads generalizing Dias and hence, to obtain generalizations of dialgebras.

Let us give some explanations about our construction of Dias. In [Gir12, Gir15], we defined a general functorial construction T producing an operad from any monoid. This construction T sends a monoid M to the operad $\mathsf{T}M$ of all words on M, where M is seen as an alphabet. The arity of a word is its length and the operadic partial composition $u \circ_i v$ of two words u and v of $\mathsf{T}M$ consists in replacing the ith letter u_i of u by a version of v obtained by multpliying to the left all its letters by u_i. The operad Dias is the suboperad of $\mathsf{T}M$, where M is the multiplicative monoid on $\{0, 1\}$, generated by the two words 01 and 10 of arity two. In the

Operad	Objects	Dimensions
Dias_γ	Words on $\{0, 1, \dots, \gamma\}$ with exactly one 0	$n\gamma^{n-1}$
As_γ	γ-corollas	γ
Trias_γ	Words on $\{0, 1, \dots, \gamma\}$ with at least one 0	$(\gamma + 1)^n - \gamma^n$

Table 1. The main operads defined in this paper. All these operads depend on a nonnegative integer parameter γ. The shown dimensions are the ones of the homogeneous components of arities $n \geqslant 2$ of the operads.

present paper, we rely on T to construct a generalization on a nonnegative integer parameter γ of Dias, denoted by Dias_γ, in such a way that $\mathsf{Dias}_1 = \mathsf{Dias}$ and Dias_γ is a suboperad of $\mathsf{Dias}_{\gamma+1}$ for any $\gamma \geqslant 0$. The operads Dias_γ, called γ-pluriassociative operads, are set-operads involving words on the alphabet $\{0, 1, \dots, \gamma\}$ with exactly one occurrence of 0. Besides, this work naturally leads to the consideration and the definition of several new operads. Table 1 summarizes some information about these. We provide for instance a generalization on a nonnegative integer parameter γ of the triassociative operad Trias [LR04], denoted by Trias_γ.

The main rationale for this work is to establish the necessary foundations to propose a generalization on a nonnegative integer parameter γ of dendriform algebras [Lod01]. Since Dias is the Koszul dual [GK94] of the operad Dendr, the underlying operad of the category of dendriform algebras, our objective is to propose the definition of the operads Dendr_γ, defined each as the Koszul dual of Dias_γ. Moreover, since Dias admits a description far simpler than Dendr, starting by constructing a generalization of Dias to obtain a generalization of Dendr by Koszul duality is a convenient path to explore. This strategy is developed in the continutation of this work [Gir16], where the operads Dendr_γ are studied. This lead to new sorts of algebras, providing analogs of dendriform algebras and different from already existing ones (see for instance [LR04, AL04, Ler04, Ler07, Nov14]).

This paper is organized as follows. Section 1 contains a conspectus of the tools used in this paper. We recall here the definition of the construction T [Gir12, Gir15] and provide a reformulation of results of Hoffbeck [Hof10] and Dotsenko and Khoroshkin [DK10] to prove that an operad is Koszul by using convergent rewrite rules. Besides, this part provides self-contained definitions about nonsymmetric operads, algebras over operads, free operads, and rewrite rules on trees. This section ends by some recalls about the diassociative operad and diassociative algebras.

Section 2 is devoted to the introduction and the study of the operad Dias_γ. We begin by detailing the construction of Dias_γ as a suboperad of the operad obtained by the construction T applied on the monoid $\mathcal{M}_\gamma$ with $\{0, 1, \dots, \gamma\}$ as underlying set and with the operation max as product. More precisely, Dias_γ is defined as the suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by the words $0a$ and $a0$ for all $a \in \{1, \dots, \gamma\}$. We then provide a presentation by generators and relations of

Dias_γ (Theorem 2.2.6), and show that it is a Koszul operad (Theorem 2.3.1). We also establish some more properties of this operad: we compute its group of symmetries (Proposition 2.3.2), show that it is a basic operad in the sense of [Val07] (Proposition 2.3.3), and show that it is a rooted operad in the sense of [Cha14] (Proposition 2.3.3). We end this section by introducing an alternating basis of Dias_γ, the K-basis, defined through a partial ordering relation over the words indexing the bases of Dias_γ. After describing how the partial composition of Dias_γ expresses over the K-basis (Theorem 2.3.7), we provide a presentation of Dias_γ over this basis (Proposition 2.3.8). Despite the fact that this alternative presentation is more complex than the original one of Dias_γ provided by Theorem 2.2.6, the computation of the Koszul dual Dendr_γ of Dias_γ from this second presentation leads to a surprisingly plain presentation of Dendr_γ considered later in [Gir16].

In Section 3, algebras over Dias_γ, called γ-pluriassociative algebras, are studied. The free γ-pluriassociative algebra over one generator is described as a vector space of words on the alphabet $\{0, 1, \dots, \gamma\}$ with exactly one occurrence of 0, endowed with 2γ binary operations (Proposition 3.1.1). We next study two different notions of units in γ-pluriassociative algebras, the bar-units and the wire-units, that are generalizations of definitions of Loday introduced into the context of diassociative algebras [Lod01]. We show that the presence of a wire-unit in a γ-pluriassociative algebra leads to many consequences on its structure (Proposition 3.2.1). Besides, we describe a general construction M to obtain γ-pluriassociative algebras by starting from γ-multiprojection algebras, that are algebraic structures with γ associative products and endowed with γ endomorphisms with extra relations (Theorem 3.3.2). The main interest of the construction M is that γ-multiprojection algebras are simpler algebraic structures than γ-pluriassociative algebras. The bar-units and wire-units of the γ-pluriassociative algebras obtained by this construction are then studied (Proposition 3.3.3). We end this section by listing five examples of γ-pluriassociative algebras constructed from γ-multiprojection algebras, including the free γ-pluriassociative algebra over one generator considered in Section 3.1.3.

Finally, by using almost the same tools as the one used in Section 2, we propose in Section 4 a generalization on a nonnegative integer parameter γ of the triassociative operad Trias of Loday and Ronco [LR04], denoted by Trias_γ. This follows a very simple idea: like Dias_γ, Trias_γ is defined as a suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by the same generators as those of Dias_γ, plus the word 00. In a previous work [Gir12, Gir15], we showed that Trias_1 is the triassociative operad. We provide here an expression for the Hilbert series of Trias_γ obtained from the description of its elements (Proposition 4.1.1) and a presentation (Theorem 4.2.1).

Acknowledgements. The author would like to thank Jean-Yves-Thibon for its pertinent remarks and questions about this work when it was in progress. Thanks are addressed to Frederic Chapoton and Eric Hoffbeck for answering some questions of the author respectively about the dendriform and diassociative operads, and Koszulity of operads. The author thanks also Vladimir Dotsenko and Bruno Vallette for pertinent bibliographic suggestions. Finally, the author warmly thanks the referee for his very careful reading and his suggestions, improving the quality of the paper.

Notations and general conventions. All the algebraic structures of this article have a field of characteristic zero $\mathbb{K}$ as ground field. If S is a set, $\mathrm{Vect}(S)$ denotes the linear span of the elements of S. For any integers a and c, $[a, c]$ denotes the set $\{b \in \mathbb{N} : a \leqslant b \leqslant c\}$ and $[n]$, the set $[1, n]$. The cardinality of a finite set S is denoted by $\# S$. If u is a word, its letters are indexed from left to right from 1 to its length $|u|$. For any $i \in [|u|]$, u_i is the letter of u at position i. If a is a letter and n is a nonnegative integer, a^n denotes the word consisting in n occurrences of a. Notice that a^0 is the empty word ϵ.

1. Preliminaries: algebraic structures and main tools

This preliminary section sets our conventions and notations about operads and algebras over an operad, and describes the main tools we will use. The definitions and some properties of the diassociative operad are also recalled. This section does not contains new results but it is a self-contained set of definitions about operads intended to readers familiar with algebra or combinatorics but not necessarily with operadic theory.

1.1. **Operads and algebras over an operad.**

We list here several staple definitions about operads and algebras over an operad. We present also an important tool for this work: the construction T producing operads from monoids.

1.1.1. *Operads.* A *nonsymmetric operad in the category of vector spaces*, or a *nonsymmetric operad* for short, is a graded vector space $\mathcal{O} := \bigoplus_{n \geqslant 1} \mathcal{O}(n)$ together with linear maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \qquad n, m \geqslant 1, i \in [n], \tag{1.1.1}$$

called *partial compositions*, and a distinguished element $\mathbb{1} \in \mathcal{O}(1)$, the *unit* of $\mathcal{O}$. This data has to satisfy the three relations

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \qquad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i \in [n], j \in [m], \tag{1.1.2a}$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \qquad x \in \mathcal{O}(n), y \in \mathcal{O}(m), z \in \mathcal{O}(k), i < j \in [n], \tag{1.1.2b}$$

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}, \qquad x \in \mathcal{O}(n), i \in [n]. \tag{1.1.2c}$$

Since we shall consider in this paper mainly nonsymmetric operads, we shall call these simply *operads*. Moreover, all considered operads are such that $\mathcal{O}(1)$ has dimension 1.

If x is an element of $\mathcal{O}$ such that $x \in \mathcal{O}(n)$ for a $n \geqslant 1$, we say that n is the *arity* of x and we denote it by $|x|$. An element x of $\mathcal{O}$ of arity 2 is *associative* if $x \circ_1 x = x \circ_2 x$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are operads, a linear map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad morphism* if it respects arities, sends the unit of $\mathcal{O}_1$ to the unit of $\mathcal{O}_2$, and commutes with partial composition maps. We say that $\mathcal{O}_2$ is a *suboperad* of $\mathcal{O}_1$ if $\mathcal{O}_2$ is a graded subspace of $\mathcal{O}_1$, and $\mathcal{O}_1$ and $\mathcal{O}_2$ have the same unit and the same partial compositions. For any set $G \subseteq \mathcal{O}$, the *operad generated by* G is the smallest suboperad of $\mathcal{O}$ containing G. When the operad generated by G is $\mathcal{O}$ itself and G is minimal with respect to inclusion among the subsets of $\mathcal{O}$ satisfying this property, G is a *generating set* of $\mathcal{O}$ and its elements are *generators* of $\mathcal{O}$. An *operad ideal* of $\mathcal{O}$ is a graded subspace I of $\mathcal{O}$ such that, for any $x \in \mathcal{O}$ and $y \in I$, $x \circ_i y$ and $y \circ_j x$ are in I for all valid integers i and j. Given an operad ideal I of $\mathcal{O}$, one can define the *quotient operad* $\mathcal{O}/_I$ of $\mathcal{O}$ by I in the usual way. When $\mathcal{O}$ is such that all $\mathcal{O}(n)$ are finite for all $n \geqslant 1$, the *Hilbert series* of $\mathcal{O}$ is the series $\mathcal{H}_{\mathcal{O}}(t)$ defined by

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n \geqslant 1} \dim \mathcal{O}(n) t^n. \tag{1.1.3}$$

Instead of working with the partial composition maps of $\mathcal{O}$, it is something useful to work with the maps

$$\circ : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n), \qquad n, m_1, \ldots, m_n \geqslant 1, \tag{1.1.4}$$

linearly defined for any $x \in \mathcal{O}$ of arity n and $y_1, \ldots, y_{n-1}, y_n \in \mathcal{O}$ by

$$x \circ (y_1, \ldots, y_{n-1}, y_n) := (\ldots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \ldots) \circ_1 y_1. \tag{1.1.5}$$

These maps are called *composition maps* of $\mathcal{O}$.

1.1.2. *Set-operads.* Instead of being a direct sum of vector spaces $\mathcal{O}(n)$, $n \geqslant 1$, $\mathcal{O}$ can be a graded disjoint union of sets. In this context, $\mathcal{O}$ is a *set-operad*. All previous definitions remain valid by replacing direct sums $\oplus$ by disjoint unions $\sqcup$, tensor products $\otimes$ by Cartesian products $\times$, and vector space dimensions dim by set cardinalities $\#$. Moreover, in the context of set-operads, we work with *operad congruences* instead of operad ideals. An operad congruence on a set-operad $\mathcal{O}$ is an equivalence relation $\equiv$ on $\mathcal{O}$ such that all elements of a same $\equiv$-equivalence class have the same arity and for all elements x, x', y, and y' of $\mathcal{O}$, $x \equiv x'$ and $y \equiv y'$ imply $x \circ_i y \equiv x' \circ_i y'$ for all valid integers i. The *quotient operad* $\mathcal{O}/_{\equiv}$ of $\mathcal{O}$ by $\equiv$ is the set-operad defined in the usual way.

Any set-operad $\mathcal{O}$ gives naturally rise to an operad on $\mathrm{Vect}(\mathcal{O})$ by extending the partial compositions of $\mathcal{O}$ by linearity. Besides this, any equivalence relation $\leftrightarrow$ of $\mathcal{O}$ such that all elements of a same $\leftrightarrow$-equivalence class have the same arity induces a subspace of $\mathrm{Vect}(\mathcal{O})$ generated by all $x - x'$ such that $x \leftrightarrow x'$, called *space induced* by $\leftrightarrow$. In particular, any operad congruence $\equiv$ on $\mathcal{O}$ induces an operad ideal of $\mathrm{Vect}(\mathcal{O})$.

1.1.3. *From monoids to operads.* In a previous work [Gir12, Gir15], the author introduced a construction which, from any monoid, produces an operad. This construction is described as follows. Let $\mathcal{M}$ be a monoid with an associative product $\bullet$ admitting a unit 1. We denote by $\mathsf{T}\mathcal{M}$ the operad $\mathsf{T}\mathcal{M} := \bigoplus_{n \geqslant 1} \mathsf{T}\mathcal{M}(n)$ where for all $n \geqslant 1$,

$$\mathsf{T}\mathcal{M}(n) := \mathrm{Vect}\left(\{u_1 \ldots u_n : u_i \in \mathcal{M} \text{ for all } i \in [n]\}\right). \tag{1.1.6}$$

The partial composition of two words $u \in \mathsf{T}\mathcal{M}(n)$ and $v \in \mathsf{T}\mathcal{M}(m)$ is linearly defined by

$$u \circ_i v := u_1 \ldots u_{i-1} \, (u_i \bullet v_1) \, \ldots \, (u_i \bullet v_m) \, u_{i+1} \ldots u_n, \qquad i \in [n]. \tag{1.1.7}$$

The unit of $\mathsf{T}\mathcal{M}$ is $\mathbb{1} := 1$. In other words, $\mathsf{T}\mathcal{M}$ is the vector space of words on $\mathcal{M}$ seen as an alphabet and the partial composition returns to insert a word v onto the ith letter u_i of a word u together with a left multiplication by u_i.

1.1.4. *Algebras over an operad.* Any operad $\mathcal{O}$ encodes a category of algebras whose objects are called *$\mathcal{O}$-algebras*. An $\mathcal{O}$-algebra $\mathcal{A}_\mathcal{O}$ is a vector space endowed with a right action

$$\cdot : \mathcal{A}_\mathcal{O}^{\otimes n} \otimes \mathcal{O}(n) \to \mathcal{A}_\mathcal{O}, \qquad n \geqslant 1, \tag{1.1.8}$$

satisfying the relations imposed by the structure of $\mathcal{O}$, that are

$$\begin{aligned}(e_1 \otimes \cdots \otimes e_{n+m-1}) \cdot (x \circ_i y) = \\ (e_1 \otimes \cdots \otimes e_{i-1} \otimes (e_i \otimes \cdots \otimes e_{i+m-1}) \cdot y \otimes e_{i+m} \otimes \cdots \otimes e_{n+m-1}) \cdot x,\end{aligned} \tag{1.1.9}$$

for all $e_1 \otimes \cdots \otimes e_{n+m-1} \in \mathcal{A}_\mathcal{O}^{\otimes n+m-1}$, $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$. Notice that, by (1.1.9), if G is a generating set of $\mathcal{O}$, it is enough to define the action of each $x \in G$ on $\mathcal{A}_\mathcal{O}^{\otimes |x|}$ to wholly define $\cdot$.

In other words, any element x of $\mathcal{O}$ of arity n plays the role of a linear operation

$$x : \mathcal{A}_\mathcal{O}^{\otimes n} \to \mathcal{A}_\mathcal{O}, \tag{1.1.10}$$

taking n elements of $\mathcal{A}_\mathcal{O}$ as inputs and computing an element of $\mathcal{A}_\mathcal{O}$. By a slight but convenient abuse of notation, for any $x \in \mathcal{O}(n)$, we shall denote by $x(e_1, \dots, e_n)$, or by $e_1 \, x \, e_2$ if x has arity 2, the element $(e_1 \otimes \cdots \otimes e_n) \cdot x$ of $\mathcal{A}_\mathcal{O}$, for any $e_1 \otimes \cdots \otimes e_n \in \mathcal{A}_\mathcal{O}^{\otimes n}$. Observe that by (1.1.9), any associative element of $\mathcal{O}$ gives rise to an associative operation on $\mathcal{A}_\mathcal{O}$.

