Generating Asymptotics for factorially divergent sequences

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Abstract

The algebraic properties of formal power series with factorial growth which admit a certain well-behaved asymptotic expansion are discussed. It is shown that these series form a subring of $\mathbb{R}[[x]]$ which is also closed under composition of power series. An 'asymptotic derivation' is defined which maps a power series to its asymptotic expansion. Leibniz and chain rules for this derivation are deduced. With these rules asymptotic expansions of implicitly defined power series can be obtained. The full asymptotic expansions of the number of connected chord diagrams and the number of simple permutations are given as examples.

Keywords: Asymptotic expansions; Formal power series; Chord diagrams; Simple permutations

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1 Introduction

This article is concerned with sequences a_n, which admit an asymptotic expansion of the form,

$$a_n \sim \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \ldots\right), \tag{1.1}$$

for some $\alpha, \beta \in \mathbb{R}_{>0}$ and $c_k \in \mathbb{R}$. Sequences of this type appear in many enumeration problems, which deal with coefficients of factorial growth. For instance, generating functions of subclasses of permutations and graphs of fixed valence show this behaviour [1, 7]. Furthermore, there are countless examples where *perturbative expansions* of physical quantities admit asymptotic expansions of this kind [4, 22, 15].

The restriction to this specific class of power series is inspired by the work of Bender [6]. In this work the asymptotic behaviour of the composition of a power series, which has mildly growing coefficients, with a power series, which has rapidly growing coefficients, is analyzed. Bender's results are extended into a complete algebraic framework. This is achieved making heavy use of generating functions in the spirit of the analytic combinatorics or 'generatingfunctionology' approach [18, 29]. The key step in this direction is to interpret the *coefficients of the asymptotic expansion as another power series.*

These structures bear many resemblances to the theory of resurgence, which was established by Jean Ecalle [16]. Resurgence assigns a special role to power series which diverge factorially, as they offer themselves to be Borel transformed. Jean Ecalle's theory can be used to assign a unique function to a factorially divergent series. This function could be interpreted as the series' generating function. Moreover, resurgence provides a promising approach to cope with divergent perturbative expansions in physics. Its application to these problems is an active field of research [15, 2].

During a conversation with David Sauzin it became plausible that the presented methods can also be derived from resurgence. In fact, the formalism can be seen as a toy model of resurgence's *calcul différentiel étranger* [16, Vol. 1] also called *alien calculus* [23, II.6]. This toy model is unable to fully reconstruct functions from asymptotic expansions, but does not rely on analytic properties of Borel transformed functions and therefore offers itself for combinatorial applications. A detailed and illuminating account on resurgence theory is given in David Sauzin's review [23, Part II] or [24]. For a review focused on applications to problems from physics consult [2].

1.1 Statement of results

Power series with well-behaved asymptotic expansions, as in eq. (1.1), form a subring of $\mathbb{R}[[x]]$, which will be denoted as $\mathbb{R}[[x]]^\alpha_\beta$. This subring is also closed under composition and inversion of power series. A linear map, $\mathcal{A}^\alpha_\beta : \mathbb{R}[[x]]^\alpha_\beta \to \mathbb{R}[[x]]$, can be defined which *maps a power series to the asymptotic expansion of its coefficients.* A natural way to define such a map is to associate the power series $\sum_{n=0} c_n x^n$ to the series $\sum_{n=0} a_n x^n$ both related as in eq. (1.1). This map turns out to be a *derivation*, that means it fulfills a *Leibniz rule*

$$\text{with } f,g \in \mathbb{R}[[x]]^\alpha_\beta \qquad (\mathcal{A}^\alpha_\beta(f\cdot g))(x) = f(x)(\mathcal{A}^\alpha_\beta g)(x) + g(x)(\mathcal{A}^\alpha_\beta f)(x)$$

$$\text{and a \textit{chain rule},} \qquad (\mathcal{A}^\alpha_\beta(f\circ g))(x) = f'(g(x))(\mathcal{A}^\alpha_\beta g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}}(\mathcal{A}^\alpha_\beta f)(g(x)),$$

where $(f\cdot g)(x) = f(x)g(x)$ and $(f\circ g)(x) = f(g(x))$. These statements will be derived from elementary properties of the Gamma function. Note that the chain rule involves a peculiar correction term if f has a non-trivial asymptotic expansion. The fact that the chain rule cannot be simple, that means for general f, g: $(\mathcal{A}^\alpha_\beta (f \circ g))(x) \neq f'(g(x))(\mathcal{A}^\alpha_\beta g)(x)$, is obvious. The reasonable requirement that the function $g(x) = x$ has a trivial asymptotic expansion, $(\mathcal{A}^\alpha_\beta g)(x) = 0$, would otherwise imply that all $f \in \mathbb{R}[[x]]^\alpha_\beta$ have trivial asymptotic expansions. The formalism can be applied to calculate the asymptotic expansions of implicitly defined power series. This procedure is similar to the extraction of the derivative of an implicitly defined function using the implicit function theorem. In sections 2-4 the derivation ring $\mathbb{R}[[x]]^\alpha_\beta$ will be described and the main Theorem 4.4, which establishes the chain rule for the asymptotic derivation, will be proven. In section 5, the apparatus will be applied to the calculation of the full asymptotic expansions of the number of *connected chord diagrams* and of the number of *simple permutations*.

1.2 Notation

A (formal) power series $f \in \mathbb{R}[[x]]$ will be denoted in the usual 'functional' notation $f(x) = \sum_{n=0}^\infty f_n x^n$. The coefficients of a power series f will be expressed by the same symbol with the index attached as a subscript f_n or with the coefficient extraction operator $[x^n]f(x) = f_n$. Ordinary (formal) derivatives are expressed as $f'(x) = \sum_{n=0}^\infty n f_n x^{n-1}$. The ring of power series, restricted to expansions of functions which are analytic at the origin, or equivalently power series with non-vanishing radius of convergence, will be denoted as $\mathbb{R}\{x\}$. The $\mathcal{O}$-notation will be used: $\mathcal{O}(a_n)$ denotes the set of all sequences b_n such that $\limsup_{n\to\infty} |\frac{b_n}{a_n}| < \infty$ and $o(a_n)$ denotes all sequences b_n such that $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$. Equations of the form $a_n = b_n + \mathcal{O}(c_n)$ are to be interpreted as statements $a_n - b_n \in \mathcal{O}(c_n)$ as usual. See [5] for an introduction to this notation. Tuples of non-negative integers will be denoted by bold letters $\mathbf{t} = (t_1, \ldots, t_L) \in \mathbb{N}_0^L$. The notation $|\mathbf{t}|$ will be used as a short form for $\sum_{l=1}^L t_l$.

2 Prerequisites

The first step is to establishing a suitable notion of a power series with a well-behaved asymptotic expansion.

Definition 2.1. For given $\alpha, \beta \in \mathbb{R}_{>0}$ let $\mathbb{R}[[x]]^\alpha_\beta$ be the subset of $\mathbb{R}[[x]]$, such that $f \in \mathbb{R}[[x]]^\alpha_\beta$ if and only if there exists a sequence of real numbers $(c_k^f)_{k\in\mathbb{N}_0}$, which fulfills

$$f_n = \sum_{k=0}^{R-1} c_k^f \alpha^{n+\beta-k}\Gamma(n+\beta-k) + \mathcal{O}\left(\alpha^n \Gamma(n+\beta-R)\right) \qquad \forall R \in \mathbb{N}_0. \tag{2.1}$$

Corollary 2.2. *$\mathbb{R}[[x]]^\alpha_\beta$ is a linear subspace of $\mathbb{R}[[x]]$.*

Remark 2.3. The expression in eq. (2.1) represents an asymptotic expansion or Poincaré expansion with the asymptotic scale $\alpha^n\Gamma(n+\beta)$ [14, Ch. 1.5].

Remark 2.4. For fixed R, an expansion as above with R explicit summands will be called an asymptotic expansion up to order $R-1$.

Remark 2.5. The subspace $\mathbb{R}[[x]]^\alpha_\beta$ includes all (real) power series whose coefficients only grow exponentially: $\mathbb{R}\{x\} \subset \mathbb{R}[[x]]^\alpha_\beta$.