Arrows in the category of $\mathcal{O}$-algebras are *$\mathcal{O}$-algebra morphisms*, that are linear maps $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ between two $\mathcal{O}$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that

$$\phi(x(e_1, \dots, e_n)) = x(\phi(e_1), \dots, \phi(e_n)), \tag{1.1.11}$$

for all $e_1, \dots, e_n \in \mathcal{A}_1$ and $x \in \mathcal{O}(n)$. We say that $\mathcal{A}_2$ is an *$\mathcal{O}$-subalgebra* of $\mathcal{A}_1$ if $\mathcal{A}_2$ is a subspace of $\mathcal{A}_1$ and $\mathcal{A}_1$ and $\mathcal{A}_2$ are endowed with the same right action of $\mathcal{O}$. If G is a set of elements of an $\mathcal{O}$-algebra $\mathcal{A}$, the *$\mathcal{O}$-algebra generated by G* is the smallest $\mathcal{O}$-subalgebra of $\mathcal{A}$ containing G. When the $\mathcal{O}$-algebra generated by G is $\mathcal{A}$ itself and G is minimal with respect to inclusion among the subsets of $\mathcal{A}$ satisfying this property, G is a *generating set* of $\mathcal{A}$ and its elements are *generators* of $\mathcal{A}$. An *$\mathcal{O}$-algebra ideal* of $\mathcal{A}$ is a subspace I of $\mathcal{A}$ such that for all operation x of $\mathcal{O}$ of arity n and elements e_1, …, e_n of $\mathcal{O}$, $x(e_1, \dots, e_n)$ is in I whenever there is a $i \in [n]$ such that e_i is in I.

The *free $\mathcal{O}$-algebra over one generator* is the $\mathcal{O}$-algebra $\mathcal{F}_\mathcal{O}$ defined in the following way. We set $\mathcal{F}_\mathcal{O} := \oplus_{n \geqslant 1} \mathcal{F}_\mathcal{O}(n) := \oplus_{n \geqslant 1} \mathcal{O}(n)$, and for any $e_1, \dots, e_n \in \mathcal{F}_\mathcal{O}$ and $x \in \mathcal{O}(n)$, the right action of x on $e_1 \otimes \cdots \otimes e_n$ is defined by

$$x(e_1, \dots, e_n) := x \circ (e_1, \dots, e_n). \tag{1.1.12}$$

Then, any element x of $\mathcal{O}(n)$ endows $\mathcal{F}_\mathcal{O}$ with an operation

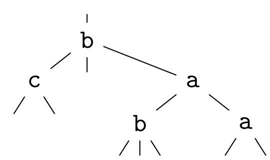
$$x : \mathcal{F}_\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{F}_\mathcal{O}(m_n) \to \mathcal{F}_\mathcal{O}(m_1 + \cdots + m_n) \tag{1.1.13}$$

respecting the graduation of $\mathcal{F}_\mathcal{O}$.

1.2. **Free operads, rewrite rules, and Koszulity.** We recall here a description of free operads through syntax trees and presentations of operads by generators and relations. The Koszul property for operads is a very important notion in this paper and its sequel [Gir16]. We recall it and describe an already known criterion to prove that a set-operad is Koszul by passing by rewrite rules on syntax trees.

1.2.1. *Syntax trees.* Unless otherwise specified, we use in the sequel the standard terminology (*i.e.*, *node*, *edge*, *root*, *parent*, *child*, *path*, *ancestor*, *etc.*) about planar rooted trees [Knu97]. Let $\mathfrak{t}$ be a planar rooted tree. The *arity* of a node of $\mathfrak{t}$ is its number of children. An *internal node* (resp. a *leaf*) of $\mathfrak{t}$ is a node with a nonzero (resp. null) arity. Given an internal node x of $\mathfrak{t}$, due to the planarity of $\mathfrak{t}$, the children of x are totally ordered from left to right and are thus indexed from 1 to the arity of x. If y is a child of x, y defines a *subtree* of $\mathfrak{t}$, that is the planar rooted tree with root y and consisting in the nodes of $\mathfrak{t}$ that have y as ancestor. We shall call *ith subtree* of x the subtree of $\mathfrak{t}$ rooted at the ith child of x. A *partial subtree* of $\mathfrak{t}$ is a subtree of $\mathfrak{t}$ in which some internal nodes have been replaced by leaves and its descendants has been forgotten. Besides, due to the planarity of $\mathfrak{t}$, its leaves are totally ordered from left to right and thus are indexed from 1 to the arity of $\mathfrak{t}$. In our graphical representations, each tree is depicted so that its root is the uppermost node.

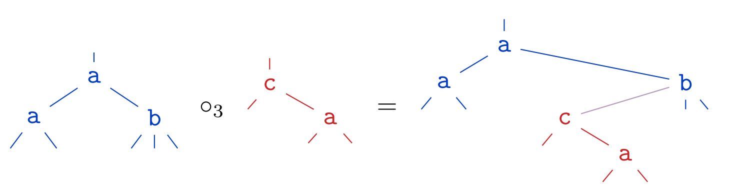
Let $S := \sqcup_{n \geqslant 1} S(n)$ be a graded set. By extension, we say that the *arity* of an element x of S is n provided that $x \in S(n)$. A *syntax tree on S* is a planar rooted tree such that its internal nodes of arity n are labeled on elements of arity n of S. The *degree* (resp. *arity*) of a syntax tree is its number of internal nodes (resp. leaves). For instance, if $S := S(2) \sqcup S(3)$ with $S(2) := \{\mathtt{a}, \mathtt{c}\}$ and $S(3) := \{\mathtt{b}\}$,



(1.2.1)

is a syntax tree on S of degree 5 and arity 8. Its root is labeled by $\mathtt{b}$ and has arity 3.

1.2.2. *Free operads.* Let S be a graded set. The *free operad* $\mathbf{Free}(S)$ *over* S is the operad wherein for any $n \geqslant 1$, $\mathbf{Free}(S)(n)$ is the vector space of syntax trees on S of arity n, the partial composition $\mathfrak{s} \circ_i \mathfrak{t}$ of two syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ on S consists in grafting the root of $\mathfrak{t}$ on the ith leaf of $\mathfrak{s}$, and its unit is the tree consisting in one leaf. For instance, if $S := S(2) \sqcup S(3)$ with $S(2) := \{\mathtt{a}, \mathtt{c}\}$ and $S(3) := \{\mathtt{b}\}$, one has in $\mathbf{Free}(S)$,



(1.2.2)

We denote by $\operatorname{cor} : S \to \mathbf{Free}(S)$ the inclusion map, sending any x of S to the *corolla* labeled by x, that is the syntax tree consisting in one internal node labeled by x attached to a required number of leaves. In the sequel, if required by the context, we shall implicitly see any element x of S as the corolla $\operatorname{cor}(x)$ of $\mathbf{Free}(S)$. For instance, when x and y are two elements of S, we shall simply denote by $x \circ_i y$ the syntax tree $\operatorname{cor}(x) \circ_i \operatorname{cor}(y)$ for all valid integers i.

For any operad $\mathcal{O}$, by seeing $\mathcal{O}$ as a graded set, $\mathbf{Free}(\mathcal{O})$ is the free operad of the syntax trees linearly labeled by elements of $\mathcal{O}$. The *evaluation map* of $\mathcal{O}$ is the map

$$\mathrm{eval}_{\mathcal{O}} : \mathbf{Free}(\mathcal{O}) \to \mathcal{O}, \tag{1.2.3}$$

recursively defined by

$$\mathrm{eval}_{\mathcal{O}}(\mathfrak{t}) := \begin{cases} \mathbb{1} & \text{if } \mathfrak{t} \text{ is the leaf,} \\ x \circ (\mathrm{eval}_{\mathcal{O}}(\mathfrak{s}_1), \dots, \mathrm{eval}_{\mathcal{O}}(\mathfrak{s}_n)) & \text{otherwise,} \end{cases} \tag{1.2.4}$$

where $\mathbb{1}$ is the unit of $\mathcal{O}$, x is the label of the root of $\mathfrak{t}$, and $\mathfrak{s}_1$, ..., $\mathfrak{s}_n$ are, from left to right, the subtrees of the root of $\mathfrak{t}$. In other words, any tree $\mathfrak{t}$ of $\mathbf{Free}(\mathcal{O})$ can be seen as a tree-like expression for an element $\mathrm{eval}_{\mathcal{O}}(\mathfrak{t})$ of $\mathcal{O}$. Moreover, by induction on the degree of $\mathfrak{t}$, it appears that $\mathrm{eval}_{\mathcal{O}}$ is a well-defined surjective operad morphism.

1.2.3. *Presentations by generators and relations.* A *presentation* of an operad $\mathcal{O}$ consists in a pair $(\mathfrak{G}, \mathfrak{R})$ such that $\mathfrak{G} := \sqcup_{n \geqslant 1} \mathfrak{G}(n)$ is a graded set, $\mathfrak{R}$ is a subspace of $\mathbf{Free}(\mathfrak{G})$, and $\mathcal{O}$ is isomorphic to $\mathbf{Free}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$, where $\langle \mathfrak{R} \rangle$ is the operad ideal of $\mathbf{Free}(\mathfrak{G})$ generated by $\mathfrak{R}$. We call $\mathfrak{G}$ the *set of generators* and $\mathfrak{R}$ the *space of relations* of $\mathcal{O}$. We say that $\mathcal{O}$ is *quadratic* if one can exhibit a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathcal{O}$ such that $\mathfrak{R}$ is a homogeneous subspace of $\mathbf{Free}(\mathfrak{G})$ consisting in syntax trees of degree 2. Besides, we say that $\mathcal{O}$ is *binary* if one can exhibit a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathcal{O}$ such that $\mathfrak{G}$ is concentrated in arity 2.

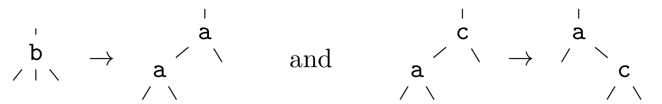
With knowledge of a presentation $(\mathfrak{G}, \mathfrak{R})$ of $\mathcal{O}$, it is easy to describe the category of the $\mathcal{O}$-algebras. Indeed, by denoting by $\pi : \mathbf{Free}(\mathfrak{G}) \to \mathbf{Free}(\mathfrak{G})/_{\langle \mathfrak{R} \rangle}$ the canonical surjection map, the category of $\mathcal{O}$-algebras is the category of vector spaces $\mathcal{A}_{\mathcal{O}}$ endowed with maps $\pi(g)$, $g \in \mathfrak{G}$, satisfying for all $r \in \mathfrak{R}$ the relations

$$r(e_1, \dots, e_n) = 0, \tag{1.2.5}$$

for all $e_1, \dots, e_n \in \mathcal{A}_{\mathcal{O}}$, where n is the arity of r.

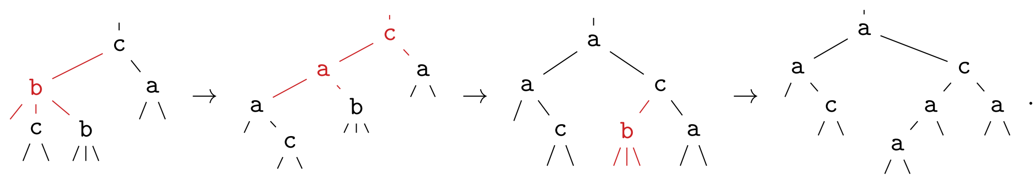
1.2.4. *Rewrite rules.* Let S be a graded set. A *rewrite rule* on syntax trees on S is a binary relation $\to$ on $\mathbf{Free}(S)$ whenever for all trees $\mathfrak{s}$ and $\mathfrak{t}$ of $\mathbf{Free}(S)$, $\mathfrak{s} \to \mathfrak{t}$ only if $\mathfrak{s}$ and $\mathfrak{t}$ have the same arity. When $\to$ involves only syntax trees of degree two, $\to$ is *quadratic*. We say that a syntax tree $\mathfrak{s}'$ can be *rewritten* by $\to$ into $\mathfrak{t}'$ if there exist two syntax trees $\mathfrak{s}$ and $\mathfrak{t}$ satisfying $\mathfrak{s} \to \mathfrak{t}$ and $\mathfrak{s}'$ has a partial subtree equal to $\mathfrak{s}$ such that, by replacing it by $\mathfrak{t}$ in $\mathfrak{s}'$, we obtain $\mathfrak{t}'$. By a slight but convenient abuse of notation, we denote by $\mathfrak{s}' \to \mathfrak{t}'$ this property. When a syntax tree $\mathfrak{t}$ can be obtained by performing a sequence of $\to$-rewritings from a syntax tree $\mathfrak{s}$, we say that $\mathfrak{s}$ is *rewritable* by $\to$ into $\mathfrak{t}$ and we denote this property by $\mathfrak{s} \overset{*}{\to} \mathfrak{t}$. For instance, for

$S := S(2) \sqcup S(3)$ with $S(2) := \{\mathtt{a}, \mathtt{c}\}$ and $S(3) := \{\mathtt{b}\}$, consider the rewrite rule $\to$ on $\mathbf{Free}(S)$ satisfying



(1.2.6)

We then have the following sequence of rewritings



(1.2.7)

We shall use the standard terminology (*confluent*, *terminating*, *convergent*, *normal form*, *critical pair*, *etc.*) about rewrite rules (see [BN98]).

Any rewrite rule $\to$ on $\mathbf{Free}(S)$ defines an operad congruence $\equiv_{\to}$ on $\mathbf{Free}(S)$ seen as a set-operad, the *operad congruence induced* by $\to$, as the finest operad congruence on $\mathbf{Free}(S)$ containing the reflexive, symmetric, and transitive closure of $\to$.

1.2.5. *Koszulity.* A quadratic operad $\mathcal{O}$ is *Koszul* if its Koszul complex is acyclic [GK94, LV12]. In this work, to prove the Koszulity of an operad $\mathcal{O}$, we shall make use of a combinatorial tool introduced by Hoffbeck [Hof10] (see also [LV12]) consisting in exhibiting a particular basis of $\mathcal{O}$, a so-called *Poincaré-Birkhoff-Witt basis*.

In this paper, we shall use this tool only in the context of set-operads, which reformulates, thanks to the work of Dotsenko and Khoroshkin [DK10], as follows. A set-operad $\mathcal{O}$ is Kosuzl if there is a graded set S and a rewrite rule $\to$ on $\mathbf{Free}(S)$ such that $\mathcal{O}$ is isomorphic to $\mathbf{Free}(S)/_{\equiv_{\to}}$ and $\to$ is a convergent quadratic rewrite rule. Moreover, the set of normal forms of $\to$ forms a Poincaré-Birkhoff-Witt basis of $\mathcal{O}$.

1.3. **Diassociative operad.**

We recall here, by using the notions presented during the previous sections, the definition and some properties of the diassociative operad.

The *diassociative operad* Dias was introduced by Loday [Lod01] as the operad admitting the presentation $(\mathfrak{G}_{\mathsf{Dias}}, \mathfrak{R}_{\mathsf{Dias}})$ where $\mathfrak{G}_{\mathsf{Dias}} := \mathfrak{G}_{\mathsf{Dias}}(2) := \{\dashv, \vdash\}$ and $\mathfrak{R}_{\mathsf{Dias}}$ is the space induced by the equivalence relation $\equiv$ satisfying

$$\dashv \circ_1 \vdash \equiv \vdash \circ_2 \dashv, \tag{1.3.1a}$$

$$\dashv \circ_1 \dashv \equiv \dashv \circ_2 \dashv \equiv \dashv \circ_2 \vdash, \tag{1.3.1b}$$

$$\vdash \circ_1 \dashv \equiv \vdash \circ_1 \vdash \equiv \vdash \circ_2 \vdash. \tag{1.3.1c}$$

Note that Dias is a binary and quadratic operad.

This operad admits the following realization [Cha05]. For any $n \geqslant 1$, $\mathsf{Dias}(n)$ is the linear span of the $\mathfrak{e}_{n,k}$, $k \in [n]$, and the partial compositions linearly satisfy, for all $n, m \geqslant 1$, $k \in [n]$, $\ell \in [m]$, and $i \in [n]$,

$$\mathfrak{e}_{n,k} \circ_i \mathfrak{e}_{m,\ell} = \begin{cases} \mathfrak{e}_{n+m-1,k+m-1} & \text{if } i < k, \\ \mathfrak{e}_{n+m-1,k+\ell-1} & \text{if } i = k, \\ \mathfrak{e}_{n+m-1,k} & \text{otherwise } (i > k). \end{cases} \tag{1.3.2}$$

Since the partial composition of two basis elements of Dias produces exactly one basis element, Dias is well-defined as a set-operad. Moreover, this realization shows that $\dim \mathsf{Dias}(n) = n$ and hence, the Hilbert series of Dias satisfies

$$\mathcal{H}_{\mathsf{Dias}}(t) = \frac{t}{(1-t)^2}. \tag{1.3.3}$$

From the presentation of Dias, we deduce that any Dias-algebra, also called *diassociative algebra*, is a vector space $\mathcal{A}_{\mathsf{Dias}}$ endowed with linear operations $\dashv$ and $\vdash$ satisfying the relations encoded by (1.3.1a)—(1.3.1c).