Remark 2.6. These with other series, which are in $o(\alpha^n\Gamma(n+\beta-R))$ for all fixed $R \geq 0$, have an asymptotic expansion of the form in eq. (2.1) with all the $c_k^f = 0$.

Remark 2.7. Definition 2.1 implies that $f_n \in \mathcal{O}\left(\alpha^n\Gamma(n+\beta)\right)$. Accordingly, the power series in $\mathbb{R}[[x]]^\alpha_\beta$ are a subset of *Gevrey-1* sequences [20, Ch XI-2]. Being *Gevrey-1* is not sufficient for a power series to be in $\mathbb{R}[[x]]^\alpha_\beta$. For instance, a sequence which behaves for large n as $a_n \sim n!(1 + \frac{1}{\sqrt{n}} + \mathcal{O}(\frac{1}{n}))$ is *Gevrey-1*, but not in $\mathbb{R}[[x]]^\alpha_\beta$.

Remark 2.8. In resurgence theory further restrictions on the allowed power series are imposed, which ensure that the Borel transformations of the sequences have proper analytic continuations or are 'endless continuable' [23, II.6]. These restrictions are analogous to the requirement that, apart from f_n, also c_k has to have a well-behaved asymptotic expansion. The coefficients of this asymptotic expansion are also required to have a well-behaved asymptotic expansion and so on. These kind of restrictions will not be necessary for the presented algebraic considerations, which are aimed at combinatorial applications.

The central theme of this article is to *interpret the coefficients* c_k^f *of the asymptotic expansion as another power series.* In fact, Definition 2.1 immediately suggests to define the following map:

Definition 2.9. Let $\mathcal{A}_\beta^\alpha : \mathbb{R}[[x]]_\beta^\alpha \to \mathbb{R}[[x]]$ be the map that associates a power series $\mathcal{A}_\beta^\alpha f \in \mathbb{R}[[x]]$ to every power series $f \in \mathbb{R}[[x]]_\beta^\alpha$ such that,

$$f_n = \sum_{k=0}^{R-1} \alpha^{n+\beta-k}\Gamma(n+\beta-k)[x^k](\mathcal{A}_\beta^\alpha f)(x) + \mathcal{O}\left(\alpha^n\Gamma(n+\beta-R)\right) \qquad \forall R \in \mathbb{N}_0. \tag{2.2}$$

Corollary 2.10. $\mathcal{A}_\beta^\alpha$ *is linear.*

Remark 2.11. In the realm of resurgence such an operator is called *alien derivative* or *alien operator* [23, II.6].

Example 2.12. The power series $f \in \mathbb{R}[[x]]$ associated to the sequence $f_n = n!$ clearly fulfills the requirements of Definition 2.1 with $\alpha = 1$ and $\beta = 1$. Therefore, $f \in \mathbb{R}[[x]]_1^1$ and $(\mathcal{A}_1^1 f)(x) = 1$.

The asymptotic expansion in eq. (2.2) is normalized such that shifts in n can be absorbed by shifts in β. More specifically,

Proposition 2.13.

- *If* $f \in \mathbb{R}[[x]]_\beta^\alpha$ *and the first* m *coefficients of* $f(x)$ *vanish, then* $\frac{f(x)}{x^m} \in \mathbb{R}[[x]]_{\beta+m}^\alpha$ *and*

$$(\mathcal{A}_\beta^\alpha f)(x) = \left(\mathcal{A}_{\beta+m}^\alpha \frac{f(x)}{x^m}\right)(x). \tag{2.3}$$

- *If* $f \in \mathbb{R}[[x]]_\beta^\alpha$ *and* $m \in \mathbb{N}_0$ *with* $\beta > m$*, then* $x^m f(x) \in \mathbb{R}[[x]]_{\beta-m}^\alpha$ *and*

$$(\mathcal{A}_\beta^\alpha f)(x) = \left(\mathcal{A}_{\beta-m}^\alpha x^m f(x)\right)(x). \tag{2.4}$$

Proof. Suppose the first m coefficients of f vanish. Set $g(x) = \frac{f(x)}{x^m}$ or $g_n = f_{n+m}$. Eq. (2.2) gives,

$$g_n = \sum_{k=0}^{R-1} \alpha^{n+m+\beta-k}\Gamma(n+m+\beta-k)[x^k](\mathcal{A}_\beta^\alpha f)(x) + \mathcal{O}\left(\alpha^{n+m}\Gamma(n+m+\beta-R)\right) \quad \forall R \in \mathbb{N}_0.$$

Therefore $(\mathcal{A}_{\beta+m}^\alpha g)(x) = \left(\mathcal{A}_{\beta+m}^\alpha \frac{f(x)}{x^m}\right)(x) = (\mathcal{A}_\beta^\alpha f)(x)$. The second statement follows analogously. □

Remark 2.14. Note that this gives an ascending chain of subspaces of $\mathbb{R}[[x]]$:

$$\mathbb{R}[[x]]_\beta^\alpha \subset \mathbb{R}[[x]]_{\beta+1}^\alpha \subset \mathbb{R}[[x]]_{\beta+2}^\alpha \subset \ldots \subset \mathbb{R}[[x]]_{\beta+m}^\alpha \subset \ldots$$

Remark 2.15. The requirement $\beta > 0$ is therefore only a spurious restriction. The ideal $x^m\mathbb{R}[[x]]_\beta^\alpha$ can be associated with $\mathbb{R}[[x]]_{\beta-m}^\alpha$. An alternative way to think about $\mathbb{R}[[x]]_\beta^\alpha$ with $\beta \le 0$ is to use the field of (formal) Laurent series $\mathbb{R}((x))$ as the target space for $\mathcal{A}_\beta^\alpha$ and demand that negative powers of x commute with the $\mathcal{A}_\beta^\alpha$ operator.

3 A derivation for asymptotics

The following lemma forms the foundation for most conclusions in this article. It provides an estimate for sums over Γ functions. Moreover, it ensures that the subspace $\mathbb{R}[[x]]^{\alpha}_{\beta}$ of formal power series falls into a large class of sequences studied by Bender [6]. From another perspective the lemma can be seen as an entry point to resurgence, which bypasses the necessity for analytic continuations and Borel transformations.

Lemma 3.1. *If $\beta \in \mathbb{R}_{>0}$, then there exists a $C \in \mathbb{R}$ such that*

$$\sum_{m=0}^{n} \Gamma(m+\beta)\Gamma(n-m+\beta) \leq C\Gamma(n+\beta) \qquad \forall n \in \mathbb{N}_0. \tag{3.1}$$

Proof. Recall that Γ is a *log-convex* function in $\mathbb{R}_{>0}$. The functions $\Gamma(m+\beta)$ and $\Gamma(n-m+\beta)$ are also log-convex functions in m on the interval $[0,n]$, as log-convexity is preserved under shifts and reflections. Furthermore, *log*-convexity closes under multiplication. This implies that $G^n_m := \Gamma(m+\beta)\Gamma(n-m+\beta)$ is a log-convex function on the interval $m \in [1, n-1]$. A convex function always attains its maximum on the boundary of its domain. Accordingly, $G^n_m \leq G^n_1$ for $m \in [1, n-1]$. This way, the sum $\sum_{m=0}^{n} G^n_m$ can be estimated after stripping off the two boundary terms:

$$\sum_{m=0}^{n} G^n_m \leq 2G^n_0 + (n-1)G^n_1 \leq 2G^n_0 + (n-1+\beta)G^n_1 \qquad \forall n \geq 1.$$

It follows from $n\Gamma(n) = \Gamma(n+1)$ that $G^n_1 = G^n_0 \frac{\beta}{n-1+\beta}$ for all $n \geq 1$. Substituting this into eq. (3.2) gives the estimate in eq. (3.1) with $C = (2+\beta)\Gamma(\beta)$. The remaining case $n = 0$ is trivial. □

Corollary 3.2. *If $\beta \in \mathbb{R}_{>0}$, then*

$$\sum_{m=R}^{n-R} \Gamma(m+\beta)\Gamma(n-m+\beta) \in \mathcal{O}(\Gamma(n-R+\beta)) \qquad \forall R \in \mathbb{N}_0. \tag{3.2}$$