From the realization of Dias, we deduce that the free diassociative algebra $\mathcal{F}_{\mathsf{Dias}}$ over one generator is the vector space Dias endowed with the linear operations

$$\dashv, \vdash \colon \mathcal{F}_{\mathsf{Dias}} \otimes \mathcal{F}_{\mathsf{Dias}} \to \mathcal{F}_{\mathsf{Dias}}, \tag{1.3.4}$$

satisfying, for all $n, m \geqslant 1$, $k \in [n]$, $\ell \in [m]$,

$$\mathfrak{e}_{n,k} \dashv \mathfrak{e}_{m,\ell} = (\mathfrak{e}_{n,k} \otimes \mathfrak{e}_{m,\ell}) \cdot \mathfrak{e}_{2,1} = (\mathfrak{e}_{2,1} \circ_2 \mathfrak{e}_{m,\ell}) \circ_1 \mathfrak{e}_{n,k} = \mathfrak{e}_{n+m,k}, \tag{1.3.5}$$

and

$$\mathfrak{e}_{n,k} \vdash \mathfrak{e}_{m,\ell} = (\mathfrak{e}_{n,k} \otimes \mathfrak{e}_{m,\ell}) \cdot \mathfrak{e}_{2,2} = (\mathfrak{e}_{2,2} \circ_2 \mathfrak{e}_{m,\ell}) \circ_1 \mathfrak{e}_{n,k} = \mathfrak{e}_{n+m,n+\ell}. \tag{1.3.6}$$

As shown in [Gir12, Gir15], the diassociative operad is isomorphic to the suboperad of $\mathsf{T}\mathcal{M}$ generated by 01 and 10 where $\mathcal{M}$ is the multiplicative monoid on $\{0, 1\}$. The concerned isomorphism sends any $\mathfrak{e}_{n,k}$ of Dias to the word $0^{k-1}\,1\,0^{n-k}$ of $\mathsf{T}\mathcal{M}$.

2. Pluriassociative operads

In this section, we define the main object of this work: a generalization on a nonnegative integer parameter γ of the diassociative operad. We provide a complete study of this new operad.

2.1. Construction and first properties.

2.1. Construction and first properties. We define here our generalization of the diassociative operad using the functor T (whose definition is recalled in Section 1.1.3). We then describe the elements and establish the Hilbert series of our generalization.

2.1.1. *Construction.* For any integer $\gamma \geqslant 0$, let $\mathcal{M}_\gamma$ be the monoid $\{0\} \cup [\gamma]$ with the binary operation max as product, denoted by $\uparrow$. We define Dias_γ as the suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by

$$\{0a, a0 : a \in [\gamma]\}. \tag{2.1.1}$$

By definition, Dias_γ is the vector space of words that can be obtained by partial compositions of words of (2.1.1). We have, for instance,

$$\mathsf{Dias}_2(1) = \mathrm{Vect}(\{0\}), \tag{2.1.2}$$

$$\mathsf{Dias}_2(2) = \mathrm{Vect}(\{01, 02, 10, 20\}), \tag{2.1.3}$$

$$\mathsf{Dias}_2(3) = \mathrm{Vect}(\{011, 012, 021, 022, 101, 102, 201, 202, 110, 120, 210, 220\}), \tag{2.1.4}$$

and

$$211\mathbf{2}01 \circ_4 31103 = 2113222301, \tag{2.1.5}$$

$$111\mathbf{1}01 \circ_3 20 = 1121101, \tag{2.1.6}$$

$$1\mathbf{0}13 \circ_2 210 = 121013. \tag{2.1.7}$$

It follows immediately from the definition of Dias_γ as a suboperad of $\mathsf{T}\mathcal{M}_\gamma$ that Dias_γ is a set-operad. Indeed, any partial composition of two basis elements of Dias_γ gives rises to exactly one basis element. We then shall see Dias_γ as a set-operad over all Section 2.

Notice that $\mathsf{Dias}_\gamma(2)$ is the set (2.1.1) of generators of Dias_γ. Besides, observe that Dias_0 is the trivial operad and that Dias_γ is a suboperad of $\mathsf{Dias}_{\gamma+1}$. We call Dias_γ the *γ-pluriassociative operad.*

2.1.2. *Elements and dimensions.*

Proposition 2.1.1. *For any integer $\gamma \geqslant 0$, as a set-operad, the underlying set of Dias_γ is the set of the words on the alphabet $\{0\} \cup [\gamma]$ containing exactly one occurrence of 0.*

Proof. Let us show that any word x of Dias_γ satisfies the statement of the proposition by induction on the length n of x. This is true when $n = 1$ because we necessarily have $x = 0$. Otherwise, when $n \geqslant 2$, there is a word y of Dias_γ of length $n-1$ and a generator g of Dias_γ such that $x = y \circ_i g$ for a $i \in [n-1]$. Then, x is obtained by replacing the ith letter a of y by the factor $u := u_1u_2$ where $u_1 := a \uparrow g_1$ and $u_2 := a \uparrow g_2$. Since g contains exactly one 0, this operation consists in inserting a nonzero letter of $[\gamma]$ into y. Since by induction hypothesis y contains exactly one 0, it follows that x satisfies the statement of the proposition.

Conversely, let us show that any word x satisfying the statement of the proposition belongs to Dias_γ by induction on the length n of x. This is true when $n = 1$ because we necessarily have $x = 0$ and 0 belongs to Dias_γ since it is its unit. Otherwise, when $n \geqslant 2$, there is an integer $i \in [n-1]$ such that $x_ix_{i+1} \in \{0a, a0\}$ for an $a \in [\gamma]$. Let us suppose without loss of generality that $x_ix_{i+1} = a0$. By setting y as the word obtained by erasing the ith letter of x, we have $x = y \circ_i a0$. Thus, since by induction hypothesis y is an element of Dias_γ, it follows that x also is. $\square$

We deduce from Proposition 2.1.1 that the Hilbert series of Dias_γ satisfies

$$\mathcal{H}_{\mathsf{Dias}_\gamma}(t) = \frac{t}{(1-\gamma t)^2} \tag{2.1.8}$$

and that for all $n \geqslant 1$, $\dim \mathsf{Dias}_\gamma(n) = n\gamma^{n-1}$. For instance, the first dimensions of Dias_1, Dias_2, Dias_3, and Dias_4 are respectively

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \tag{2.1.9}$$

$$1, 4, 12, 32, 80, 192, 448, 1024, 2304, 5120, 11264, \tag{2.1.10}$$

$$1, 6, 27, 108, 405, 1458, 5103, 17496, 59049, 196830, 649539, \tag{2.1.11}$$

$$1, 8, 48, 256, 1280, 6144, 28672, 131072, 589824, 2621440, 11534336. \tag{2.1.12}$$

The second one is Sequence **A001787**, the third one is Sequence **A027471**, and the last one is Sequence **A002697** of [Slo].

2.2. **Presentation by generators and relations.** To establish a presentation of Dias_γ, we shall start by defining a morphism word_γ from a free operad to Dias_γ. Then, after showing that word_γ is a surjection, we will show that word_γ induces an operad isomorphism between a quotient of a free operad by a certain operad congruence $\equiv_\gamma$ and Dias_γ. The space of relations of Dias_γ of its presentation will be induced by $\equiv_\gamma$.

2.2.1. *From syntax trees to words.* For any integer $\gamma \geqslant 0$, let $\mathfrak{G}_{\mathsf{Dias}_\gamma} := \mathfrak{G}_{\mathsf{Dias}_\gamma}(2)$ be the graded set where

$$\mathfrak{G}_{\mathsf{Dias}_\gamma}(2) := \{\dashv_a, \vdash_a : a \in [\gamma]\}. \tag{2.2.1}$$

Let $\mathfrak{t}$ be a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ and x be a leaf of $\mathfrak{t}$. We say that an integer $a \in \{0\} \cup [\gamma]$ is *eligible* for x if $a = 0$ or there is an ancestor y of x labeled by $\dashv_a$ (resp. $\vdash_a$) and x is in the right (resp. left) subtree of y. The *image* of x is its greatest eligible integer. Moreover, let

$$\mathrm{word}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n) \to \mathsf{Dias}_\gamma(n), \qquad n \geqslant 1, \tag{2.2.2}$$

the map where $\mathrm{word}_\gamma(\mathfrak{t})$ is the word obtained by considering, from left to right, the images of the leaves of $\mathfrak{t}$ (see Figure 1).

Lemma 2.2.1. *For any integer $\gamma \geqslant 0$, the map word_γ is an operad morphism from $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ to Dias_γ.*

Proof. Let us first show that word_γ is a well-defined map. Let $\mathfrak{t}$ be a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ of arity n. Observe that by starting from the root of $\mathfrak{t}$, there is a unique maximal path obtained by following the directions specified by its internal nodes (a $\dashv_a$ means to go the left child while a $\vdash_a$ means to go to the right child). Then, the leaf at the end of this path is the only leaf with 0 as image. Others $n-1$ leaves have integers of $[\gamma]$ as images. By Proposition 2.1.1, this implies that $\mathrm{word}_\gamma(\mathfrak{t})$ is an element of $\mathsf{Dias}_\gamma(n)$.

To prove that word_γ is an operad morphism, we consider its following alternative description. If $\mathfrak{t}$ is a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$, we can consider the tree $\mathfrak{t}'$ obtained by replacing in $\mathfrak{t}$ each label $\dashv_a$ (resp. $\vdash_a$) by the word $0a$ (resp. $a0$), where $a \in [\gamma]$. Then, by a straightforward

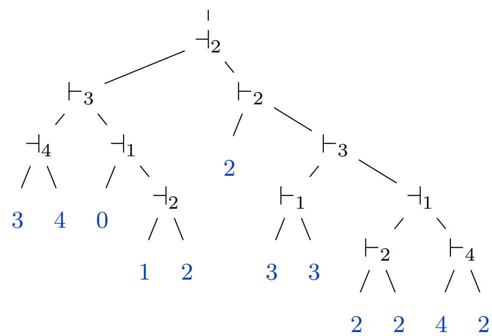


FIGURE 1. A syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ where images of its leaves are shown. This tree satisfies $\mathrm{word}_\gamma(\mathfrak{t}) = 340122332242$.

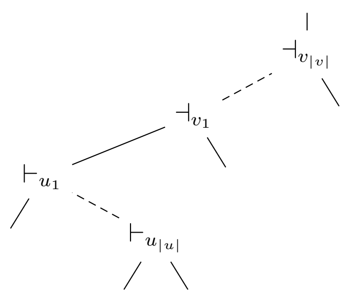
induction on the number of internal nodes of $\mathfrak{t}$, we obtain that $\mathrm{eval}_{\mathsf{Dias}_\gamma}(\mathfrak{t}')$, where $\mathfrak{t}'$ is seen as a syntax tree of $\mathbf{Free}\left(\mathsf{Dias}_\gamma(2)\right)$, is $\mathrm{word}_\gamma(\mathfrak{t})$. It then follows that word_γ is an operad morphism. $\square$

2.2.2. *Hook syntax trees.* Let us now consider the map

$$\mathrm{hook}_\gamma : \mathsf{Dias}_\gamma(n) \to \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n), \qquad n \geqslant 1, \tag{2.2.3}$$

defined for any word x of Dias_γ by

$$\mathrm{hook}_\gamma(x) :=$$



$$, \tag{2.2.4}$$

where x decomposes, by Proposition 2.1.1, uniquely in $x = u0v$ where u and v are words on the alphabet $[\gamma]$. The dashed edges denote, depending on their orientation, a right comb (wherein internal nodes are labeled, from top to bottom by $\vdash_{u_1}, \ldots, \vdash_{u_{|u|}}$) or a left comb (wherein internal nodes are labeled, from bottom to top, by $\dashv_{v_1}, \ldots, \dashv_{v_{|v|}}$). We shall call any syntax tree of the form (2.2.4) a *hook syntax tree*.

Lemma 2.2.2. *For any integer $\gamma \geqslant 0$, the map* word_γ *is a surjective operad morphism from* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ *onto* Dias_γ*. Moreover, for any element x of* Dias_γ*,* $\mathrm{hook}_\gamma(x)$ *belongs to the fiber of x under* word_γ*.*

Proof. The fact that x belongs to the fiber of x under word_γ is an immediate consequence of the definitions of word_γ and hook_γ, and the fact that by Proposition 2.1.1, any word x of Dias_γ decomposes uniquely in $x = u0v$ where u and v are words on the alphabet $[\gamma]$. Then, word_γ

is surjective as a map. Moreover, since by Lemma 2.2.1, word_γ is an operad morphism, it is a surjective operad morphism. □

2.2.3. *A rewrite rule on syntax trees.* Let $\to_\gamma$ be the quadratic rewrite rule on $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ satisfying

$$\vdash_{a'} \circ_2 \dashv_a \ \to_\gamma \ \dashv_a \circ_1 \vdash_{a'}, \qquad a, a' \in [\gamma], \tag{2.2.5a}$$

$$\dashv_a \circ_2 \vdash_b \ \to_\gamma \ \dashv_a \circ_1 \dashv_b, \qquad a < b \in [\gamma], \tag{2.2.5b}$$

$$\vdash_a \circ_1 \dashv_b \ \to_\gamma \ \vdash_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{2.2.5c}$$

$$\dashv_a \circ_2 \dashv_b \ \to_\gamma \ \dashv_b \circ_1 \dashv_a, \qquad a < b \in [\gamma], \tag{2.2.5d}$$

$$\vdash_a \circ_1 \vdash_b \ \to_\gamma \ \vdash_b \circ_2 \vdash_a, \qquad a < b \in [\gamma], \tag{2.2.5e}$$

$$\dashv_d \circ_2 \dashv_c \ \to_\gamma \ \dashv_d \circ_1 \dashv_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5f}$$

$$\dashv_d \circ_2 \vdash_c \ \to_\gamma \ \dashv_d \circ_1 \dashv_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5g}$$

$$\vdash_d \circ_1 \dashv_c \ \to_\gamma \ \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5h}$$

$$\vdash_d \circ_1 \vdash_c \ \to_\gamma \ \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma], \tag{2.2.5i}$$

and denote by $\equiv_\gamma$ the operadic congruence on $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ induced by $\to_\gamma$.

Lemma 2.2.3. *For any integer $\gamma \geqslant 0$ and any syntax trees $\mathfrak{t}_1$ and $\mathfrak{t}_2$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$, $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$ implies $\mathrm{word}_\gamma(\mathfrak{t}_1) = \mathrm{word}_\gamma(\mathfrak{t}_2)$.*

Proof. Let us denote by $\leftrightarrow_\gamma$ the symmetric closure of $\to_\gamma$. In the first place, observe that for any relation $\mathfrak{s}_1 \leftrightarrow_\gamma \mathfrak{s}_2$ where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(3)$, for any $i \in [3]$, the eligible integers for the ith leaves of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are the same. Besides, by definition of $\equiv_\gamma$, since $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$, one can obtain $\mathfrak{t}_2$ from $\mathfrak{t}_1$ by performing a sequence of $\leftrightarrow_\gamma$-rewritings. According to the previous observation, a $\leftrightarrow_\gamma$-rewriting preserve the eligible integers of all leaves of the tree on which they are performed. Therefore, the images of the leaves of $\mathfrak{t}_2$ are, from left to right, the same as the images of the leaves of $\mathfrak{t}_1$ and hence, $\mathrm{word}_\gamma(\mathfrak{t}_1) = \mathrm{word}_\gamma(\mathfrak{t}_2)$. □

Lemma 2.2.3 implies that the map

$$\bar{\mathrm{word}}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n)/_{\equiv_\gamma} \to \mathsf{Dias}_\gamma(n), \qquad n \geqslant 1, \tag{2.2.6}$$

satisfying, for any $\equiv_\gamma$-equivalence class $[\mathfrak{t}]_{\equiv_\gamma}$,

$$\bar{\mathrm{word}}_\gamma\left([\mathfrak{t}]_\gamma\right) = \mathrm{word}_\gamma(\mathfrak{t}), \tag{2.2.7}$$

where $\mathfrak{t}$ is any tree of $[\mathfrak{t}]_{\equiv_\gamma}$ is well-defined.

Lemma 2.2.4. *For any integer $\gamma \geqslant 0$, any syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ can be rewritten, by a sequence of $\to_\gamma$-rewritings, into a hook syntax tree. Moreover, this hook syntax tree is $\mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}))$.*

Proof. In the following, to gain readability, we shall denote by $\dashv_*$ (resp. $\vdash_*$) any element $\dashv_a$ (resp. $\vdash_a$) of $\mathfrak{G}_{\mathsf{Dias}_\gamma}$ when taking into account the value of $a \in [\gamma]$ is not necessary. Using this notation, from (2.2.5a)—(2.2.5i), we observe that $\to_\gamma$ expresses as

$$\vdash_* \circ_2 \dashv_* \;\to_\gamma\; \dashv_* \circ_1 \vdash_*, \tag{2.2.8a}$$

$$\dashv_* \circ_2 \vdash_* \;\to_\gamma\; \dashv_* \circ_1 \dashv_*, \tag{2.2.8b}$$

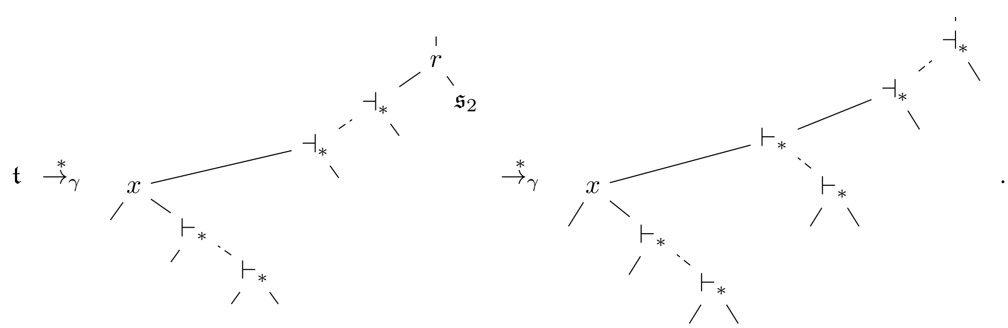
$$\vdash_* \circ_1 \dashv_* \;\to_\gamma\; \vdash_* \circ_2 \vdash_*, \tag{2.2.8c}$$

$$\dashv_* \circ_2 \dashv_* \;\to_\gamma\; \dashv_* \circ_1 \dashv_*, \tag{2.2.8d}$$

$$\vdash_* \circ_1 \vdash_* \;\to_\gamma\; \vdash_* \circ_2 \vdash_* \,. \tag{2.2.8e}$$

Let us first focus on the first part of the statement of the lemma to show that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree. We reason by induction on the arity n of $\mathfrak{t}$. When $n \leqslant 2$, $\mathfrak{t}$ is immediately a hook syntax tree. Otherwise, $\mathfrak{t}$ has at least two internal nodes. Then, $\mathfrak{t}$ is made of a root connected to a first subtree $\mathfrak{t}_1$ and a second subtree $\mathfrak{t}_2$. By induction hypothesis, $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a tree made of a root r of the same label as the one of the root of $\mathfrak{t}$, connected to a first subtree $\mathfrak{s}_1$ such that $\mathfrak{t}_1 \overset{*}{\to}_\gamma \mathfrak{s}_1$ and a second subtree $\mathfrak{s}_2$ such that $\mathfrak{t}_2 \overset{*}{\to}_\gamma \mathfrak{s}_2$, both being hook syntax trees. We have to deal two cases following the number of internal nodes of $\mathfrak{t}_1$.

Case 1. If $\mathfrak{t}_1$ has at least one internal node, we have the two $\overset{*}{\to}_\gamma$-relations



(2.2.9)

The first $\overset{*}{\to}_\gamma$-relation of (2.2.9) has just been explained. The second one comes from the application of the induction hypothesis on the upper part of the tree of the middle of (2.2.9) obtained by cutting the edge connecting the node x to its father. When the rightmost tree of (2.2.9) is not already a hook syntax tree, one has two cases following the label of x.

Case 1.1. If x is labeled by $\vdash_*$, by (2.2.8e), the bottom part of the rightmost tree of (2.2.9) consisting in internal nodes labeled by $\vdash_*$ is rewritable by $\to_\gamma$ into a right comb tree wherein internal nodes are labeled by $\vdash_*$. Then, the rightmost tree of (2.2.9) is rewritable by $\to_\gamma$ into a hook syntax tree, and then $\mathfrak{t}$ also is.

Case 1.2. Otherwise, x is labeled by $\dashv_*$. By definition of hook_γ, the second subtree of x is a leaf. By (2.2.8c), the bottom part of the rightmost tree of (2.2.9) consisting in x and internal nodes labeled by $\vdash_*$ can be rewritten by $\to_\gamma$ into a right comb tree wherein internal nodes are labeled by $\vdash_*$. Then, the rightmost tree of (2.2.9) is rewritable by $\to_\gamma$ into a hook syntax tree, and then $\mathfrak{t}$ also is.

Case 2. Otherwise, $\mathfrak{t}_1$ is the leaf. We then have the $\overset{*}{\to}_\gamma$-relation

$$\mathfrak{t} \overset{*}{\to}_\gamma \begin{array}{ccc} & | & \\ & r & \\ / & & \backslash \\ & & r' \\ & & / \quad \backslash \\ & & \mathfrak{s}_{21} \quad \mathfrak{s}_{22} \end{array}, \tag{2.2.10}$$

where $\mathfrak{s}_{21}$ is the first subtree of the root of $\mathfrak{s}_2$, $\mathfrak{s}_{22}$ is the second subtree of the root of $\mathfrak{s}_2$, and r' is a node with the same label as the root of $\mathfrak{s}_2$.

Case 2.1. If $r \circ_2 r'$ is equal to $\vdash_* \circ_2 \dashv_*$, $\dashv_* \circ_2 \vdash_*$, or $\dashv_* \circ_2 \dashv_*$, respectively by (2.2.8a), (2.2.8b), and (2.2.8d), the rightmost tree of (2.2.10) can be rewritten by $\to_\gamma$ into a tree $\mathfrak{r}$ having a first subtree with at least one internal node. Hence, $\mathfrak{r}$ is of the form required to be treated by *Case 1.*, implying that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree.

Case 2.2. Otherwise, $r \circ_2 r'$ is equal to $\vdash_* \circ_2 \vdash_*$. Since $\mathfrak{s}_2$ is by hypothesis a hook syntax tree, it is necessarily a right comb tree whose internal nodes are labeled by $\vdash_*$. Hence, the rightmost tree of (2.2.10) is already a hook syntax tree, showing that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree.

Let us finally show the last part of the statement of the lemma. Observe that, by definition of hook_γ and word_γ, if $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two different hook syntax trees, $\mathrm{word}_\gamma(\mathfrak{s}_1) \neq \mathrm{word}_\gamma(\mathfrak{s}_2)$. We have just shown that $\mathfrak{t}$ is rewritable by $\to_\gamma$ into a hook syntax tree $\mathfrak{s}$. Besides, by Lemma 2.2.3, one has $\mathrm{word}_\gamma(\mathfrak{t}) = \mathrm{word}_\gamma(\mathfrak{s})$. Then, $\mathfrak{s}$ is necessarily the hook syntax tree $\mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}))$. □

2.2.4. *Presentation by generators and relations.*

Lemma 2.2.5. *For any integers $\gamma \geqslant 0$ and $n \geqslant 1$, the map $\bar{\mathrm{word}}_\gamma$ defines a bijection between* $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n)/_{\equiv_\gamma}$ *and* $\mathsf{Dias}_\gamma(n)$.

Proof. Let us show that $\bar{\mathrm{word}}_\gamma$ is injective. Let $\mathfrak{t}_1$ and $\mathfrak{t}_2$ be two syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ such that $\mathrm{word}_\gamma(\mathfrak{t}_1) = \mathrm{word}_\gamma(\mathfrak{t}_2)$ and let $\mathfrak{s} := \mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}_1)) = \mathrm{hook}_\gamma(\mathrm{word}_\gamma(\mathfrak{t}_2))$. By Lemma 2.2.4, one has $\mathfrak{t}_1 \overset{*}{\to}_\gamma \mathfrak{s}$ and $\mathfrak{t}_2 \overset{*}{\to}_\gamma \mathfrak{s}$, and hence, $\mathfrak{t}_1 \equiv_\gamma \mathfrak{t}_2$. By the definition of the map $\bar{\mathrm{word}}_\gamma$ from the map word_γ, this show that $\bar{\mathrm{word}}_\gamma$ is injective. Besides, by Lemma 2.2.2, $\bar{\mathrm{word}}_\gamma$ is surjective, whence the statement of the lemma. □

Theorem 2.2.6. *For any integer $\gamma \geqslant 0$, the operad Dias_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{Dias}_\gamma}$ and its space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space induced by the equivalence relation $\leftrightarrow_\gamma$ satisfying*

$$\dashv_a \circ_1 \vdash_{a'} \leftrightarrow_\gamma \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma], \tag{2.2.11a}$$

$$\dashv_a \circ_1 \dashv_b \leftrightarrow_\gamma \dashv_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{2.2.11b}$$

$$\vdash_a \circ_1 \dashv_b \;\leftrightarrow_\gamma\; \vdash_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{2.2.11c}$$

$$\dashv_b \circ_1 \dashv_a \;\leftrightarrow_\gamma\; \dashv_a \circ_2 \dashv_b, \qquad a < b \in [\gamma], \tag{2.2.11d}$$

$$\vdash_a \circ_1 \vdash_b \;\leftrightarrow_\gamma\; \vdash_b \circ_2 \vdash_a, \qquad a < b \in [\gamma], \tag{2.2.11e}$$

$$\dashv_d \circ_1 \dashv_d \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \dashv_c \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \vdash_c, \qquad c \leqslant d \in [\gamma], \tag{2.2.11f}$$

$$\vdash_d \circ_1 \dashv_c \;\leftrightarrow_\gamma\; \vdash_d \circ_1 \vdash_c \;\leftrightarrow_\gamma\; \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma]. \tag{2.2.11g}$$

Proof. By Lemma 2.2.5, the map $\overline{\mathrm{word}}_\gamma$ is, for any $n \geqslant 1$, a bijection between the sets $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(n)/_{\equiv_\gamma}$ and $\mathsf{Dias}_\gamma(n)$. Moreover, by Lemma 2.2.1, word_γ is an operad morphism, and then $\overline{\mathrm{word}}_\gamma$ also is. Hence, $\overline{\mathrm{word}}_\gamma$ is an operad isomorphism between $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)/_{\equiv_\gamma}$ and Dias_γ. Therefore, since $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ is the space induced by $\equiv_\gamma$, Dias_γ admits the stated presentation. □

The space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ of Dias_γ exhibited by Theorem 2.2.6 can be rephrased in a more compact way as the space generated by

$$\dashv_a \circ_1 \vdash_{a'} - \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma], \tag{2.2.12a}$$

$$\dashv_a \circ_1 \dashv_{a\uparrow a'} - \dashv_a \circ_2 \vdash_{a'}, \qquad a, a' \in [\gamma], \tag{2.2.12b}$$

$$\vdash_a \circ_1 \dashv_{a'} - \vdash_a \circ_2 \vdash_{a\uparrow a'}, \qquad a, a' \in [\gamma], \tag{2.2.12c}$$

$$\dashv_{a\uparrow a'} \circ_1 \dashv_a - \dashv_a \circ_2 \dashv_{a'}, \qquad a, a' \in [\gamma], \tag{2.2.12d}$$

$$\vdash_a \circ_1 \vdash_{a'} - \vdash_{a\uparrow a'} \circ_2 \vdash_a, \qquad a, a' \in [\gamma]. \tag{2.2.12e}$$

Observe that, by Theorem 2.2.6, Dias_1 and the diassociative operad (see [Lod01] or Section 1.3) admit the same presentation. Then, for all integers $\gamma \geqslant 0$, the operads Dias_γ are generalizations of the diassociative operad.

2.3. Miscellaneous properties.

From the description of the elements of Dias_γ and its structure revealed by its presentation, we develop here some of its properties. Unless otherwise specified, Dias_γ is still considered in this section as a set-operad.

2.3.1. *Koszulity.*

Theorem 2.3.1. *For any integer $\gamma \geqslant 0$, Dias_γ is a Koszul operad. Moreover, the set of hook syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ forms a Poincaré-Birkhoff-Witt basis of Dias_γ.*

Proof. From the definition of hook syntax trees, it appears that no hook syntax tree can be rewritten by $\to_\gamma$ into another syntax tree. Hence, and by Lemma 2.2.4, $\to_\gamma$ is a terminating rewrite rule and its normal forms are hook syntax trees. Moreover, again by Lemma 2.2.4, since any syntax tree is rewritable by $\to_\gamma$ into a unique hook syntax tree, $\to_\gamma$ is a confluent rewrite rule, and hence, $\to_\gamma$ is convergent. Now, since by Theorem 2.2.6, the space of relations of Dias_γ is the space induced by the operad congruence induced by $\to_\gamma$, by the Koszulity criterion [Hof10, DK10, LV12] we have reformulated in Section 1.2.5, Dias_γ is a Koszul operad and the set of of hook syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ forms a Poincaré-Birkhoff-Witt basis of Dias_γ. □

2.3.2. *Symmetries.* If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two operads, a linear map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad antimorphism* if it respects arities and anticommutes with partial composition maps, that is,

$$\phi(x \circ_i y) = \phi(x) \circ_{n-i+1} \phi(y), \qquad x \in \mathcal{O}(n), y \in \mathcal{O}, i \in [n]. \tag{2.3.1}$$

A *symmetry* of an operad $\mathcal{O}$ is either an automorphism or an antiautomorphism. The set of all symmetries of $\mathcal{O}$ form a group for the composition, called the *group of symmetries* of $\mathcal{O}$.

Proposition 2.3.2. *For any integer $\gamma \geqslant 0$, the group of symmetries of Dias_γ as a set-operad contains two elements: the identity map and the linear map sending any word of Dias_γ to its mirror image.*

Proof. Let us denote by $\mathbb{G}_\gamma$ the set $\{0a, a0 : a \in [\gamma]\}$. Since Dias_γ is generated by $\mathbb{G}_\gamma$, any automorphism or antiautomorphism ϕ of Dias_γ is wholly determined by the images of the elements of $\mathbb{G}_\gamma$. Besides let us observe that ϕ is in particular a permutation of $\mathbb{G}_\gamma$.

By contradiction, assume that ϕ is an automorphism of Dias_γ different from the identity map. We have two cases to explore.

Case 1. If there are $a, a' \in [\gamma]$ satisfying $\phi(0a) = a'0$, since ϕ is a permutation of $\mathbb{G}_\gamma$, there are $b, b' \in [\gamma]$ satisfying $\phi(b0) = 0b'$. Then, we have at the same time $b0 \circ_2 0a = b0a = 0a \circ_1 b0$,

$$\phi(b0 \circ_2 0a) = \phi(b0) \circ_2 \phi(0a) = 0b' \circ_2 a'0 = 0\,(b' \uparrow a')\,b', \tag{2.3.2}$$

and

$$\phi(0a \circ_1 b0) = \phi(0a) \circ_1 \phi(b0) = a'0 \circ_1 0b' = a'\,(a' \uparrow b')\,0. \tag{2.3.3}$$

This shows that $\phi(b0 \circ_2 0a) \neq \phi(0a \circ_1 b0)$ and hence, ϕ is not an operad morphism. By a similar argument, one can show that there are no $a, a' \in [\gamma]$ such that $\phi(a0) = 0a'$.

Case 2. Otherwise, for all $a \in [\gamma]$, we have $\phi(0a) = 0a'$ and $\phi(a0) = a''0$ for some $a', a'' \in [\gamma]$. Since, by hypothesis, ϕ is not the identity map, there exist $a \neq a' \in [\gamma]$ such that $\phi(0a) = 0a'$ or $\phi(a0) = a'0$. Let us assume, without loss of generality, that $\phi(0a) = 0a'$. Since ϕ is a permutation of $\mathbb{G}_\gamma$, there exist $b \neq b' \in [\gamma]$ such that $\phi(0b) = 0b'$. One can assume, without loss of generality, that $a < b$ and $b' < a'$. Then, we have at the same time $0a \circ_2 0b = 0ab = 0b \circ_1 0a$,

$$\phi(0a \circ_2 0b) = \phi(0a) \circ_2 \phi(0b) = 0a' \circ_2 0b' = 0a'a', \tag{2.3.4}$$

and

$$\phi(0b \circ_1 0a) = \phi(0b) \circ_1 \phi(0a) = 0b' \circ_1 0a' = 0a'b'. \tag{2.3.5}$$

This shows that $\phi(0a \circ_2 0b) \neq \phi(0b \circ_1 0a)$ and hence, that ϕ is not an operad morphism. By a similar argument, one can show that there are no $a \neq a' \in [\gamma]$ such that $\phi(a0) = \phi(a'0)$.

We then have shown that if ϕ is an automorphism of Dias_γ, it is necessarily the identity map.

Finally, by Proposition 2.1.1, if x is an element of Dias_γ, its mirror image also is in Dias_γ. Moreover, it is immediate to see that the map sending a word to its mirror image is an antiautomorphism of Dias_γ. Similar arguments as the ones developed previously show that it is the only. $\square$

2.3.3. *Basic operad.* A set-operad $\mathcal{O}$ is *basic* if for all $y_1, \dots, y_n \in \mathcal{O}$, all the maps

$$\circ^{y_1,\dots,y_n} : \mathcal{O}(n) \to \mathcal{O}(|y_1| + \dots + |y_n|) \tag{2.3.6}$$

defined by

$$\circ^{y_1,\dots,y_n}(x) := x \circ (y_1, \dots, y_n), \qquad x \in \mathcal{O}(n), \tag{2.3.7}$$

are injective. This property for set-operads introduced by Vallette [Val07] is a very relevant one since there is a general construction producing a family of posets (see [MY91] and [CL07]) from a basic set-operad. This family of posets leads to the definition of an incidence Hopf algebra by a construction of Schmitt [Sch94].