Proof. Rewrite the left hand side as $\sum_{m=0}^{n-2R} \Gamma(m+R+\beta)\Gamma(n-R-m+\beta)$. Lemma 3.1 can be applied with the substitutions $\beta \to \beta + R$, $n \to n-2R$ to obtain the required estimate. □

Corollary 3.3. *If $\beta \in \mathbb{R}_{>0}$, $C \in \mathbb{R}$ and $P \in \mathbb{R}[m]$ some polynomial in m, then*

$$\sum_{m=R}^{n} C^m P(m)\Gamma(n-m+\beta) \in \mathcal{O}(\Gamma(n-R+\beta)) \qquad \forall R \in \mathbb{N}_0. \tag{3.3}$$

Proof. There is a $C' \in \mathbb{R}$ such that $|C^m P(m)|$ is bounded by $C'\Gamma(m+\beta)$ for all $m \in \mathbb{N}_0$. Corollary 3.2 ensures that $\sum_{m=R}^{n-R} C^m P(m)\Gamma(n-m+\beta) \in \mathcal{O}(\Gamma(n-R+\beta))$. The remainder $\sum_{m=n-R+1}^{n} C^m P(m)\Gamma(n-m+\beta) = \sum_{m=0}^{R-1} C^{n-m} P(n-m)\Gamma(m+\beta)$ is obviously in $\mathcal{O}(\Gamma(n-R+\beta))$. □

Corollary 3.4. *If $\beta \in \mathbb{R}_{>0}$, then there exists a $C \in \mathbb{R}$ such that*

$$\sum_{\substack{\boldsymbol{m}\in\mathbb{N}_0^L \\ |\boldsymbol{m}|=n}} \prod_{l=1}^{L} \Gamma(m_l+\beta) \leq C^L \Gamma(n+\beta) \qquad \forall n, L \in \mathbb{N}_0 \text{ with } L \geq 1, \tag{3.4}$$

where $\substack{\boldsymbol{m}\in\mathbb{N}_0^L \\ |\boldsymbol{m}|=n}$ *denotes the simplex* $\{(m_1,\ldots,m_L) \in \mathbb{N}_0^L | \sum_{l=1}^{L} m_l = n\}$.

Proof. This inequality is merely an iterated version of Lemma 3.1. It can be proven by induction in L. The case $L = 1$ is trivial. Lemma 3.1 guarantees the existence of a $C \in \mathbb{R}$ such that

$$C^L \sum_{m_{L+1}=0}^{n} \Gamma(m_{L+1} + \beta)\Gamma(n - m_{L+1} + \beta) \leq C^{L+1}\Gamma(n + \beta) \qquad \forall n, L \in \mathbb{N}_0.$$

Suppose the statement holds for L. Using the statement on $C^L\Gamma(n - m_{L+1} + \beta)$ on the left hand side results in

$$\sum_{m_{L+1}=0}^{n} \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^L \\ |\mathbf{m}|=n-m_{L+1}}} \prod_{l=1}^{L+1} \Gamma(m_l + \beta) \leq C^{L+1}\Gamma(n + \beta) \qquad \forall n \in \mathbb{N}_0,$$

which is the statement for $L + 1$. □

An immediate consequence of Corollary 3.2 is

Proposition 3.5. *$\mathbb{R}[[x]]^\alpha_\beta$ forms a subring of $\mathbb{R}[[x]]$: If $f, g \in \mathbb{R}[[x]]^\alpha_\beta$, then $f(x)g(x) \in \mathbb{R}[[x]]^\alpha_\beta$. Moreover, $\mathcal{A}^\alpha_\beta$ is a derivation, that means it fulfills the Leibniz rule*

$$(\mathcal{A}^\alpha_\beta(f \cdot g))(x) = f(x)(\mathcal{A}^\alpha_\beta g)(x) + g(x)(\mathcal{A}^\alpha_\beta f)(x). \tag{3.5}$$

Proof. Set $h(x) = (f \cdot g)(x) = f(x)g(x)$. The coefficients h_n are given as a sum by the Cauchy product formula. This sum can be written suggestively as

$$h_n = \sum_{m=0}^{n} f_{n-m}g_m = \sum_{m=0}^{R-1} f_{n-m}g_m + \sum_{m=0}^{R-1} f_m g_{n-m} + \sum_{m=R}^{n-R} f_m g_{n-m} \qquad \forall n \geq 2R. \tag{3.6}$$

Definition 2.9 guarantees that the first two sums have sound asymptotic expansions for large n. Together they constitute an asymptotic expansion of h_n up to order $R - 1$. We verify this exemplary on the first sum, where the asymptotic expansion from eq. (2.2) up to order $R-m-1$ of f_{n-m} can be substituted:

$$\begin{aligned}\sum_{m=0}^{R-1} f_{n-m}g_m &= \sum_{m=0}^{R-1} g_m \sum_{k=0}^{R-m-1} \alpha^{n-m+\beta-k}\Gamma(n - m + \beta - k)[x^k](\mathcal{A}^\alpha_\beta f)(x) + \mathcal{O}\left(\alpha^n\Gamma(n + \beta - R)\right) \\ &= \sum_{k=0}^{R-1} \alpha^{n+\beta-k}\Gamma(n + \beta - k) \sum_{m=0}^{k} g_m[x^{k-m}](\mathcal{A}^\alpha_\beta f)(x) + \mathcal{O}\left(\alpha^n\Gamma(n + \beta - R)\right).\end{aligned}$$

The inner sum $\sum_{m=0}^{k} g_m[x^{k-m}](\mathcal{A}^\alpha_\beta f)(x)$ is the k-th coefficient of the product $g(x)(\mathcal{A}^\alpha_\beta f)(x)$. It remains to be shown that the last sum in eq. (3.6) is negligible. Because $f_n, g_n \in \mathcal{O}(\alpha^n\Gamma(n+\beta))$, there is a constant C such that $|f_n| \leq C\alpha^n\Gamma(n + \beta)$ and $|g_n| \leq C\alpha^n\Gamma(n + \beta)$ for all $n \geq 0$. Hence,

$$\sum_{m=R}^{n-R} |f_m g_{n-m}| \leq C^2\alpha^n \sum_{m=R}^{n-R} \Gamma(m + \beta)\Gamma(n - m + \beta) \qquad \forall n \geq 2R$$

shows that this sum is in $\mathcal{O}(\alpha^n\Gamma(n - R + \beta))$ by Corollary 3.2. □

Corollary 3.6. *If $g^1, \dots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$, then*

$$\left(\mathcal{A}^\alpha_\beta \left(\prod_{l=1}^{L} g^l(x)\right)\right)(x) = \sum_{l=1}^{L} \left(\prod_{\substack{m=1\\ m\neq l}}^{L} g^m(x)\right) (\mathcal{A}^\alpha_\beta g^l)(x). \tag{3.7}$$

Proof. Proof by induction on L using the Leibniz rule. □

Corollary 3.7. *If $g^1, \dots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$ and $\boldsymbol{t} = (t_1, \dots, t_L) \in \mathbb{N}_0^L$, then*

$$\left(\mathcal{A}^\alpha_\beta \left(\prod_{l=1}^{L} (g^l(x))^{t_l}\right)\right)(x) = \sum_{l=1}^{L} t_l (g^l(x))^{t_l - 1} \left(\prod_{\substack{m=1\\ m\neq l}}^{L} (g^m(x))^{t_m}\right) (\mathcal{A}^\alpha_\beta g^l)(x). \tag{3.8}$$

Corollary 3.8. *If $g^1, \dots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$ and $p \in \mathbb{R}[y_1, \dots, y_L]$ is polynomial in L variables, then $h(x) = p(g^1(x), \dots, g^L(x)) \in \mathbb{R}[[x]]^\alpha_\beta$ and*

$$(\mathcal{A}^\alpha_\beta h)(x) = (\mathcal{A}^\alpha_\beta (p(g^1, \dots, g^L)))(x) = \sum_{l=1}^{L} \frac{\partial p}{\partial g^l}(g^1, \dots, g^L)(\mathcal{A}^\alpha_\beta g^l)(x). \tag{3.9}$$

Although the last three statements are only basic general properties of commutative derivation rings, they suggest that $\mathcal{A}^\alpha_\beta$ fulfills a simple chain rule. In fact, Corollary 3.8 can still be generalized from polynomials to analytic functions, but, as already mentioned, this intuition turns out to be false in general.