Proposition 2.3.3. *For any integer $\gamma \geqslant 0$, Dias_γ is a basic operad.*

Proof. Let $n \geqslant 1$, $y_1, \dots, y_n$ be words of Dias_γ, and x and x' be two words of $\mathsf{Dias}_\gamma(n)$ such that $\circ^{y_1,\dots,y_n}(x) = \circ^{y_1,\dots,y_n}(x')$. Then, for all $i \in [n]$ and $j \in [|y_i|]$, we have $x_i \uparrow y_{i,j} = x'_i \uparrow y_{i,j}$ where $y_{i,j}$ is the jth letter of y_i. Since by Proposition 2.1.1, any word y_i contains a 0, we have in particular $x_i \uparrow 0 = x'_i \uparrow 0$ for all $i \in [n]$. This implies $x = x'$ and thus, that $\circ^{y_1,\dots,y_n}$ is injective. □

2.3.4. *Rooted operad.* We restate here a property on operads introduced by Chapoton [Cha14]. An operad $\mathcal{O}$ is *rooted* if there is a map

$$\mathrm{root} : \mathcal{O}(n) \to [n], \qquad n \geqslant 1, \tag{2.3.8}$$

satisfying, for all $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $i \in [n]$,

$$\mathrm{root}(x \circ_i y) = \begin{cases} \mathrm{root}(x) + m - 1 & \text{if } i \leqslant \mathrm{root}(x) - 1, \\ \mathrm{root}(x) + \mathrm{root}(y) - 1 & \text{if } i = \mathrm{root}(x), \\ \mathrm{root}(x) & \text{otherwise } (i \geqslant \mathrm{root}(x) + 1). \end{cases} \tag{2.3.9}$$

We call such a map a *root map*. More intuitively, the root map of a rooted operad associates a particular input with any of its elements and this input is preserved by partial compositions.

It is immediate that any operad $\mathcal{O}$ is a rooted operad for the root maps $\mathrm{root}_{\mathrm{L}}$ and $\mathrm{root}_{\mathrm{R}}$, which send respectively all elements x of arity n to 1 or to n. For this reason, we say that an operad $\mathcal{O}$ is *nontrivially rooted* if it can be endowed with a root map different from $\mathrm{root}_{\mathrm{L}}$ and $\mathrm{root}_{\mathrm{R}}$.

Proposition 2.3.4. *For any integer $\gamma \geqslant 0$, Dias_γ is a nontrivially rooted operad for the root map sending any word of Dias_γ to the position of its 0.*

Proof. Thanks to Proposition 2.1.1, the map of the statement of the proposition is well-defined. The fact that 0 is the neutral element for the $\uparrow$ operation and the fact that any word of Dias_γ contains exactly one 0 imply that this map satisfies (2.3.9). Finally, this map is obviously different from $\mathrm{root}_{\mathrm{L}}$ and $\mathrm{root}_{\mathrm{R}}$, whence the statement of the proposition. □

2.3.5. *Alternative basis.* In this section, Dias_γ is considered as an operad in the category of vector spaces.

Let $\preccurlyeq_\gamma$ be the order relation on the underlying set of $\mathsf{Dias}_\gamma(n)$, $n \geqslant 1$, where for all words x and y of Dias_γ of a same arity n, we have

$$x \preccurlyeq_\gamma y \qquad \text{if } x_i \leqslant y_i \text{ for all } i \in [n]. \tag{2.3.10}$$

This order relation allows to define for all word x of Dias_γ the elements

$$\mathsf{K}_x^{(\gamma)} := \sum_{x \preccurlyeq_\gamma x'} \mu_\gamma(x, x')\, x', \tag{2.3.11}$$

where μ_γ is the Möbius function of the poset defined by $\preccurlyeq_\gamma$. For instance,

$$\mathsf{K}_{102}^{(2)} = 102 - 202, \tag{2.3.12}$$

$$\mathsf{K}_{102}^{(3)} = \mathsf{K}_{102}^{(4)} = 102 - 103 - 202 + 203, \tag{2.3.13}$$

$$\mathsf{K}_{23102}^{(3)} = 23102 - 23103 - 23202 + 23203 - 33102 + 33103 + 33202 - 33203. \tag{2.3.14}$$

Since, by Möbius inversion, for any word x of Dias_γ one has

$$x = \sum_{x \preccurlyeq_\gamma x'} \mathsf{K}_{x'}^{(\gamma)}, \tag{2.3.15}$$

the family of all $\mathsf{K}_x^{(\gamma)}$, where the x are words of Dias_γ, forms by triangularity a basis of Dias_γ, called the K-*basis*.

If u and v are two words of a same length n, we denote by $\operatorname{ham}(u, v)$ the *Hamming distance* between u and v that is the number of positions $i \in [n]$ such that $u_i \neq v_i$. Moreover, for any word x of Dias_γ of length n and any subset J of $[n]$, we denote by $\operatorname{Incr}_\gamma(x, J)$ the set of words obtained by incrementing by one some letters of x smaller than γ and greater than 0 whose positions are in J. We shall simply denote by $\operatorname{Incr}_\gamma(x)$ the set $\operatorname{Incr}_\gamma(x, [n])$. Proposition 2.1.1 ensures that all $\operatorname{Incr}_\gamma(x, J)$ are sets of words of Dias_γ.

Lemma 2.3.5. *For any integer $\gamma \geqslant 0$ and any word x of Dias_γ,*

$$\mathsf{K}_x^{(\gamma)} = \sum_{x' \in \operatorname{Incr}_\gamma(x)} (-1)^{\operatorname{ham}(x, x')}\, x'. \tag{2.3.16}$$

Proof. Let n be the arity of x. To compute $\mathsf{K}_x^{(\gamma)}$ from its definition (2.3.11), it is enough to know the Möbius function μ_γ of the poset $\mathbb{P}_x^{(\gamma)}$ consisting in the words x' of Dias_γ satisfying $x \preccurlyeq_\gamma x'$. Immediately from the definition of $\preccurlyeq_\gamma$, it appears that $\mathbb{P}_x^{(\gamma)}$ is isomorphic to the Cartesian product poset

$$\mathbb{T}_x^{(\gamma)} := \mathbb{T}\left(\gamma - x_1\right) \times \cdots \times \mathbb{T}\left(\gamma - x_{r-1}\right) \times \mathbb{T}(0) \times \mathbb{T}\left(\gamma - x_{r+1}\right) \times \cdots \times \mathbb{T}\left(\gamma - x_n\right), \tag{2.3.17}$$

where for any nonnegative integer k, $\mathbb{T}(k)$ denotes the poset over $\{0\} \cup [k]$ with the natural total order relation, and r is the position of, by Proposition 2.1.1, the only 0 of x. The map $\phi_x^{(\gamma)} : \mathbb{P}_x^{(\gamma)} \to \mathbb{T}_x^{(\gamma)}$ defined for all words x' of $\mathbb{P}_x^{(\gamma)}$ by

$$\phi_x^{(\gamma)}(x') := \left(x'_1 - x_1, \ldots, x'_{r-1} - x_{r-1}, 0, x'_{r+1} - x_{r+1}, \ldots, x'_n - x_n\right) \tag{2.3.18}$$

is an isomorphism of posets.

Recall that the Möbius function μ of $\mathbb{T}(k)$ satisfies, for all $a, a' \in \mathbb{T}(k)$,

$$\mu(a, a') = \begin{cases} 1 & \text{if } a' = a, \\ -1 & \text{if } a' = a + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.19}$$

Moreover, since by [Sta11], the Möbius function of a Cartesian product poset is the product of the Möbius functions of the posets involved in the product, through the isomorphism $\phi_x^{(\gamma)}$, we obtain that when x' is in $\mathrm{Incr}_\gamma(x)$, $\mu_\gamma(x, x') = (-1)^{\mathrm{ham}(x,x')}$ and that when x' is not in Incr_γ, $\mu_\gamma(x, x') = 0$. Therefore, (2.3.16) is established. □

Lemma 2.3.6. *For any integer $\gamma \geqslant 0$, any word x of Dias_γ, and any nonempty set J of positions of letters of x that are greater than 0 and smaller than γ,*

$$\sum_{x' \in \mathrm{Incr}_\gamma(x, J)} (-1)^{\mathrm{ham}(x,x')} = 0. \tag{2.3.20}$$

Proof. The statement of the lemma follows by induction on the nonzero cardinality of J. □

To compute a direct expression for the partial composition of Dias_γ over the K-basis, we have to introduce two notations. If x is a word of Dias_γ of length nonsmaller than 2, we denote by $\min(x)$ the smallest letter of x among its letters different from 0. Proposition 2.1.1 ensures that $\min(x)$ is well-defined. Moreover, for all words x and y of Dias_γ, a position i such that $x_i \neq 0$, and $a \in [\gamma]$, we denote by $x \circ_{a,i} y$ the word $x \circ_i y$ in which the 0 coming from y is replaced by a instead of x_i.

Theorem 2.3.7. *For any integer $\gamma \geqslant 0$, the partial composition of Dias_γ over the K-basis satisfies, for all words x and y of Dias_γ of arities nonsmaller than 2,*

$$\mathsf{K}_x^{(\gamma)} \circ_i \mathsf{K}_y^{(\gamma)} = \begin{cases} \mathsf{K}_{x \circ_i y}^{(\gamma)} & \text{if } \min(y) > x_i, \\ \sum_{a \in [x_i, \gamma]} \mathsf{K}_{x \circ_{a,i} y}^{(\gamma)} & \text{if } \min(y) = x_i, \\ 0 & \text{otherwise } (\min(y) < x_i). \end{cases} \tag{2.3.21}$$

Proof. First of all, by Lemma 2.3.5 together with (2.3.15), we obtain

$$\begin{aligned} \mathsf{K}_x^{(\gamma)} \circ_i \mathsf{K}_y^{(\gamma)} &= \sum_{\substack{x' \in \mathrm{Incr}_\gamma(x) \\ y' \in \mathrm{Incr}_\gamma(y)}} (-1)^{\mathrm{ham}(x,x') + \mathrm{ham}(y,y')} \left(\sum_{x' \circ_i y' \preccurlyeq_\gamma z} \mathsf{K}_z^{(\gamma)} \right) \\ &= \sum_{x \circ_i y \preccurlyeq_\gamma z} \sum_{\substack{x' \in \mathrm{Incr}_\gamma(x) \\ y' \in \mathrm{Incr}_\gamma(y) \\ x' \circ_i y' \preccurlyeq_\gamma z}} (-1)^{\mathrm{ham}(x,x') + \mathrm{ham}(y,y')} \mathsf{K}_z^{(\gamma)}. \end{aligned} \tag{2.3.22}$$

Let us denote by n (resp. m) the arity of x (resp. y) and let z be a word of Dias_γ such that $x \circ_i y \preccurlyeq_\gamma z$. Let $x' \in \mathrm{Incr}_\gamma(x)$ and $y' \in \mathrm{Incr}_\gamma(y)$. We have, by definition of the partial composition of Dias_γ,

$$x \circ_i y = x_1 \dots x_{i-1}\, t_1 \dots t_{r-1}\, x_i\, t_{r+1} \dots t_m\, x_{i+1} \dots x_n, \tag{2.3.23}$$

and

$$x' \circ_i y' = x'_1 \dots x'_{i-1}\, t'_1 \dots t'_{r-1}\, x'_i\, t'_{r+1} \dots t'_m\, x'_{i+1} \dots x'_n, \tag{2.3.24}$$

where r denotes the position of the only, by Proposition 2.1.1, 0 of y and for all $j \in [m] \setminus \{r\}$, $t_j := x_i \uparrow y_j$ and $t'_j := x'_i \uparrow y'_j$. By (2.3.22), the pair (x', y') contributes to the coefficient of $\mathsf{K}_z^{(\gamma)}$ in (2.3.22) if and only if $x \circ_i y \preccurlyeq_\gamma x' \circ_i y' \preccurlyeq z$. To compute this coefficient, we have three cases to consider following the value of $\min(y)$ compared to the value of x_i.

Case 1. Assume first that $\min(y) < x_i$. Then, there is at least a $s \in [m] \setminus \{r\}$ such that $y_s < x_i$. This implies that $t_s = x_i$ and that y'_s has no influence on t'_s and then, on $x' \circ_i y'$. Thus, the word $y'' := y'_1 \dots y'_{s-1} a y'_{s+1} \dots y'_m$ where a is the only possible letter such that $y'' \in \mathrm{Incr}_\gamma(y)$ and $a \neq y'_s$ satisfies $x' \circ_i y'' = x' \circ_i y'$. Therefore, since $\mathrm{ham}(y', y'') = 1$, the contribution of the pair (x', y') for the coefficient of $\mathsf{K}_z^{(\gamma)}$ in (2.3.22) is compensated by the contribution of the pair (x', y''). This shows that this coefficient is 0 and hence, $\mathsf{K}_x^{(\gamma)} \circ_i \mathsf{K}_y^{(\gamma)} = 0$.

Case 2. Assume now that $\min(y) > x_i$. Then, for all $j \in [m] \setminus \{r\}$, we have $y_j > x_i$ and thus, $t_j = y_j$. When $z = x \circ_i y$, we necessarily have $x' = x$ and $y' = y$. Hence, the coefficient of $\mathsf{K}_{x \circ_i y}^{(\gamma)}$ in (2.3.22) is 1. Else, when $z \neq x \circ_i y$, we have $x' \circ_i y' \in \mathrm{Incr}_\gamma(x \circ_i y, J)$, where J is the nonempty set of the positions of letters of z different from letters of $x \circ_i y$. Now, from (2.3.22), the coefficient of $\mathsf{K}_z^{(\gamma)}$ in (2.3.22) is

$$\sum_{x' \circ_i y' \in \mathrm{Incr}_\gamma(x \circ_i y, J)} (-1)^{\mathrm{ham}(x,x') + \mathrm{ham}(y,y')}. \tag{2.3.25}$$

Lemma 2.3.6 implies that this coefficient is 0. This shows that $\mathsf{K}_x^{(\gamma)} \circ_i \mathsf{K}_y^{(\gamma)} = \mathsf{K}_{x \circ_i y}^{(\gamma)}$.

Case 3. The last case occurs when $\min(y) = x_i$. Then, for all $j \in [m] \setminus \{r\}$, we have $y_j \geqslant x_i$ and thus, $t_j = y_j$. Moreover, there is at least a $s \in [m] \setminus \{r\}$ such that $y_s = x_i$. When $z = x \circ_{a,i} y$ with $a \in [x_i, \gamma]$, we necessarily have $x' = x$ and $y' = y$. Therefore, for all $a \in [x_i, \gamma]$, the $\mathsf{K}_{x \circ_{a,i}}^{(\gamma)}$ have coefficient 1 in (2.3.22). The same argument as the one exposed for *Case 2.* shows that when $z \neq x \circ_{a,i} y$ for all $a \in [x_i, \gamma]$, the coefficient of $\mathsf{K}_z^{(\gamma)}$ is zero. Hence, $\mathsf{K}_x^{(\gamma)} \circ_i \mathsf{K}_y^{(\gamma)} = \sum_{a \in [x_i, \gamma]} \mathsf{K}_{x \circ_{a,i} y}^{(\gamma)}$.

□

We have for instance

$$\mathsf{K}_{20413}^{(5)} \circ_1 \mathsf{K}_{304}^{(5)} = \mathsf{K}_{3240413}^{(5)}, \tag{2.3.26}$$

$$\mathsf{K}_{20413}^{(5)} \circ_2 \mathsf{K}_{304}^{(5)} = \mathsf{K}_{2304413}^{(5)}, \tag{2.3.27}$$

$$\mathsf{K}_{20413}^{(5)} \circ_3 \mathsf{K}_{304}^{(5)} = 0, \tag{2.3.28}$$

$$\mathsf{K}_{20413}^{(5)} \circ_4 \mathsf{K}_{304}^{(5)} = \mathsf{K}_{2043143}^{(5)}, \tag{2.3.29}$$

$$\mathsf{K}_{20413}^{(5)} \circ_5 \mathsf{K}_{304}^{(5)} = \mathsf{K}_{2041334}^{(5)} + \mathsf{K}_{2041344}^{(5)} + \mathsf{K}_{2041354}^{(5)}. \tag{2.3.30}$$

Theorem 2.3.7 implies in particular that the structure coefficients of the partial composition of Dias_γ over the K-basis are 0 or 1. It is possible to define another bases of Dias_γ by reversing in (2.3.11) the relation $\preccurlyeq_\gamma$ and by suppressing or keeping the Möbius function μ_γ. This gives obviously rise to three other bases. It worth to note that, as small computations reveal, over all these additional bases, the structure coefficients of the partial composition of Dias_γ can be negative or different from 1. This observation makes the K-basis even more particular and interesting. It has some other properties, as next section will show.

2.3.6. *Alternative presentation.* The K-basis introduced in the previous section leads to state a new presentation for Dias_γ in the following way.