4 Composition

4.1 Composition by analytic functions

Theorem 4.1. *If $g^1, \dots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$ with each $g^l_0 = 0$ and $f \in \mathbb{R}\{y_1, \dots, y_L\}$, a function in L variables, which is analytic at the origin, then $h(x) = f(g^1(x), \dots, g^L(x)) \in \mathbb{R}[[x]]^\alpha_\beta$ and*

$$(\mathcal{A}^\alpha_\beta h)(x) = (\mathcal{A}^\alpha_\beta (f(g^1, \dots, g^L)))(x) = \sum_{l=1}^{L} \frac{\partial f}{\partial g^l}(g^1, \dots, g^L)(\mathcal{A}^\alpha_\beta g^l)(x). \tag{4.1}$$

In [6] Edward Bender established this theorem for the case $L = 1$ in a less 'generatingfunctionology' biased notation. If for example $g \in \mathbb{R}[[x]]^\alpha_\beta$ and $f \in \mathbb{R}\{x, y\}$, then his Theorem 1 allows us to calculate the asymptotics of the power series $f(g(x), x)$. In fact, Bender analyzed more general power series including sequences with even more rapid than factorial growth.

The following proof of Theorem 4.1 is a straightforward generalization of Bender's Lemma 2 and Theorem 1 in [6] to the multivariate case $f \in \mathbb{R}\{y_1, \dots, y_L\}$.

Lemma 4.2. *If $g^1, \dots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$, then there exists a constant $C \in \mathbb{R}$ such that*

$$\left| [x^n] \prod_{l=1}^{L} \left(g^l(x)\right)^{t_l} \right| \leq C^{|\boldsymbol{t}|} \alpha^n \Gamma(n + \beta) \qquad \forall \boldsymbol{t} \in \mathbb{N}_0^L, |\boldsymbol{t}| \geq 1 \text{ and } n \in \mathbb{N}_0. \tag{4.2}$$

Proof. The proof is a straightforward application of Corollary 3.4. There is a constant C such that $g^l_n \leq C\alpha^n\Gamma(n+\beta)$ for all $n \in \mathbb{N}_0$ and $l \in [1, L]$. Accordingly,

$$\left|[x^n]\prod_{l=1}^{L}\left(g^l(x)\right)^{t_l}\right| \leq \alpha^n C^{|\mathbf{t}|}\sum_{\substack{\mathbf{m}\in\mathbb{N}_0^{|\mathbf{t}|}\\ |\mathbf{m}|=n}}\prod_{r=1}^{|\mathbf{t}|}\Gamma(m_r+\beta) \qquad \forall \mathbf{t}\in\mathbb{N}_0^L \text{ and } n\in\mathbb{N}_0.$$

An application of Corollary 3.4 results in the lemma. □

Corollary 4.3. *If $g^1, \ldots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$ with $g^l_0 = 0$, then there exists a constant $C \in \mathbb{R}$ such that*

$$\left|[x^n]\prod_{l=1}^{L}\left(g^l(x)\right)^{t_l}\right| \leq C^{|\boldsymbol{t}|}\alpha^n\Gamma(n+\beta-|\boldsymbol{t}|+1) \qquad \forall \boldsymbol{t}\in\mathbb{N}_0^L, n\in\mathbb{N}_0 \text{ with } 1\leq|\boldsymbol{t}|\leq n. \tag{4.3}$$

Proof. As a consequence of Proposition 2.13, $\frac{g^l(x)}{x} \in \mathbb{R}[[x]]^\alpha_{\beta+1}$. Hence,

$$\left|[x^{n-|\mathbf{t}|}]\prod_{l=1}^{L}\left(\frac{g^l(x)}{x}\right)^{t_l}\right| \leq C^{|\mathbf{t}|}\alpha^{n-|\mathbf{t}|}\Gamma(n-|\mathbf{t}|+\beta+1) \qquad \forall \mathbf{t}\in\mathbb{N}_0^L, n\in\mathbb{N}_0 \text{ with } 1\leq|\mathbf{t}|\leq n.$$

□

Proof of Theorem 4.1. The composition of two power series can be expressed as the sum

$$h(x) = \sum_{\mathbf{t}\in\mathbb{N}_0^L} f_{t_1,\ldots,t_L}\prod_{l=1}^{L}\left(g^l(x)\right)^{t_l},$$

which can be split in preparation for the extraction of asymptotics:

$$= \sum_{\substack{\mathbf{t}\in\mathbb{N}_0^L\\ |\mathbf{t}|\leq R}} f_{t_1,\ldots,t_L}\prod_{l=1}^{L}\left(g^l(x)\right)^{t_l} + \sum_{\substack{\mathbf{t}\in\mathbb{N}_0^L\\ |\mathbf{t}|>R}} f_{t_1,\ldots,t_L}\prod_{l=1}^{L}\left(g^l(x)\right)^{t_l} \qquad \forall R\in\mathbb{N}_0.$$

The left sum is just the composition by a polynomial. Corollary 3.8 asserts that this sum is in $\mathbb{R}[[x]]^\alpha_\beta$. It has the asymptotic expansion given in eq. (3.9) which agrees with the right hand side of eq. (4.1) up to order $R-1$, because the partial derivative reduces the order of a polynomial by one and $g_l = 0$. It is left to prove that the coefficients of the power series given by the right sum are in $\mathcal{O}(\alpha^n\Gamma(n-R+\beta))$. Corollary 4.3 and the fact that there is a constant C, such that $|f_{t_1,\ldots,t_L}| \leq C^{|\mathbf{t}|}$ for all $\mathbf{t}\in\mathbb{N}_0^L$, due to the analyticity of f, ensure that there is a constant $C' \in \mathbb{R}$ such that

$$\sum_{\substack{\mathbf{t}\in\mathbb{N}_0^L\\ n\geq|\mathbf{t}|>R}}\left|f_{t_1,\ldots,t_L}[x^n]\prod_{l=1}^{L}\left(g^l(x)\right)^{t_l}\right| \leq \alpha^n\sum_{t=R+1}^{n}C'^t\Gamma(n+\beta-t+1)\sum_{\substack{\mathbf{t}\in\mathbb{N}_0^L\\ |\mathbf{t}|=t}}1,$$

for all $n \geq R+1$. The result of the last sum $|\{t_1,\ldots,t_L \in \mathbb{N}_0 | t_1+\ldots+t_L = t\}| = \binom{t+L-1}{L-1}$ is a polynomial in t. Corollary 3.3 asserts that the remainder sum is in $\mathcal{O}\left(\alpha^n\Gamma(n+\beta-R)\right)$. □

4.2 Proof of the main theorem: Composition of power series in $\mathbb{R}[[x]]^\alpha_\beta$

Despite the fact that Bender's theorem applies to a broader range of compositions $f \circ g$, where f does not need to be analytic and g does not need to be an element of $\mathbb{R}[[x]]^\alpha_\beta$, it cannot be used in the case $f, g \in \mathbb{R}[[x]]^\alpha_\beta$, where $f \notin \ker \mathcal{A}^\alpha_\beta$. The problem is that we cannot truncate the sum $\sum_{k=0}^\infty f_k g(x)^k$ without losing significant information. In this section, this obstacle will be confronted and the general chain rule for the asymptotic derivative will be proven. Let $\operatorname{Diff}_{\mathrm{id}}(\mathbb{R}, 0) := (\{f \in \mathbb{R}[[x]] : f_0 = 0 \text{ and } f_1 = 1\}, \circ)$ denote the group of *formal diffeomorphisms tangent to the identity*. It is the group of all power series with $f_0 = 0$ and $f_1 = 1$ and with composition as group operation. The restriction of this group to elements in $\mathbb{R}[[x]]^\alpha_\beta$ is of special interest to us.

Theorem 4.4. $\operatorname{Diff}_{\mathrm{id}}(\mathbb{R}, 0)^\alpha_\beta := (\{f \in \mathbb{R}[[x]]^\alpha_\beta : f_0 = 0 \text{ and } f_1 = 1\}, \circ)$ *is a subgroup*[1] *of* $\operatorname{Diff}_{\mathrm{id}}(\mathbb{R}, 0)$. *Moreover,* $\mathcal{A}^\alpha_\beta$ *fulfills a chain rule: If* $f, g \in \mathbb{R}[[x]]^\alpha_\beta$ *with* $g_0 = 0, g_1 = 1$, *then* $f \circ g, g^{-1} \in \mathbb{R}[[x]]^\alpha_\beta$ *and*

$$(\mathcal{A}^\alpha_\beta(f \circ g))(x) = f'(g(x))(\mathcal{A}^\alpha_\beta g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}^\alpha_\beta f)(g(x)), \tag{4.4}$$

$$(\mathcal{A}^\alpha_\beta g^{-1})(x) = -{g^{-1}}'(x) \left(\frac{x}{g^{-1}(x)}\right)^\beta e^{\frac{g^{-1}(x)-x}{\alpha x g^{-1}(x)}} (\mathcal{A}^\alpha_\beta g)(g^{-1}(x)). \tag{4.5}$$

This theorem will be proven by ensuring that if $f, g \in \mathbb{R}[[x]]^\alpha_\beta$, then $f \circ g^{-1} \in \mathbb{R}[[x]]^\alpha_\beta$ and by constructing the asymptotic expansion of $f \circ g^{-1}$. For this, it turns out to be convenient to work in the ring $\mathbb{R}[[x]]^\alpha_{\beta+2}$ that contains $\mathbb{R}[[x]]^\alpha_\beta$ as a subring.

Lemma 4.5. *If* f, g *as above and* $A(x) := \frac{\frac{x}{g(x)} - 1}{x}$ *as well as* $B(x) := f(x) g'(x) \left(\frac{g(x)}{x}\right)^\beta$, *then* $A, B \in \mathbb{R}[[x]]^\alpha_{\beta+2}$ *and*

$$[x^n] f(g^{-1}(x)) = \sum_{m=0}^n \binom{n+\beta+1}{m} [x^{n-m}] B(x) A(x)^m \qquad \forall n \in \mathbb{N}_0. \tag{4.6}$$

Proof. The statement $A \in \mathbb{R}[[x]]^\alpha_{\beta+2}$ follows from $g_0 = 0$, $g_1 = 1$, Theorem 4.1 and Proposition 2.13. The fact that $B \in \mathbb{R}[[x]]^\alpha_{\beta+2}$ follows additionally from the result $f(x) \in \mathbb{R}[[x]]^\alpha_\beta \Rightarrow f'(x) \in \mathbb{R}[[x]]^\alpha_{\beta+2}$ from Proposition A.1. Eq. (4.6) can be derived using the Lagrange inversion theorem [18, A.6]:

$$\begin{aligned}
[x^n] f(g^{-1}(x)) &= \frac{1}{n}[x^{n-1}] f'(x) \left(\frac{x}{g(x)}\right)^n = [x^n] f(x) \left(\frac{x}{g(x)}\right)^{n-1} \left(\frac{x}{g(x)} - x \frac{\partial}{\partial x} \frac{x}{g(x)}\right) \\
&= [x^n] f(x) g'(x) \left(\frac{g(x)}{x}\right)^\beta \left(\frac{x}{g(x)}\right)^{n+\beta+1} = [x^n] B(x) \left(1 + x A(x)\right)^{n+\beta+1} \\
&= \sum_{m=0}^n \binom{n+\beta+1}{m} [x^{n-m}] B(x) A(x)^m.
\end{aligned}$$

□

[1] Julien Courtiel remarked that the statement $f \in \operatorname{Diff}_{\mathrm{id}}(\mathbb{R}, 0)^\alpha_\beta \Rightarrow f^{-1} \in \operatorname{Diff}_{\mathrm{id}}(\mathbb{R}, 0)^\alpha_\beta$ was not obvious in a previous version of this manuscript. The argument in this version was modified to be more transparent in this respect.

The tail of the sum in eq. (4.6) over m turns out to be asymptotically negligible. However, in contrast to the preceding arguments, the sum cannot be truncated at a fixed value of m independent of n. A cutoff that grows slowly with n has to be introduced. More specifically,

Lemma 4.6. *If f, g, A, B as above, then there is a constant $C \in \mathbb{R}$ such that*

$$[x^n]f(g^{-1}(x)) = \sum_{m=0}^{s(n)} \binom{n+\beta+1}{m} [x^{n-m}]B(x)A(x)^m + \mathcal{O}(\alpha^n \Gamma(n+\beta+2-R)) \quad \forall R \in \mathbb{N}_0, \tag{4.7}$$

where $s(n) = \lceil 4(R \log n + C) \rceil$.

Proof. It needs to be proven that the partial sum over $m \in (s(n), \infty)$ in eq. (4.6) is in $\mathcal{O}(\alpha^n \Gamma(n+\beta+2-R))$ for an appropriate function $s(n)$. Recall that $A, B \in \mathbb{R}[[x]]^\alpha_{\beta+2}$. Lemma 4.2 guarantees the existence of a $C \in \mathbb{R}$ such that

$$\sum_{m=s(n)}^{n} \binom{n+\beta+1}{m} |[x^{n-m}]B(x)A(x)^m| \leq \sum_{m=s(n)}^{n} \binom{n+\beta+1}{m} C^{m+1} \alpha^n \Gamma(n-m+\beta+2),$$

for all $n \geq s(n) \geq 0$. The binomial can be expressed in terms of Γ-functions $\binom{n+\beta+1}{m} = \frac{\Gamma(n+\beta+2)}{\Gamma(n-m+\beta+2)m!}$. Subsequently, an elementary inequality, which quantifies the fast convergence of the Taylor expansion of the exponential, proven in Lemma B.1, can be applied with $s(n) = \lceil 4(R \log n + C) \rceil$:

$$= \sum_{m=s(n)}^{n} \frac{C^{m+1}}{m!} \alpha^n \Gamma(n+\beta+2) \leq C\alpha^n \frac{\Gamma(n+\beta+2)}{n^R} \in \mathcal{O}(\alpha^n \Gamma(n+\beta+2-R)).$$

□

Lemma 4.7. *If f, g, A, B as above, then $f \circ g^{-1} \in \mathbb{R}[[x]]^\alpha_{\beta+2}$ and*

$$[x^k]\left(\mathcal{A}^\alpha_{\beta+2} f \circ g^{-1}\right)(x) = [x^k]\left(\mathcal{A}^\alpha_{\beta+2} B(x)(1+xA(x))^{k-1} e^{\frac{A(x)}{\alpha}}\right)(x) \qquad \forall k \in \mathbb{N}_0. \tag{4.8}$$

Proof. The proof proceeds by substitution of the asymptotic expansion up to order $R-1$ of $B(x)A(x)^m$ into eq. (4.7). This can be done, because $n-m$ is large. The result is $[x^n]f(g^{-1}(x)) =$

$$\sum_{m=0}^{s(n)} \binom{n+\beta+1}{m} \sum_{k=0}^{R-1} \alpha^{n+\beta+2-k-m} \Gamma(n+\beta+2-k-m)[x^k]\left(\mathcal{A}^\alpha_{\beta+2} B(x)A(x)^m\right)(x),$$

where the rest term, $\mathcal{O}(\alpha^n \Gamma(n+\beta+2-R))$, was omitted. An elementary variant of the Chu-Vandermonde identity (Lemma B.2) can be used to expand the product of the binomial and Γ-function. This expansion allows us to perform the summation over m behind the coefficient extraction:

$$= \sum_{k=0}^{R-1} \sum_{m=0}^{s(n)} \sum_{l=0}^{m} \alpha^{n+\beta+2-k} \binom{l+k-1}{l} \frac{\Gamma(n+\beta+2-k-l)}{(m-l)!} [x^k]\left(\mathcal{A}^\alpha_{\beta+2} B(x) \left(\frac{A(x)}{\alpha}\right)^m\right)(x)$$

$$= \sum_{k=0}^{R-1} \sum_{l=0}^{s(n)} \alpha^{n+\beta+2-k-l} \binom{l+k-1}{l} \Gamma(n+\beta+2-k-l)[x^k]\left(\mathcal{A}^\alpha_{\beta+2} B(x)A(x)^l \sum_{m=0}^{s(n)-l} \frac{\left(\frac{A(x)}{\alpha}\right)^m}{m!}\right)(x)$$