For any integer $\gamma \geqslant 0$, let $\dashv\!\!\Vert_a$ and $\Vert\!\!\vdash_a$, $a \in [\gamma]$, be the elements of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(2)$ defined by

$$\dashv\!\!\Vert_a := \begin{cases} \dashv_\gamma & \text{if } a = \gamma, \\ \dashv_a - \dashv_{a+1} & \text{otherwise,} \end{cases} \tag{2.3.31a}$$

and

$$\Vert\!\!\vdash_a := \begin{cases} \vdash_\gamma & \text{if } a = \gamma, \\ \vdash_a - \vdash_{a+1} & \text{otherwise.} \end{cases} \tag{2.3.31b}$$

Then, since for all $a \in [\gamma]$ we have

$$\dashv_a = \sum_{a \leqslant b \in [\gamma]} \dashv\!\!\Vert_b \tag{2.3.32a}$$

and

$$\vdash_a = \sum_{a \leqslant b \in [\gamma]} \Vert\!\!\vdash_b, \tag{2.3.32b}$$

by triangularity, the family $\mathfrak{G}'_{\mathsf{Dias}_\gamma} := \{\dashv\!\!\Vert_a, \Vert\!\!\vdash_a : a \in [\gamma]\}$ forms a basis of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(2)$ and then, generates $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ as an operad. This change of basis from $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)$ to $\mathbf{Free}(\mathfrak{G}'_{\mathsf{Dias}_\gamma})$ comes from the change of basis from the usual basis of Dias_γ to the K-basis. Let us now express a presentation of Dias_γ through the family $\mathfrak{G}'_{\mathsf{Dias}_\gamma}$.

Proposition 2.3.8. *For any integer $\gamma \geqslant 0$, the operad Dias_γ admits the following presentation. It is generated by $\mathfrak{G}'_{\mathsf{Dias}_\gamma}$ and its space of relations is $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is generated by*

$$\dashv\!\!\Vert_a \circ_1 \Vert\!\!\vdash_{a'} - \Vert\!\!\vdash_{a'} \circ_2 \dashv\!\!\Vert_a, \qquad a, a' \in [\gamma], \tag{2.3.33a}$$

$$\Vert\!\!\vdash_b \circ_1 \Vert\!\!\vdash_a, \qquad a < b \in [\gamma], \tag{2.3.33b}$$

$$\dashv\!\!\Vert_b \circ_2 \dashv\!\!\Vert_a, \qquad a < b \in [\gamma], \tag{2.3.33c}$$

$$\Vert\!\!\vdash_b \circ_1 \dashv\!\!\Vert_a, \qquad a < b \in [\gamma], \tag{2.3.33d}$$

$$\dashv\!\!\Vert_b \circ_2 \Vert\!\!\vdash_a, \qquad a < b \in [\gamma], \tag{2.3.33e}$$

$$\Vert\!\!\vdash_a \circ_1 \Vert\!\!\vdash_b - \Vert\!\!\vdash_b \circ_2 \Vert\!\!\vdash_a, \qquad a < b \in [\gamma], \tag{2.3.33f}$$

$$\dashv\!\!\Vert_b \circ_1 \dashv\!\!\Vert_a - \dashv\!\!\Vert_a \circ_2 \dashv\!\!\Vert_b, \qquad a < b \in [\gamma], \tag{2.3.33g}$$

$$\Vert\!\!\vdash_a \circ_1 \dashv\!\!\Vert_b - \Vert\!\!\vdash_a \circ_2 \Vert\!\!\vdash_b, \qquad a < b \in [\gamma], \tag{2.3.33h}$$

$$\dashv\!\!\Vert_a \circ_1 \dashv\!\!\Vert_b - \dashv\!\!\Vert_a \circ_2 \Vert\!\!\vdash_b, \qquad a < b \in [\gamma], \tag{2.3.33i}$$

$$\Vdash_a \circ_1 \Vdash_a - \left(\sum_{a \leqslant b \in [\gamma]} \Vdash_a \circ_2 \Vdash_b \right), \qquad a \in [\gamma], \tag{2.3.33j}$$

$$\left(\sum_{a \leqslant b \in [\gamma]} \dashV_a \circ_1 \dashV_b \right) - \dashV_a \circ_2 \dashV_a, \qquad a \in [\gamma], \tag{2.3.33k}$$

$$\Vdash_a \circ_1 \dashV_a - \left(\sum_{a \leqslant b \in [\gamma]} \Vdash_b \circ_2 \Vdash_a \right), \qquad a \in [\gamma], \tag{2.3.33l}$$

$$\left(\sum_{a \leqslant b \in [\gamma]} \dashV_b \circ_1 \dashV_a \right) - \dashV_a \circ_2 \Vdash_a, \qquad a \in [\gamma]. \tag{2.3.33m}$$

Proof. Let us show that $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is equal to the space of relations $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ of Dias_γ defined in the statement of Theorem 2.2.6. First of all, recall that the map $\mathrm{word}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right) \to \mathsf{Dias}_\gamma$ defined in Section 2.2.1 satisfies $\mathrm{word}_\gamma(\dashv_a) = 0a$ and $\mathrm{word}_\gamma(\vdash_a) = a0$ for all $a \in [\gamma]$. By Theorem 2.2.6, for any $x \in \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Dias}_\gamma}\right)(3)$, x is in $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ if and only if $\mathrm{word}_\gamma(x) = 0$.

Besides, by definition of $\dashV_a$, $\Vdash_a$, $a \in [\gamma]$, and by making use of the K-basis of Dias_γ, we have $\mathrm{word}_\gamma(\dashV_a) = \mathsf{K}^{(\gamma)}_{0a}$ and $\mathrm{word}_\gamma(\Vdash_a) = \mathsf{K}^{(\gamma)}_{a0}$. By using the partial composition rules for Dias_γ over the K-basis of Theorem 2.3.7, straightforward computations show that $\mathrm{word}_\gamma(x) = 0$ for all elements x among (2.3.33a)—(2.3.33m). This implies that $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ is a subspace of $\mathfrak{R}_{\mathsf{Dias}_\gamma}$.

Now, one can observe that elements (2.3.33a)—(2.3.33m) are linearly independent. Then, $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ has dimension $5\gamma^2$ which is also, by Theorem 2.2.6, the dimension of $\mathfrak{R}_{\mathsf{Dias}_\gamma}$. Hence, $\mathfrak{R}'_{\mathsf{Dias}_\gamma}$ and $\mathfrak{R}_{\mathsf{Dias}_\gamma}$ are equal. The statement of the proposition follows. □

Despite the apparent complexity of the presentation of Dias_γ exhibited by Proposition 2.3.8, as we will see in Section 2 of [Gir16], the Koszul dual of Dias_γ computed from this presentation has a very simple and manageable expression.

3. Pluriassociative algebras

We now focus on algebras over γ-pluriassociative operads. For this purpose, we construct free Dias_γ-algebras over one generator, and define and study two notions of units for Dias_γ-algebras. We end this section by introducing a convenient way to define Dias_γ-algebras and give several examples of such algebras.

3.1. **Category of pluriassociative algebras and free objects.** Let us study the category of Dias_γ-algebras and the units for algebras in this category.

3.1.1. *Pluriassociative algebras.* We call *γ-pluriassociative algebra* any Dias_γ-algebra. From the presentation of Dias_γ provided by Theorem 2.2.6, any γ-pluriassociative algebra is a vector space endowed with linear operations $\dashv_a, \vdash_a$, $a \in [\gamma]$, satisfying the relations encoded by (2.2.12a)—(2.2.12e).

3.1.2. *General definitions.* Let $\mathcal{P}$ be a γ-pluriassociative algebra. We say that $\mathcal{P}$ is *commutative* if for all $x, y \in \mathcal{P}$ and $a \in [\gamma]$, $x \dashv_a y = y \vdash_a x$. Besides, $\mathcal{P}$ is *pure* for all $a, a' \in [\gamma]$, $a \neq a'$ implies $\dashv_a \neq \dashv_{a'}$ and $\vdash_a \neq \vdash_{a'}$.

Given a subset C of $[\gamma]$, one can keep on the vector space $\mathcal{P}$ only the operations $\dashv_a$ and $\vdash_a$ such that $a \in C$. By renumbering the indexes of these operations from 1 to $\# C$ by respecting their former relative numbering, we obtain a $\# C$-pluriassociative algebra. We call it the $\# C$-*pluriassociative subalgebra induced by* C of $\mathcal{P}$.

3.1.3. *Free pluriassociative algebras.* Recall that $\mathcal{F}_{\mathsf{Dias}_\gamma}$ denotes the free Dias_γ-algebra over one generator. By definition, $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is the linear span of the set of the words on $\{0\} \cup [\gamma]$ with exactly one occurrence of 0. Let us endow this space with the linear operations

$$\dashv_a, \vdash_a \colon \mathcal{F}_{\mathsf{Dias}_\gamma} \otimes \mathcal{F}_{\mathsf{Dias}_\gamma} \to \mathcal{F}_{\mathsf{Dias}_\gamma}, \qquad a \in [\gamma], \tag{3.1.1}$$

satisfying, for any such words u and v,

$$u \dashv_a v := u \, \mathrm{h}_a(v) \tag{3.1.2a}$$

and

$$u \vdash_a v := \mathrm{h}_a(u) \, v, \tag{3.1.2b}$$

where $\mathrm{h}_a(u)$ (resp. $\mathrm{h}_a(v)$) is the word obtained by replacing in u (resp. v) any occurrence of a letter smaller than a by a.

Proposition 3.1.1. *For any integer $\gamma \geqslant 0$, the vector space $\mathcal{F}_{\mathsf{Dias}_\gamma}$ of nonempty words on $\{0\} \cup [\gamma]$ containing exactly one occurrence of 0 endowed with the operations $\dashv_a$, $\vdash_a$, $a \in [\gamma]$, is the free γ-pluriassociative algebra over one generator.*

Proof. The fact that $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is the stated vector space is a consequence of the description of the elements of Dias_γ provided by Proposition 2.1.1. Since Dias_γ is by definition the suboperad of $\mathsf{T}\mathcal{M}_\gamma$ generated by $\{0a, a0 : a \in [\gamma]\}$, $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is endowed with 2γ binary operations where any generator $0a$ (resp. $a0$) gives rise to the operation $\dashv_a$ (resp. $\vdash_a$) of $\mathcal{F}_{\mathsf{Dias}_\gamma}$. Moreover, by making use of the realization of Dias_γ, we have for all $u, v \in \mathcal{F}_{\mathsf{Dias}_\gamma}$ and $a \in [\gamma]$,

$$u \dashv_a v = (u \otimes v) \cdot 0a = (0a \circ_2 v) \circ_1 u = u \, \mathrm{h}_a(v) \tag{3.1.3a}$$

and

$$u \vdash_a v = (u \otimes v) \cdot a0 = (a0 \circ_2 v) \circ_1 u = \mathrm{h}_a(u) \, v. \tag{3.1.3b}$$

□

One has for instance in $\mathcal{F}_{\mathsf{Dias}_4}$,

$$101241 \dashv_2 203 = 101241\mathbf{2}23 \tag{3.1.4}$$

and

$$101241 \vdash_3 203 = \mathbf{3333}4\mathbf{3}203. \tag{3.1.5}$$

3.2. **Bar and wire-units.** Loday has defined in [Lod01] some notions of units in diassociative algebras. We generalize here these definitions to the context of γ-pluriassociative algebras.

3.2.1. *Bar-units.* Let $\mathcal{P}$ be a γ-pluriassociative algebra and $a \in [\gamma]$. We say that an element e of $\mathcal{P}$ is an *a-bar-unit*, or simply a *bar-unit* when taking into account the value of a is not necessary, of $\mathcal{P}$ if for all $x \in \mathcal{P}$,

$$x \dashv_a e = x = e \vdash_a x. \tag{3.2.1}$$

As we shall see below, a γ-pluriassociative algebra can have, for a given $a \in [\gamma]$, several a-bar-units. The *a-halo* of $\mathcal{P}$, denoted by $\mathrm{Halo}_a(\mathcal{P})$, is the set of the a-bar-units of $\mathcal{P}$.

3.2.2. *Wire-units.* Let $\mathcal{P}$ be a γ-pluriassociative algebra and $a \in [\gamma]$. We say that an element e of $\mathcal{P}$ is an *a-wire-unit*, or simply a *wire-unit* when taking into account the value of a is not necessary, of $\mathcal{P}$ if for all $x \in \mathcal{P}$,

$$e \dashv_a x = x = x \vdash_a e. \tag{3.2.2}$$

As shows the following proposition, the presence of a wire-unit in $\mathcal{P}$ has some implications.

Proposition 3.2.1. *Let $\gamma \geqslant 0$ be an integer and $\mathcal{P}$ be a γ-pluriassociative algebra admitting a b-wire-unit e for a $b \in [\gamma]$. Then*

(i) for all $a \in [b]$, the operations $\dashv_a$, $\dashv_b$, $\vdash_a$, and $\vdash_b$ of $\mathcal{P}$ are equal;
(ii) e is also an a-wire-unit for all $a \in [b]$;
(iii) e is the only wire-unit of $\mathcal{P}$;
(iv) if e' is an a-bar unit for a $a \in [b]$, then $e' = e$.

Proof. Let us show part *(i)*. By Relation (2.2.12d) of γ-pluriassociative algebras and by the fact that e is a b-wire-unit of $\mathcal{P}$, we have for all elements y and z of $\mathcal{P}$ and all $a \in [b]$,

$$y \dashv_a z = e \dashv_b (y \dashv_a z) = e \dashv_b (y \vdash_a z) = y \vdash_a z. \tag{3.2.3}$$

Thus, the operations $\dashv_a$ and $\vdash_a$ of $\mathcal{P}$ are equal. Moreover, for the same reasons, we have

$$y \dashv_a z = e \dashv_b (y \dashv_a z) = (e \dashv_b y) \dashv_b z = y \dashv_b z. \tag{3.2.4}$$

Then, the operations $\dashv_a$ and $\dashv_b$ of $\mathcal{P}$ are equal, whence *(i)*.

Now, by *(i)* and by the fact that e is a b-wire-unit, we have for all elements x of $\mathcal{P}$ and all $a \in [b]$,

$$e \dashv_a x = e \dashv_b x = x = x \vdash_b e = x \vdash_a e, \tag{3.2.5}$$

showing *(ii)*.

To prove *(iii)*, assume that e' is a b'-wire-unit of $\mathcal{P}$ for a $b' \in [\gamma]$. By *(i)* and by the fact that e is a b-wire-unit, one has

$$e = e \vdash_{b'} e' = e \dashv_b e' = e', \tag{3.2.6}$$

showing *(iii)*.

To establish *(iv)*, let us first prove that e is a b-bar-unit. By *(i)* and by the fact that e is a b-wire-unit, we have for all elements x of $\mathcal{P}$,

$$e \vdash_b x = e \dashv_b x = x = x \vdash_b e = x \dashv_b e. \tag{3.2.7}$$

Now, since e' is an a-bar-unit for an $a \in [b]$, by *(i)* and by the fact that e is a b-wire-unit,

$$e = e' \vdash_a e = e' \vdash_b e = e'. \tag{3.2.8}$$

This shows *(iv)*. □

Relying on Proposition 3.2.1, we define the *height* of a γ-pluriassociative algebra $\mathcal{P}$ as zero if $\mathcal{P}$ has no wire-unit, otherwise as the greatest integer $h \in [\gamma]$ such that the unique wire-unit e of $\mathcal{P}$ is a h-wire-unit. Observe that any pure γ-pluriassociative algebra has height 0 or 1.

3.3. **Construction of pluriassociative algebras.** We now present a general way to construct γ-pluriassociative algebras. Our construction is a natural generalization of some constructions introduced by Loday [Lod01] in the context of diassociative algebras. We introduce in this section new algebraic structures, the so-called γ-multiprojection algebras, which are inputs of our construction.

3.3.1. *Multiassociative algebras.* For any integer $\gamma \geqslant 0$, a *γ-multiassociative algebra* is a vector space $\mathcal{M}$ endowed with linear operations

$$\star_a : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}, \qquad a \in [\gamma], \tag{3.3.1}$$

satisfying, for all $x, y, z \in \mathcal{M}$, the relations

$$(x \star_a y) \star_b z = (x \star_b y) \star_{a'} z = x \star_{a''} (y \star_b z) = x \star_b (y \star_{a'''} z), \qquad a, a', a'', a''' \leqslant b \in [\gamma]. \tag{3.3.2}$$

These algebras are obvious generalizations of associative algebras since all of its operations are associative. Observe that by (3.3.2), all bracketings of an expression involving elements of a γ-multiassociative algebra and some of its operations are equal. Then, since the bracketings of such expressions are not significant, we shall denote these without parenthesis. In Section 3 of [Gir16], we will study the underlying operads of the category of γ-multiassociative algebras, called As_γ, for a very specific purpose.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two γ-multiassociative algebras, a linear map $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a *γ-multiassociative algebra morphism* if it commutes with the operations of $\mathcal{M}_1$ and $\mathcal{M}_2$. We say that $\mathcal{M}$ is *commutative* when all operations of $\mathcal{M}$ are commutative. Besides, for an $a \in [\gamma]$, an element $\mathbb{1}$ of $\mathcal{M}$ is an *a-unit*, or simply a *unit* when taking into account the value of a is not necessary, of $\mathcal{M}$ if for all $x \in \mathcal{M}$, $\mathbb{1} \star_a x = x = x \star_a \mathbb{1}$. When $\mathcal{M}$ admits a unit, we say that $\mathcal{M}$ is *unital*. As shows the following proposition, the presence of a unit in $\mathcal{M}$ has some implications.