The sum over l can be truncated at order $R-1-k$, because the summands can be estimated by $C^l P(l)\alpha^n\Gamma(n+\beta+2-k-l)$ with some $C \in \mathbb{R}$ and $P \in \mathbb{R}[l]$ each depending on k. Corollary 3.3 asserts that the truncated part is in $\mathcal{O}(\alpha^n\Gamma(n+\beta+2-R))$. The subsequent change in summation variables $k \to k+l$ gives rise to,

$$= \sum_{k=0}^{R-1} \alpha^{n+\beta+2-k}\Gamma(n+\beta+2-k)[x^k] \left(\mathcal{A}^\alpha_{\beta+2} B(x) \sum_{l=0}^{k} \binom{k-1}{l} (xA(x))^l \sum_{m=0}^{s(n)-l} \frac{\left(\frac{A(x)}{\alpha}\right)^m}{m!} \right)(x),$$

which results in the statement after noting that the sums over l and m can be completed, because $\lim_{n\to\infty} s(n) = \infty$. □

Proof of Theorem 4.4. The rest of the proof is merely an algebraic exercise. We start with the expression from Lemma 4.7 for $[x^k]\left(\mathcal{A}^\alpha_{\beta+2} f \circ g^{-1}\right)(x)$ and use the chain and product rules from Proposition 3.5, Theorem 4.1 as well as $\mathcal{A}^\alpha_\beta x^2\partial_x = \mathcal{A}^\alpha_{\beta+2}\partial_x = (\alpha^{-1} - x\beta + x^2\partial_x)\mathcal{A}^\alpha_\beta$ from Proposition A.1 to expand it. Accordingly, for all $k \in \mathbb{N}_0$: $[x^k]\left(\mathcal{A}^\alpha_{\beta+2} f \circ g^{-1}\right)(x) =$

$$\begin{aligned}
\left[x^k\right] \left(\mathcal{A}^\alpha_{\beta+2} B(x)(1+xA(x))^{k-1} e^{\frac{A(x)}{\alpha}}\right)(x) &= \left[x^k\right] \left(\mathcal{A}^\alpha_{\beta+2} f(x) g'(x) \left(\frac{g(x)}{x}\right)^{\beta-k+1} e^{\frac{\frac{x}{g(x)}-1}{\alpha x}} \right)(x) \\
&= \left[x^k\right] e^{\frac{\frac{x}{g(x)}-1}{\alpha x}} \left(\frac{g(x)}{x}\right)^{\beta-k+1} \Bigg(g'(x)(\mathcal{A}^\alpha_{\beta+2} f)(x) \\
+f(x)g'(x)\frac{1}{g(x)} \left((\beta-k+1) - \alpha^{-1}\frac{1}{g(x)}\right) (\mathcal{A}^\alpha_{\beta+2} g)(x) &+ f(x)(\alpha^{-1} - \beta x + x^2\frac{\partial}{\partial x})(\mathcal{A}^\alpha_\beta g)(x) \Bigg).
\end{aligned}$$

Using $\mathcal{A}^\alpha_{\beta+2} = x^2\mathcal{A}^\alpha_\beta$ (Proposition 2.13) as well as $[x^k]f(x)\left(x\frac{\partial}{\partial x}g(x)\right) = k[x^k]f(x)g(x) - [x^k]\left(x\frac{\partial}{\partial x}f(x)\right)g(x)$ to reexpress the derivative of $(\mathcal{A}^\alpha_\beta g)(x)$ gives,

$$= [x^k]x^2 e^{\frac{\frac{x}{g(x)}-1}{\alpha x}} \left(\frac{g(x)}{x}\right)^{\beta-k+1} \left(g'(x)(\mathcal{A}^\alpha_\beta f)(x) - f'(x)(\mathcal{A}^\alpha_\beta g)(x)\right).$$

The x^2 prefactor shows that $f \circ g^{-1}$ is actually in the subspace $\mathbb{R}[[x]]^\alpha_\beta \subset \mathbb{R}[[x]]^\alpha_{\beta+2}$. In accord with Proposition 2.13:

$$\left[x^k\right]\left(\mathcal{A}^\alpha_\beta f \circ g^{-1}\right)(x) = [x^k] e^{\frac{\frac{x}{g(x)}-1}{\alpha x}} \left(\frac{g(x)}{x}\right)^{\beta-k-1} \left(g'(x)(\mathcal{A}^\alpha_\beta f)(x) - f'(x)(\mathcal{A}^\alpha_\beta g)(x)\right).$$

As $f \circ g^{-1} \in \mathbb{R}[[x]]^\alpha_\beta$, the subset $\mathrm{Diff}_{\mathrm{id}}(\mathbb{R},0)^\alpha_\beta$ is a subgroup of $\mathrm{Diff}_{\mathrm{id}}(\mathbb{R},0)$. Another application of the Lagrange inversion theorem, $[x^n]p(q^{-1}(x)) = [x^n]p(x)q'(x)\left(\frac{x}{q(x)}\right)^{n+1}$, transforms the expression for $\left[x^k\right]\left(\mathcal{A}^\alpha_\beta f \circ g^{-1}\right)(x)$ into an explicit generating function:

$$(\mathcal{A}^\alpha_\beta f \circ g^{-1})(x) = e^{\frac{\frac{g^{-1}(x)}{x}-1}{\alpha g^{-1}(x)}} \left(\frac{x}{g^{-1}(x)}\right)^\beta \left((\mathcal{A}^\alpha_\beta f)(g^{-1}(x)) - \frac{f'(g^{-1}(x))}{g'(g^{-1}(x))}(\mathcal{A}^\alpha_\beta g)(g^{-1}(x)) \right). \quad (4.9)$$

The special case $f(x) = x$ with application of $g'(g^{-1}(x)) = \frac{1}{{g^{-1}}'(x)}$ results in eq. (4.5). Solving eq. (4.5) for $(\mathcal{A}^\alpha_\beta g)(g^{-1}(x))$ and substituting the result into eq. (4.9) gives eq. (4.4) with the substitution $g \to g^{-1}$. □

Remark 4.8. Bender and Richmond [8] established that $[x^n](1+g(x))^{\gamma n+\delta} = n\gamma e^{\frac{\gamma g_1}{\alpha}} g_n + \mathcal{O}(g_n)$ if $g_n \sim \alpha n g_{n-1}$ and $g_0 = 0$. Using Lagrange inversion the first coefficient in the expansion of the compositional inverse in eq. (4.5) can be obtained from this. In this way, Theorem 4.4 is a generalization of Bender and Richmond's result. In the same article Bender and Richmond proved a theorem similar to Theorem 4.4 for the class of power series f which grow more rapidly than factorial such that $n f_{n-1} \in o(f_n)$. Theorem 4.4 establishes a link to the excluded case $n f_{n-1} = \mathcal{O}(f_n)$.

Remark 4.9. The chain rule in eq. (4.4) exposes a peculiar algebraic structure. It would be useful to have a combinatorial interpretation of the $e^{\frac{g(x)-x}{\alpha x g(x)}}$ term.

5 Applications

5.1 Connected chord diagrams

A chord diagram with n-chords is a circle with $2n$ points, which are labeled by integers $1,\ldots,2n$ and connected in disjoint pairs by n-chords. There are $(2n-1)!!$ of such diagrams. A chord diagram is *connected* if no set of chords can be separated from the remaining chords by a line which does not cross any chords. Let $I(x) = \sum_{n=0}(2n-1)!!x^n$ and $C(x) = \sum_{n=0} C_n x^n$, where C_n is the number of connected chord diagrams with n chords. Following [17], the power series $I(x)$ and $C(x)$ are related by,

$$I(x) = 1 + C(xI(x)^2). \tag{5.1}$$

This functional equation can be solved for the coefficients of $C(x)$ by basic iterative methods. The first few terms are,

$$C(x) = x + x^2 + 4x^3 + 27x^4 + 248x^5 + \ldots \tag{5.2}$$

This sequence is entry `A000699` in Neil Sloane's integer sequence on-line encyclopedia [26]. Because $(2n-1)!! = \frac{2^{n+\frac{1}{2}}}{\sqrt{2\pi}}\Gamma(n+\frac{1}{2})$, the power series I is in $\mathbb{R}[[x]]^2_{\frac{1}{2}}$ and $(\mathcal{A}^2_{\frac{1}{2}} I)(x) = \frac{1}{\sqrt{2\pi}}$. Theorem 4.4 guarantees that also $C \in \mathbb{R}[[x]]^2_{\frac{1}{2}}$, because of the closure properties of $\mathbb{R}[[x]]^2_{\frac{1}{2}}$. Moreover, an application of the general chain rule from Theorem 4.4 on the functional eq. (5.1) results in

$$\begin{aligned}(\mathcal{A}^2_{\frac{1}{2}} I)(x) &= \mathcal{A}^2_{\frac{1}{2}}(1 + C(xI(x)^2)) = \mathcal{A}^2_{\frac{1}{2}} C(xI(x)^2) \\ &= 2xI(x)C'(xI(x)^2)(\mathcal{A}^2_{\frac{1}{2}} I)(x) + \left(\frac{x}{xI(x)^2}\right)^{\frac{1}{2}} e^{\frac{xI(x)^2-x}{2x^2I(x)^2}} (\mathcal{A}^2_{\frac{1}{2}} C)(xI(x)^2),\end{aligned} \tag{5.3}$$

which can be solved for $(\mathcal{A}^2_{\frac{1}{2}} C)(x)$ and simplified using eq. (5.1),

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{1 + C(x) - 2xC'(x)}{\sqrt{2\pi}} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}. \tag{5.4}$$

A further simplification can be achieved by utilizing the linear differential equation $2x^2 I'(x) + xI(x) + 1 = I(x)$ from which the differential equation $C'(x) = \frac{C(x)(1+C(x))-x}{2xC(x)}$ [17] can be deduced:

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}. \tag{5.5}$$

sequence	0	1	2	3	4	5	6	7	8
$\frac{\sqrt{2\pi}}{e^{-1}}(\mathcal{A}^2_{\frac{1}{2}} C)$	1	$-\frac{5}{2}$	$-\frac{43}{8}$	$-\frac{579}{16}$	$-\frac{44477}{128}$	$-\frac{5326191}{1280}$	$-\frac{180306541}{3072}$	$-\frac{203331297947}{215040}$	$-\frac{58726239094693}{3440640}$
$\frac{\sqrt{2\pi}}{e^{-1}}(\mathcal{A}^2_{\frac{1}{2}} M)$	1	-4	-6	$-\frac{154}{3}$	$-\frac{1610}{3}$	$-\frac{34588}{5}$	$-\frac{4666292}{45}$	$-\frac{553625626}{315}$	$-\frac{1158735422}{35}$

Table 1: First coefficients of the asymptotic expansions of C_n and M_n.

This is the generating function of the full asymptotic expansion of C_n. The first few terms are,

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{e^{-1}}{\sqrt{2\pi}} \left(1 - \frac{5}{2}x - \frac{43}{8}x^2 - \frac{579}{16}x^3 - \frac{44477}{128}x^4 - \frac{5326191}{1280}x^5 + \dots\right). \tag{5.6}$$

Expressed in the traditional way using eq. (2.2) from Definition 2.9 this becomes

$$\begin{aligned} C_n &\sim \sum_{k\geq 0} 2^{n+\frac{1}{2}-k}\Gamma(n+\frac{1}{2}-k)[x^k](\mathcal{A}^2_{\frac{1}{2}} C)(x) = \sqrt{2\pi}\sum_{k\geq 0}(2(n-k)-1)!![x^k](\mathcal{A}^2_{\frac{1}{2}} C)(x) \\ &= e^{-1}\left((2n-1)!! - \frac{5}{2}(2n-3)!! - \frac{43}{8}(2n-5)!! - \frac{579}{16}(2n-7)!! + \dots\right) \end{aligned} \tag{5.7}$$

The first term, e^{-1}, in this expansion has been computed by Kleitman [21], Stein and Everett [28] and Bender and Richmond [8] each using different methods. With the presented method an arbitrary number of coefficients can be computed. Some additional coefficients are given in Table 1.

The probability of a random chord diagram with n chords to be connected is therefore $e^{-1}(1-\frac{5}{4n}) + \mathcal{O}(\frac{1}{n^2})$.

5.2 Monolithic chord diagrams

A chord diagram is called monolithic if it consists only of a connected component and of isolated chords which do not 'contain' each other [17]. That means with (a, b) and (c, d) the labels of two chords, it is not allowed that $a < c < d < b$ and $c < a < b < d$. Let $M(x) = \sum_{n=0} M_n x^n$ be the generating function of monolithic chord diagrams. Following [17], $M(x)$ fulfills

$$M(x) = C\left(\frac{x}{(1-x)^2}\right). \tag{5.8}$$

Using the $\mathcal{A}^2_{\frac{1}{2}}$ derivative on both sides of this equation together with the result for $(\mathcal{A}^2_{\frac{1}{2}} C)(x)$ in eq. (5.5) gives

$$\begin{aligned} (\mathcal{A}^2_{\frac{1}{2}} M)(x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{(1-x)}\frac{x}{M(x)}e^{1-\frac{x}{2}-\frac{(1-x)^2}{2x}(2M(x)+M(x)^2)} \\ &= \frac{1}{\sqrt{2\pi}}\left(1 - 4x - 6x^2 - \frac{154}{3}x^3 - \frac{1610}{3}x^4 - \frac{34588}{5}x^5 + \dots\right). \end{aligned} \tag{5.9}$$

Some additional coefficients are given in Table 1. The probability of a random chord diagram with n chords to be non-monolithic is therefore $1 - \left(1 - \frac{4}{2n-1} + \mathcal{O}(\frac{1}{n^2})\right) = \frac{2}{n} + \mathcal{O}(\frac{1}{n^2})$.

5.3 Simple permutations

A permutation is called simple if it does not map a non-trivial interval to another interval. Expressed formally, the permutation $\pi \in S_n^{\text{simple}} \subset S_n$ if and only if $\pi([i,j]) \neq [k,l]$ for all

sequence	0	1	2	3	4	5	6	7	8	9
$\frac{1}{e^{-2}}(\mathcal{A}_1^1 S)$	1	-4	2	$-\frac{40}{3}$	$-\frac{182}{3}$	$-\frac{7624}{15}$	$-\frac{202652}{45}$	$-\frac{14115088}{315}$	$-\frac{30800534}{63}$	$-\frac{16435427656}{2835}$

Table 2: First coefficients of the asymptotic expansion of S_n.

$i, j, k, l \in [0, n]$ with $2 \leq |[i, j]| \leq n - 1$. See Albert et al. [1] for a detailed exposition of simple permutations. Set $S(x) = \sum_{n=4}^{\infty} |S_n^{\text{simple}}| x^n$, the generating function of *simple* permutations, and $F(x) = \sum_{n=1}^{\infty} n! x^n$, the generating function of *all* permutations. Following [1], $S(x)$ and $F(x)$ are related by the equation,

$$\frac{F(x) - F(x)^2}{1 + F(x)} = x + S(F(x)). \tag{5.10}$$

This can be solved iteratively for the coefficients of $S(x)$:

$$S(x) = 2x^4 + 6x^5 + 46x^6 + 338x^7 + 2926x^8 + \ldots \tag{5.11}$$

This sequence is entry `A111111` [27] with the slightly different convention, $\texttt{A111111} = 1 + 2x + S(x)/x$ of Neil Sloane's online encyclopedia.