Proposition 3.3.1. *Let $\gamma \geqslant 0$ be an integer and $\mathcal{M}$ be a γ-multiassociative algebra admitting a b-unit $\mathbb{1}$ for a $b \in [\gamma]$. Then*

- *(i) for all $a \in [b]$, the operations $\star_a$ and $\star_b$ of $\mathcal{M}$ are equal;*
- *(ii) $\mathbb{1}$ is also an a-unit for all $a \in [b]$;*
- *(iii) $\mathbb{1}$ is the only unit of $\mathcal{M}$.*

Proof. By Relation (3.3.2) of γ-multiassociative algebras and by the fact that $\mathbb{1}$ is a b-unit of $\mathcal{M}$, we have for all elements y and z of $\mathcal{M}$ and all $a \in [b]$,

$$y \star_a z = y \star_a z \star_b \mathbb{1} = y \star_b z \star_b \mathbb{1} = y \star_b z. \tag{3.3.3}$$

Therefore, $\star_a = \star_b$, showing *(i)*.

Now, by *(i)* and by the fact that $\mathbb{1}$ is a b-unit, we have for all elements x of $\mathcal{M}$ and all $a \in [b]$,

$$\mathbb{1} \star_a x = \mathbb{1} \star_b x = x = x \star_b \mathbb{1} = x \star_a \mathbb{1}, \tag{3.3.4}$$

showing *(ii)*.

To prove *(iii)*, assume that $\mathbb{1}'$ is a b'-unit of $\mathcal{M}$ for a $b' \in [\gamma]$. By *(i)* and by the fact that $\mathbb{1}$ is a b-unit, one has

$$\mathbb{1} = \mathbb{1} \star_{b'} \mathbb{1}' = \mathbb{1} \star_b \mathbb{1}' = \mathbb{1}', \tag{3.3.5}$$

establishing *(iii)*. □

Relying on Proposition 3.3.1, similarly to the case of γ-pluriassociative algebras, we define the *height* of a γ-multiassociative algebra $\mathcal{M}$ as zero if $\mathcal{M}$ has no unit, otherwise as the greatest integer $h \in [\gamma]$ such that the unit $\mathbb{1}$ of $\mathcal{M}$ is an h-unit.

3.3.2. *Multiprojection algebras.* We call *γ-multiprojection algebra* any γ-multiassociative algebra $\mathcal{M}$ endowed with endomorphisms

$$\pi_a : \mathcal{M} \to \mathcal{M}, \qquad a \in [\gamma], \tag{3.3.6}$$

satisfying

$$\pi_a \circ \pi_{a'} = \pi_{a \uparrow a'}, \qquad a, a' \in [\gamma]. \tag{3.3.7}$$

By extension, the *height* of $\mathcal{M}$ is its height as a γ-multiassociative algebra. We say that $\mathcal{M}$ is *unital* as a γ-multiprojection algebra if $\mathcal{M}$ is unital as a γ-multiassociative algebra and its only, by Proposition 3.3.1, unit $\mathbb{1}$ satisfies $\pi_a(\mathbb{1}) = \mathbb{1}$ for all $a \in [h]$ where h is the height of $\mathcal{M}$.

3.3.3. *From multiprojection algebras to pluriassociative algebras.* Next result describes how to construct γ-pluriassociative algebras from γ-multiprojection algebras.

Theorem 3.3.2. *For any integer $\gamma \geqslant 0$ and any γ-multiprojection algebra $\mathcal{M}$, the vector space $\mathcal{M}$ endowed with binary linear operations $\dashv_a$, $\vdash_a$, $a \in [\gamma]$, defined for all $x, y \in \mathcal{M}$ by*

$$x \dashv_a y := x \star_a \pi_a(y) \tag{3.3.8a}$$

and

$$x \vdash_a y := \pi_a(x) \star_a y, \tag{3.3.8b}$$

where the $\star_a$, $a \in [\gamma]$, are the operations of $\mathcal{M}$ and the π_a, $a \in [\gamma]$, are its endomorphisms, is a γ-pluriassociative algebra, denoted by $\mathrm{M}(\mathcal{M})$.

Proof. This is a verification of the relations of γ-pluriassociative algebras in $\mathrm{M}(\mathcal{M})$. Let x, y, and z be three elements of $\mathrm{M}(\mathcal{M})$ and $a, a' \in [\gamma]$.

By (3.3.2), we have

$$(x \vdash_{a'} y) \dashv_a z = \pi_{a'}(x) \star_{a'} y \star_a \pi_a(z) = x \vdash_{a'} (y \dashv_a z), \tag{3.3.9}$$

showing that (2.2.12a) is satisfied in $\mathrm{M}(\mathcal{M})$.

Moreover, by (3.3.2) and (3.3.7), we have

$$\begin{aligned} x \dashv_a (y \vdash_{a'} z) &= x \star_a \pi_a(\pi_{a'}(y) \star_{a'} z) \\ &= x \star_a \pi_{a\uparrow a'}(y) \star_{a'} \pi_a(z) \\ &= x \star_{a\uparrow a'} \pi_{a\uparrow a'}(y) \star_a \pi_a(z) \\ &= (x \dashv_{a\uparrow a'} y) \dashv_a z, \end{aligned} \tag{3.3.10}$$

so that (2.2.12b), and for the same reasons (2.2.12c), check out in $\mathrm{M}(\mathcal{M})$.

Finally, again by (3.3.2) and (3.3.7), we have

$$\begin{aligned} x \dashv_a (y \dashv_{a'} z) &= x \star_a \pi_a(y \star_{a'} \pi_{a'}(z)) \\ &= x \star_a \pi_a(y) \star_{a'} \pi_{a\uparrow a'}(z) \\ &= x \star_a \pi_a(y) \star_{a\uparrow a'} \pi_{a\uparrow a'}(z) \\ &= (x \dashv_a y) \dashv_{a\uparrow a'} z, \end{aligned} \tag{3.3.11}$$

showing that (2.2.12d), and for the same reasons (2.2.12e), are satisfied in $\mathrm{M}(\mathcal{M})$. □

When $\mathcal{M}$ is commutative, since for all $x, y \in \mathrm{M}(\mathcal{M})$ and $a \in [\gamma]$,

$$x \dashv_a y = x \star_a \pi_a(y) = \pi_a(y) \star_a x = y \vdash_a x, \tag{3.3.12}$$

it appears that $\mathrm{M}(\mathcal{M})$ is a commutative γ-pluriassociative algebra.

When $\mathcal{M}$ is unital, $\mathrm{M}(\mathcal{M})$ has several properties, summarized in the next proposition.

Proposition 3.3.3. *Let $\gamma \geqslant 0$ be an integer, $\mathcal{M}$ be a unital γ-multiprojection algebra of height h. Then, by denoting by $\mathbb{1}$ the unit of $\mathcal{M}$ and by π_a, $a \in [\gamma]$, its endomorphisms,*

(i) *for any $a \in [h]$, $\mathbb{1}$ is an a-bar-unit of* $\mathrm{M}(\mathcal{M})$*;*
(ii) *for any $a \leqslant b \in [h]$,* $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ *is a subset of* $\mathrm{Halo}_b(\mathrm{M}(\mathcal{M}))$*;*
(iii) *for any $a \in [h]$, the linear span of* $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ *forms an $h-a+1$-pluriassociative subalgebra of the $h-a+1$-pluriassociative subalgebra of* $\mathrm{M}(\mathcal{M})$ *induced by $[a, h]$;*
(iv) *for any $a \in [h]$, π_a is the identity map if and only if $\mathbb{1}$ is an a-wire-unit of* $\mathrm{M}(\mathcal{M})$*.*

Proof. Let us denote by $\star_a$, $a \in [\gamma]$, the operations of $\mathcal{M}$.

Since $\mathbb{1}$ is an h-unit of $\mathcal{M}$, for all elements x of $\mathrm{M}(\mathcal{M})$ and all $a \in [h]$,

$$x \dashv_a \mathbb{1} = x \star_a \pi_a(\mathbb{1}) = x \star_a \mathbb{1} = x = \mathbb{1} \star_a x = \pi_a(\mathbb{1}) \star_a x = \mathbb{1} \vdash_a x, \tag{3.3.13}$$

showing *(i)*.

Assume that e is an element of $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ for an $a \in [h]$, that is, e is an a-bar-unit of $\mathrm{M}(\mathcal{M})$. Then, for all elements x of $\mathrm{M}(\mathcal{M})$,

$$x \dashv_a e = x \star_a \pi_a(e) = x = \pi_a(e) \star_a x = e \vdash_a x, \tag{3.3.14}$$

showing that $\pi_a(e)$ is the unit for the operation $\star_a$ on $\mathrm{M}(\mathcal{M})$ and therefore, $\pi_a(e) = \mathbb{1}$. Since $\mathcal{M}$ is unital, we have $\pi_b(\mathbb{1}) = \mathbb{1}$ for all $b \in [h]$. Hence, and by (3.3.7), for all $a \leqslant b \in [h]$,

$$\pi_b(e) = \pi_b(\pi_a(e)) = \pi_b(\mathbb{1}) = \mathbb{1}. \tag{3.3.15}$$

Then, for all elements x of $\mathrm{M}(\mathcal{M})$ and all $a \leqslant b \in [h]$,

$$x \dashv_b e = x \star_b \pi_b(e) = x \star_b \mathbb{1} = x = \mathbb{1} \star_b x = \pi_b(e) \star_b x = e \vdash_b x, \tag{3.3.16}$$

showing that e is also a b-bar-unit of $\mathrm{M}(\mathcal{M})$, whence *(ii)*.

Let $a \in [\gamma]$ and e and e' be elements of $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$. By *(ii)*, e and e' are b-bar-units of $\mathrm{M}(\mathcal{M})$ for all $a \leqslant b \in [h]$ and hence,

$$e \dashv_b e' = e = e' \vdash_b e. \tag{3.3.17}$$

Therefore, the linear span of $\mathrm{Halo}_a(\mathrm{M}(\mathcal{M}))$ is stable for the operations $\dashv_b$ and $\vdash_b$. This implies *(iii)*.

Finally, assume that π_a is the identity map for an $a \in [h]$. Then, for all elements x of $\mathrm{M}(\mathcal{M})$,

$$\mathbb{1} \dashv_a x = \mathbb{1} \star_a \pi_a(x) = \mathbb{1} \star_a x = x = x \star_a \mathbb{1} = \pi_a(x) \star_a \mathbb{1} = x \vdash_a \mathbb{1}, \tag{3.3.18}$$

showing that $\mathbb{1}$ is an a-wire unit of $\mathrm{M}(\mathcal{M})$. Conversely, if $\mathbb{1}$ is an a-wire unit of $\mathrm{M}(\mathcal{M})$, for all elements x of $\mathrm{M}(\mathcal{M})$, the relations $\mathbb{1} \dashv_a x = x = x \vdash_a \mathbb{1}$ imply $\mathbb{1} \star_a \pi_a(x) = x = \pi_a(x) \star_a \mathbb{1}$ and hence, $\pi_a(x) = x$. This shows *(iv)*. □

3.3.4. *Examples of constructions of pluriassociative algebras.* The construction M of Theorem 3.3.2 allows to build several γ-pluriassociative algebras. Here follows few examples.

The γ-pluriassociative algebra of positive integers. Let $\gamma \geqslant 1$ be an integer and consider the vector space Pos of positive integers, endowed with the operations $\star_a$, $a \in [\gamma]$, all equal to the operation $\uparrow$ extended by linearity and with the endomorphisms π_a, $a \in [\gamma]$, linearly defined for any positive integer x by $\pi_a(x) := a \uparrow x$. Then, Pos is a non-unital γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{Pos})$ is a γ-pluriassociative algebra. We have for instance

$$2 \dashv_3 5 = 5, \tag{3.3.19}$$

and

$$1 \vdash_3 2 = 3. \tag{3.3.20}$$

We can observe that $\mathrm{M}(\mathsf{Pos})$ is commutative, pure, and its 1-halo is $\{1\}$. Moreover, when $\gamma \geqslant 2$, $\mathrm{M}(\mathsf{Pos})$ has no wire-unit and no a-bar-unit for $a \geqslant 2 \in [\gamma]$. This example is important because it provides a counterexample for *(ii)* of Proposition 3.3.3 in the case when the construction M is applied to a non-unital γ-multiprojection algebra.

The γ-pluriassociative algebra of finite sets. Let $\gamma \geqslant 1$ be an integer and consider the vector space Sets of finite sets of positive integers, endowed with the operations $\star_a$, $a \in [\gamma]$, all equal to the union operation $\cup$ extended by linearity and with the endomorphisms π_a, $a \in [\gamma]$, linearly defined for any finite set of positive integers x by $\pi_a(x) := x \cap [a, \gamma]$. Then, Sets is a γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{Sets})$ is a γ-pluriassociative algebra. We have for instance

$$\{2, 4\} \dashv_3 \{1, 3, 5\} = \{2, 3, 4, 5\}, \tag{3.3.21}$$

and

$$\{1, 2, 4\} \vdash_3 \{1, 3, 5\} = \{1, 3, 4, 5\}. \tag{3.3.22}$$

We can observe that $\mathrm{M}(\mathsf{Sets})$ is commutative and pure. Moreover, $\emptyset$ is a 1-wire-unit of $\mathrm{M}(\mathsf{Sets})$ and, by Proposition 3.2.1, it is its only wire-unit. Therefore, $\mathrm{M}(\mathsf{Sets})$ has height 1. Observe that for any $a \in [\gamma]$, the a-halo of $\mathrm{M}(\mathsf{Sets})$ consists in the subsets of $[a-1]$. Besides, since Sets is a unital γ-multiprojection algebra, $\mathrm{M}(\mathsf{Sets})$ satisfies all properties exhibited by Proposition 3.3.3.

The γ-pluriassociative algebra of words. Let $\gamma \geqslant 1$ be an integer and consider the vector space Words of the words of positive integers. Let us endow Words with the operations $\star_a$, $a \in [\gamma]$, all equal to the concatenation operation extended by linearity and with the endomorphisms π_a, $a \in [\gamma]$, where for any word x of positive integers, $\pi_a(x)$ is the longest subword of x consisting in letters greater than or equal to a. Then, Words is a γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{Words})$ is a γ-pluriassociative algebra. We have for instance

$$412 \dashv_3 14231 = 41243, \tag{3.3.23}$$

and

$$11 \vdash_2 323 = 323. \tag{3.3.24}$$

We can observe that $\mathrm{M}(\mathsf{Words})$ is not commutative and is pure. Moreover, ϵ is a 1-wire-unit of $\mathrm{M}(\mathsf{Words})$ and by Proposition 3.2.1, it is its only wire-unit. Therefore, $\mathrm{M}(\mathsf{Words})$ has height 1. Observe that for any $a \in [\gamma]$, the a-halo of $\mathrm{M}(\mathsf{Words})$ consists in the words on the alphabet $[a-1]$. Besides, since Words is a unital γ-multiprojection algebra, $\mathrm{M}(\mathsf{Words})$ satisfies all properties exhibited by Proposition 3.3.3.

The γ-pluriassociative algebras $\mathrm{M}(\mathsf{Sets})$ and $\mathrm{M}(\mathsf{Words})$ are related in the following way. Let I_{com} be the subspace of $\mathrm{M}(\mathsf{Words})$ generated by the $x - x'$ where x and x' are words of positive integers and have the same commutative image. Since I_{com} is a γ-pluriassociative algebra ideal of $\mathrm{M}(\mathsf{Words})$, one can consider the quotient γ-pluriassociative algebra $\mathsf{CWords} := \mathrm{M}(\mathsf{Words})/_{I_{\mathrm{com}}}$. Its elements can be seen as commutative words of positive integers.

Moreover, let I_{occ} be the subspace of $\mathrm{M}(\mathsf{CWords})$ generated by the $x - x'$ where x and x' are commutative words of positive integers and for any letter $a \in [\gamma]$, a appears in x if and only if a appears in x'. Since I_{occ} is a γ-pluriassociative algebra ideal of $\mathrm{M}(\mathsf{CWords})$, one can consider the quotient γ-pluriassociative algebra $\mathrm{M}(\mathsf{CWords})/_{I_{\mathrm{occ}}}$. Its elements can be seen as finite subsets of positive integers and we observe that $\mathrm{M}(\mathsf{CWords})/_{I_{\mathrm{occ}}} = \mathrm{M}(\mathsf{Sets})$.

The γ-pluriassociative algebra of marked words. Let $\gamma \geqslant 1$ be an integer and consider the vector space MWords of the words of positive integers where letters can be marked or not, with at least one occurrence of a marked letter. We denote by $\bar{a}$ any *marked letter* a and we say that the *value* of $\bar{a}$ is a. Let us endow MWords with the linear operations $\star_a$, $a \in [\gamma]$, where for all words u and v of MWords, $u \star_a v$ is obtained by concatenating u and v, and by replacing therein all marked letters by $\bar{c}$ where $c := \max(u) \uparrow a \uparrow \max(v)$ where $\max(u)$ (resp. $\max(v)$) denotes the greatest value among the marked letters of u (resp. v). For instance,

$$2\bar{1}31\bar{3} \star_2 3\bar{4}\bar{1}21 = 2\bar{\mathbf{4}}31\bar{\mathbf{4}}3\bar{4}\bar{\mathbf{4}}21, \tag{3.3.25}$$

and

$$\bar{2}11\bar{1} \star_3 34\bar{2} = \bar{\mathbf{3}}11\bar{\mathbf{3}}34\bar{\mathbf{3}}. \tag{3.3.26}$$

We also endow MWords with the endomorphisms π_a, $a \in [\gamma]$, where for any word u of MWords, $\pi_a(u)$ is obtained by replacing in u any occurrence of a nonmarked letter smaller than a by a. For instance,

$$\pi_3\left(2\bar{2}14\bar{4}3\bar{5}\right) = \mathbf{3}\bar{2}\mathbf{3}4\bar{4}\mathbf{3}\bar{5}. \tag{3.3.27}$$

One can show without difficulty that MWords is a γ-multiprojection algebra. By Theorem 3.3.2, $\mathrm{M}(\mathsf{MWords})$ is a γ-pluriassociative algebra. We have for instance

$$3\bar{2}5 \dashv_3 4\bar{4}1 = 3\bar{\mathbf{4}}54\bar{4}\mathbf{3}, \tag{3.3.28}$$

and

$$1\bar{3}4\bar{1}3 \vdash_2 31\bar{2}3\bar{1}1 = \mathbf{2}\bar{3}4\bar{\mathbf{3}}331\bar{\mathbf{3}}3\bar{\mathbf{3}}1. \tag{3.3.29}$$

We can observe that $\mathrm{M}(\mathsf{MWords})$ is not commutative, pure, and has no wire-units neither bar-units.

The free γ-pluriassociative algebra over one generator. Let $\gamma \geqslant 0$ be an integer. We give here a construction of the free γ-pluriassociative algebra $\mathcal{F}_{\mathsf{Dias}_\gamma}$ over one generator described in Section 3.1.3 passing through the following γ-multiprojection algebra and the construction M. Consider the vector space of nonempty words on the alphabet $\{0\} \cup [\gamma]$ with exactly one occurrence of 0, endowed with the operations $\star_a$, $a \in [\gamma]$, all equal to the concatenation operation extended by linearity and with the endomorphisms h_a, $a \in [\gamma]$, defined in Section 3.1.3. This vector space is a γ-multiprojection algebra. Therefore, by Theorem 3.3.2, it gives rise by the construction M to a γ-pluriassociative algebra and it appears that it is $\mathcal{F}_{\mathsf{Dias}_\gamma}$. Besides, we can now observe that $\mathcal{F}_{\mathsf{Dias}_\gamma}$ is not commutative, pure, and has no wire-units neither bar-units.

4. Pluritriassociative operads

Our original idea of using the T construction (see Sections 1.1.3 and 2.1.1) to obtain a generalization of the diassociative operad admits an analogue in the context of the triassociative operad [LR04]. We describe in this section a generalisation on a nonnegative integer parameter γ of the triassociative operad.

Since the proofs of the results contained in this section are very similar to the ones of Section 2, we omit proofs here.

4.1. **Construction and first properties.** For any integer $\gamma \geqslant 0$, we define Trias_γ as the suboperad of $\mathcal{M}_\gamma$ generated by

$$\{0a, 00, a0 : a \in [\gamma]\}. \tag{4.1.1}$$

By definition, Trias_γ is the vector space of words that can be obtained by partial compositions of words of (4.1.1). We have, for instance,

$$\mathsf{Trias}_2(1) = \mathrm{Vect}(\{0\}), \tag{4.1.2}$$

$$\mathsf{Trias}_2(2) = \mathrm{Vect}(\{00, 01, 02, 10, 20\}), \tag{4.1.3}$$

$$\begin{split}\mathsf{Trias}_2(3) = \mathrm{Vect}(\{&000, 001, 002, 010, 011, 012, 020, 021, \\ &022, 100, 101, 102, 110, 120, 200, 201, 202, 210, 220\}),\end{split} \tag{4.1.4}$$

It follows immediately from the definition of Trias_γ as a suboperad of $\mathsf{T}\mathcal{M}_\gamma$ that Trias_γ is a set-operad. Moreover, one can observe that Trias_γ is generated by the same generators as the ones of Dias_γ (see (2.1.1)), plus the word 00. Therefore, Dias_γ is a suboperad of Trias_γ. Besides, note that Trias_0 is the associative operad and that Trias_γ is a suboperad of $\mathsf{Trias}_{\gamma+1}$. We call Trias_γ the *γ-pluritriassociative operad*.

Proposition 4.1.1. *For any integer $\gamma \geqslant 0$, as a set-operad, the underlying set of Trias_γ is the set of the words on the alphabet $\{0\} \cup [\gamma]$ containing at least one occurrence of 0.*

We deduce from Proposition 4.1.1 that the Hilbert series of Trias_γ satisfies

$$\mathcal{H}_{\mathsf{Trias}_\gamma}(t) = \frac{t}{(1-\gamma t)(1-\gamma t - t)} \tag{4.1.5}$$

and that for all $n \geqslant 1$, $\dim \mathsf{Trias}_\gamma(n) = (\gamma+1)^n - \gamma^n$. For instance, the first dimensions of Trias_1, Trias_2, Trias_3, and Trias_4 are respectively

$$1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, \tag{4.1.6}$$

$$1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025, 175099, \tag{4.1.7}$$

$$1, 7, 37, 175, 781, 3367, 14197, 58975, 242461, 989527, 4017157, \tag{4.1.8}$$

$$1, 9, 61, 369, 2101, 11529, 61741, 325089, 1690981, 8717049, 44633821. \tag{4.1.9}$$

The first one is Sequence **A000225**, the second one is Sequence **A001047**, the third one is Sequence **A005061**, and the last one is Sequence **A005060** of [Slo].

4.2. **Presentation by generators and relations.** We follow the same strategy as the one used in Section 2.2 to establish a presentation by generators and relations of Trias_γ and prove that it is a Koszul operad. As announced above, we omit complete proofs here but we describe the analogue for Trias_γ of the maps word_γ and hook_γ defined in Section 2.2 for the operad Dias_γ.

For any integer $\gamma \geqslant 0$, let $\mathfrak{G}_{\mathsf{Trias}_\gamma} := \mathfrak{G}_{\mathsf{Trias}_\gamma}(2)$ be the graded set where

$$\mathfrak{G}_{\mathsf{Trias}_\gamma}(2) := \{\dashv_a, \perp, \vdash_a : a \in [\gamma]\}. \tag{4.2.1}$$

Let $\mathfrak{t}$ be a syntax tree of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)$ and x be a leaf of $\mathfrak{t}$. We say that an integer $a \in \{0\} \cup [\gamma]$ is *eligible* for x if $a = 0$ or there is an ancestor y of x labeled by $\dashv_a$ (resp. $\vdash_a$) and x is in the right (resp. left) subtree of y. The *image* of x is its greatest eligible integer. Moreover, let

$$\mathrm{wordt}_\gamma : \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)(n) \to \mathsf{Trias}_\gamma(n), \qquad n \geqslant 1, \tag{4.2.2}$$

the map where $\mathrm{wordt}_\gamma(\mathfrak{t})$ is the word obtained by considering, from left to right, the images of the leaves of $\mathfrak{t}$ (see Figure 2). Observe that wordt_γ is an extension of word_γ (see (2.2.2)).

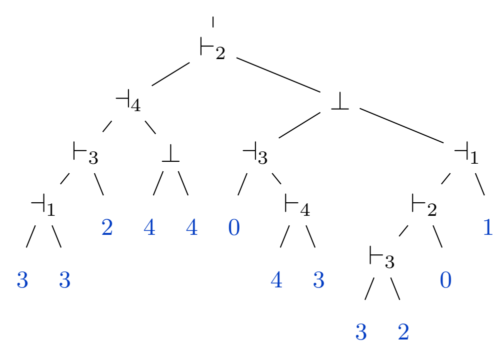


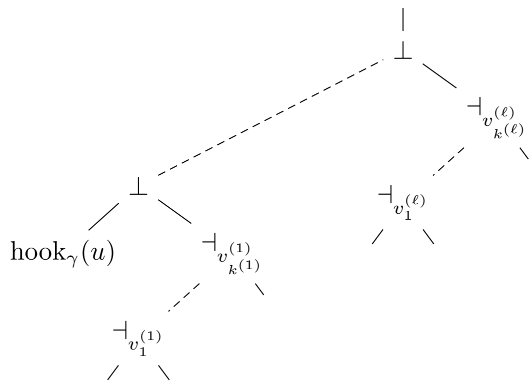
FIGURE 2. A syntax tree $\mathfrak{t}$ of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)$ where images of its leaves are shown. This tree satisfies $\mathrm{wordt}_\gamma(\mathfrak{t}) = 332440433201$.

Consider now the map

$$\mathrm{hookt}_\gamma : \mathsf{Trias}_\gamma(n) \to \mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)(n), \qquad n \geqslant 1, \tag{4.2.3}$$

defined for any word x of Trias_γ by

$$\mathrm{hookt}_\gamma(x) := \qquad (4.2.4)$$



where x decomposes, by Proposition 4.1.1, uniquely in $x = u0v^{(1)} \dots 0v^{(\ell)}$ where u is a word of Dias_γ and for all $i \in [\ell]$, the $v^{(i)}$ are words on the alphabet $[\gamma]$. The length $|v^{(i)}|$ of any v_i is denoted by $k^{(i)}$. The dashed edges denote left comb trees wherein internal nodes are labeled as specified. Observe that hookt_γ is an extension of hook_γ (see (2.2.3)). We shall call any syntax tree of the form (4.2.4) an *extended hook syntax tree.*

Theorem 4.2.1. *For any integer $\gamma \geqslant 0$, the operad Trias_γ admits the following presentation. It is generated by $\mathfrak{G}_{\mathsf{Trias}_\gamma}$ and its space of relations $\mathfrak{R}_{\mathsf{Trias}_\gamma}$ is the space induced by the equivalence relation $\leftrightarrow_\gamma$ satisfying*

$$\perp \circ_1 \perp \;\leftrightarrow_\gamma\; \perp \circ_2 \perp, \tag{4.2.5a}$$

$$\dashv_a \circ_1 \perp \;\leftrightarrow_\gamma\; \perp \circ_2 \dashv_a, \qquad a \in [\gamma], \tag{4.2.5b}$$

$$\perp \circ_1 \vdash_a \;\leftrightarrow_\gamma\; \vdash_a \circ_2 \perp, \qquad a \in [\gamma], \tag{4.2.5c}$$

$$\perp \circ_1 \dashv_a \;\leftrightarrow_\gamma\; \perp \circ_2 \vdash_a, \qquad a \in [\gamma], \tag{4.2.5d}$$

$$\dashv_a \circ_1 \vdash_{a'} \;\leftrightarrow_\gamma\; \vdash_{a'} \circ_2 \dashv_a, \qquad a, a' \in [\gamma], \tag{4.2.5e}$$

$$\dashv_a \circ_1 \dashv_b \;\leftrightarrow_\gamma\; \dashv_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{4.2.5f}$$

$$\vdash_a \circ_1 \dashv_b \;\leftrightarrow_\gamma\; \vdash_a \circ_2 \vdash_b, \qquad a < b \in [\gamma], \tag{4.2.5g}$$

$$\dashv_b \circ_1 \dashv_a \;\leftrightarrow_\gamma\; \dashv_a \circ_2 \dashv_b, \qquad a < b \in [\gamma], \tag{4.2.5h}$$

$$\vdash_a \circ_1 \vdash_b \;\leftrightarrow_\gamma\; \vdash_b \circ_2 \vdash_a, \qquad a < b \in [\gamma], \tag{4.2.5i}$$

$$\dashv_d \circ_1 \dashv_d \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \perp \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \dashv_c \;\leftrightarrow_\gamma\; \dashv_d \circ_2 \vdash_c, \qquad c \leqslant d \in [\gamma], \tag{4.2.5j}$$

$$\vdash_d \circ_1 \dashv_c \;\leftrightarrow_\gamma\; \vdash_d \circ_1 \vdash_c \;\leftrightarrow_\gamma\; \vdash_d \circ_1 \perp \;\leftrightarrow_\gamma\; \vdash_d \circ_2 \vdash_d, \qquad c \leqslant d \in [\gamma]. \tag{4.2.5k}$$

Observe that, by Theorem 4.2.1, Trias_1 and the triassociative operad [LR04] admit the same presentation. Then, for all integers $\gamma \geqslant 0$, the operads Trias_γ are generalizations of the triassociative operad.

Theorem 4.2.2. *For any integer $\gamma \geqslant 0$, Trias_γ is a Koszul operad. Moreover, the set of extended hook syntax trees of $\mathbf{Free}\left(\mathfrak{G}_{\mathsf{Trias}_\gamma}\right)$ forms a Poincaré-Birkhoff-Witt basis of Trias_γ.*

References

[AL04] M. Aguiar and J.-L Loday. Quadri-algebras. *J. Pure Appl. Algebra*, 191(3):205–221, 2004. 3

[BN98] F. Baader and T. Nipkow. *Term rewriting and all that*. Cambridge University Press, Cambridge, New York, NY, USA, 1998. 10

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, volume 347 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1973. 2

[Cha05] F. Chapoton. On some anticyclic operads. *Algebr. Geom. Topol.*, 5:53–69, 2005. 2, 11

[Cha08] F. Chapoton. Operads and algebraic combinatorics of trees. *Sém. Lothar. Combin.*, 58, 2008. 2

[Cha14] F. Chapoton. Flows on rooted trees and the Menous-Novelli-Thibon idempotents. *Math. Scand.*, 115(1), 2014. 4, 20

[CL01] F. Chapoton and M. Livernet. Pre-Lie algebras and the rooted trees operad. *Int. Math. Res. Notices*, 8:395–408, 2001. 2

[CL07] F. Chapoton and M. Livernet. Relating two Hopf algebras built from an operad. *Int. Math. Res. Notices*, 24:Art. ID rnm131, 27, 2007. 20

[DK10] V. Dotsenko and A. Khoroshkin. Gröbner bases for operads. *Duke Math. J.*, 153(2):363–396, 2010. 3, 10, 18

[Gir12] S. Giraudo. Constructing combinatorial operads from monoids. *Formal Power Series and Algebraic Combinatorics*, pages 229–240, 2012. 2, 3, 4, 6, 11

[Gir15] S. Giraudo. Combinatorial operads from monoids. *J. Algebr. Comb.*, 41(2):493–538, 2015. 2, 3, 4, 6, 11

[Gir16] S. Giraudo. Pluriassociative algebras II: The polydendriform operad and related operads. *Adv. Appl. Math.*, 77:43–85, 2016. 3, 4, 8, 25, 28

[GK94] V. Ginzburg and M. M. Kapranov. Koszul duality for operads. *Duke Math. J.*, 76(1):203–272, 1994. 3, 10

[Hof10] E. Hoffbeck. A Poincaré-Birkhoff-Witt criterion for Koszul operads. *Manuscripta Math.*, 131(1-2):87–110, 2010. 3, 10, 18

[Knu97] D. Knuth. *The Art of Computer Programming, volume 1: Fundamental Algorithms*. Addison Wesley Longman, Redwood City, CA, USA, 3rd edition, 1997. 8

[Ler04] P. Leroux. Ennea-algebras. *J. Algebra*, 281(1):287–302, 2004. 3

[Ler07] P. Leroux. A simple symmetry generating operads related to rooted planar m-ary trees and polygonal numbers. *J. Integer Seq.*, 10(4):Article 07.4.7, 23, 2007. 3

[Liv06] M. Livernet. A rigidity theorem for pre-Lie algebras. *J. Pure Appl. Algebra*, 207(1):1–18, 2006. 2

[Lod95] J.-L. Loday. Cup-product for Leibniz cohomology and dual Leibniz algebras. *Math. Scand.*, 77(2), 1995. 2

[Lod01] J.-L. Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001. 2, 3, 4, 10, 18, 27, 28

[Lod08] J.-L. Loday. Generalized bialgebras and triples of operads. *Astérisque*, 320:x+116, 2008. 2

[LR04] J.-L. Loday and M. Ronco. Trialgebras and families of polytopes. In *Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory*, volume 346 of *Contemp. Math.*, pages 369–398. Amer. Math. Soc., Providence, RI, 2004. 3, 4, 33, 36

[LV12] J.-L. Loday and B. Vallette. *Algebraic Operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften*. Springer, Heidelberg, 2012. 2, 10, 18

[May72] J. P. May. *The geometry of iterated loop spaces*. Springer-Verlag, Berlin-New York, 1972. Lectures Notes in Mathematics, Vol. 271. 2

[MY91] M. Méndez and J. Yang. Möbius Species. *Adv. Math.*, 85(1):83–128, 1991. 2, 20

[Nov14] J.-C. Novelli. m-dendriform algebras. `arXiv:1406.1616v1[math.CO]`, 2014. 3

[Sch94] W. R. Schmitt. Incidence Hopf algebras. *J. Pure Appl. Algebra*, 96(3):299–330, 1994. 20

[Slo] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. `https://oeis.org/`. 13, 34

[Sta11] R.P. Stanley. *Enumerative Combinatorics, Volume 1*. Cambridge University Press, New York, NY, USA, 2nd edition, 2011. 22

[Val07] B. Vallette. Homology of generalized partition posets. *J. Pure Appl. Algebra*, 208(2):699–725, 2007. 4, 20

Laboratoire d'Informatique Gaspard-Monge, Université Paris-Est Marne-la-Vallée, 5 boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

E-mail address: samuele.giraudo@u-pem.fr