As $F(x) \in \mathbb{R}[[x]]_1^1$ and $(\mathcal{A}_1^1 F) = 1$, the full asymptotic expansion of $S(x)$ can be obtained by applying the chain rule (Theorem 4.4) to both side of eq. (5.10). Alternatively, eq. (5.10) implies $\frac{x - x^2}{1+x} = F^{-1}(x) + S(x)$ with $F^{-1}(F(x)) = x$. Using $\mathcal{A}_1^1 \frac{x-x^2}{1+x} = 0$ together with the expression for the asymptotic expansion of $F^{-1}(x)$ in terms of $(\mathcal{A}_1^1 F)(x)$ from eq. (4.5) shows that,

$$(\mathcal{A}_1^1 S)(x) = -(\mathcal{A}_1^1 F^{-1})(x) = F^{-1\prime}(x) \frac{x}{F^{-1}(x)} e^{\frac{F^{-1}(x) - x}{x F^{-1}(x)}}. \tag{5.12}$$

This can be reexpressed using the functional equation (5.10), $F^{-1}(F(x)) = x$ as well as the differential equation $x^2 F'(x) + (x - 1)F(x) + x = 0$:

$$(\mathcal{A}_1^1 S)(x) = \frac{x F^{-1}(x)}{x - (1 + x) F^{-1}(x)} e^{\frac{F^{-1}(x) - x}{x F^{-1}(x)}} = \frac{1}{1 + x} \frac{1 - x - (1 + x)\frac{S(x)}{x}}{1 + (1 + x)\frac{S(x)}{x^2}} e^{-\frac{2 + (1+x)\frac{S(x)}{x^2}}{1 - x - (1+x)\frac{S(x)}{x}}}. \tag{5.13}$$

The coefficients of $(\mathcal{A}_1^1 S)(x)$ can be computed iteratively. The first few terms are

$$(\mathcal{A}_1^1 S)(x) = e^{-2} \left(1 - 4x + 2x^2 - \frac{40}{3} x^3 - \frac{182}{3} x^4 - \frac{7624}{15} x^5 + \ldots \right). \tag{5.14}$$

By Definition 2.9, this is an expression of the asymptotics of the number of simple permutations:

$$|S_n^{\text{simple}}| \sim e^{-2} \left(n! - 4(n-1)! + 2(n-2)! - \frac{40}{3}(n-3)! - \frac{182}{3}(n-4)! + \ldots \right). \tag{5.15}$$

Albert et al. [1] calculated the first three terms of this expansion. With the presented methods the calculation of the asymptotic expansion $(\mathcal{A}_1^1 S)(x) = -(\mathcal{A}_1^1 F^{-1})(x)$ up to order n is as easy as calculating the expansion of $S(x)$ or $F^{-1}(x)$ up to order $n + 2$. Some additional coefficients are given in Table 2.

Remark 5.1. The examples above are chosen to demonstrate that given a (functional) equation which relates two power series in $\mathbb{R}[[x]]_\beta^\alpha$, it is an easy task to calculate the full asymptotic expansion of one of the power series from the asymptotic expansion of the other power series.

Applications include functional equations for 'irreducible combinatorial objects'. The two examples fall into this category. Irreducible combinatorial objects were studied in general by Beissinger [3].

Remark 5.2. Eqs. (5.5), (5.9) and (5.13) expose another interesting algebraic property. Proposition 2.13 and the chain rule imply that $(\mathcal{A}^2_{\frac{1}{2}} C)(x) \in \mathbb{R}[[x]]^2_{\frac{3}{2}}$, $(\mathcal{A}^2_{\frac{1}{2}} M)(x) \in \mathbb{R}[[x]]^2_{\frac{3}{2}}$ and $(\mathcal{A}^1_1 S)(x) \in \mathbb{R}[[x]]^1_3$. This way, the 'higher-order' asymptotics of the asymptotic sequence can be calculated by iterating the application of the $\mathcal{A}$ map. With the powerful techniques of resurgence, it might be possible to construct *convergent* large-order expansions for these cases.

Furthermore, the fact that the asymptotics of each sequence may be expressed as a combination of polynomial and exponential expressions of the original sequence can be seen as an avatar of resurgence.

Remark 5.3. In quantum field theory the *coupling*, an expansion parameter, needs to be reparametrized in the process of *renormalization* [12]. Those reparametrizations are merely compositions of power series which are believed to be *Gevrey-1*. Theorem 4.4 might be useful for the resummation of renormalized quantities in quantum field theory. In fact, Dyson-Schwinger equations in quantum field theory can be stated as functional equations of a form similar to the above [11, 9]. These considerations will be the subject of a future publication [10], where the presented formalism will be applied to zero-dimensional quantum field theory.

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References

[1] MH Albert, MD Atkinson, and M Klazar. The enumeration of simple permutations. *Journal of Integer Sequences*, 6(4):Art–03, 2003.

[2] Inês Aniceto, Ricardo Schiappa, and Marcel Vonk. The resurgence of instantons in string theory. *arXiv preprint arXiv:1106.5922*, 2011.

[3] Janet Simpson Beissinger. The enumeration of irreducible combinatorial objects. *Journal of Combinatorial Theory, Series A*, 38(2):143 – 169, 1985.

[4] Carl M Bender and Tai Tsun Wu. Anharmonic oscillator. *Physical Review*, 184(5):1231, 1969.

[5] Edward A Bender. Asymptotic methods in enumeration. *SIAM review*, 16(4):485–515, 1974.

[6] Edward A Bender. An asymptotic expansion for the coefficients of some formal power series. *Journal of the London Mathematical Society*, 2(3):451–458, 1975.

[7] Edward A Bender and E Rodney Canfield. The asymptotic number of labeled graphs with given degree sequences. *Journal of Combinatorial Theory, Series A*, 24(3):296–307, 1978.

[8] Edward A. Bender and L.Bruce Richmond. An asymptotic expansion for the coefficients of some power series II: Lagrange inversion. *Discrete Mathematics*, 50:135 – 141, 1984.

[9] Michael Borinsky. Algebraic lattices in QFT renormalization. *Letters in Mathematical Physics*, 106(7):879–911, 2016.

[10] Michael Borinsky. Renormalized asymptotic enumeration of Feynman diagrams. In preparation, 2017.

[11] D.J. Broadhurst and D. Kreimer. Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality. *Nuclear Physics B*, 600(2):403 – 422, 2001.

[12] Alain Connes and Dirk Kreimer. Renormalization in quantum field theory and the Riemann–Hilbert problem II: The β-function, diffeomorphisms and the renormalization group. *Communications in Mathematical Physics*, 216(1):215–241, 2001.

[13] Ovidiu Costin. *Asymptotics and Borel summability.* CRC Press, 2008.

[14] N.G. de Bruijn. *Asymptotic Methods in Analysis.* Bibliotheca mathematica. Dover Publications, 1970.

[15] Gerald V Dunne and Mithat Ünsal. Resurgence and trans-series in quantum field theory: the $\mathbb{CP}^{N-1}$ model. *Journal of High Energy Physics*, 2012(11):1–86, 2012.

[16] Jean Écalle. Les fonctions résurgentes. *Publ. math. d'Orsay/Univ. de Paris, Dep. de math.*, 1981.

[17] Philippe Flajolet and Marc Noy. *Formal Power Series and Algebraic Combinatorics: 12th International Conference, FPSAC'00, Moscow, Russia, June 2000, Proceedings*, pages 191–201. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.

[18] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics.* Cambridge University press, 2009.

[19] Stavros Garoufalidis, Alexander Its, Andrei Kapaev, and Marcos Marino. Asymptotics of the instantons of Painlevé I. *International Mathematics Research Notices*, 2012(3):561–606, 2012.

[20] P.F. Hsieh and Y. Sibuya. *Basic Theory of Ordinary Differential Equations.* Universitext. Springer New York, 2012.

[21] Daniel J Kleitman. Proportions of irreducible diagrams. *Studies in Applied Mathematics*, 49(3):297–299, 1970.

[22] Jean-Claude Le Guillou and Jean Zinn-Justin. *Large-order behaviour of perturbation theory*, volume 7. Elsevier, 2012.

[23] Claude Mitschi and David Sauzin. *Divergent Series, Summability and Resurgence I.* Springer, 2016.

[24] David Sauzin. Introduction to 1-summability and resurgence. *arXiv preprint arXiv:1405.0356*, 2014.

[25] Ricardo Schiappa and Ricardo Vaz. The resurgence of instantons: multi-cut Stokes phases and the Painleve II equation. *Communications in Mathematical Physics*, 330(2):655–721, 2014.

[26] N. J. A. Sloane. The on-line encyclopedia of integer sequences. http://oeis.org. *Sequence A000699*, 1999.

[27] N. J. A. Sloane. The on-line encyclopedia of integer sequences. http://oeis.org. *Sequence A111111*, 2005.

[28] PR Stein and CJ Everett. On a class of linked diagrams II. asymptotics. *Discrete Mathematics*, 21(3):309–318, 1978.

[29] Herbert S Wilf. *generatingfunctionology*. Elsevier, 2013.

A Some remarks on differential equations

Differential equations arising from physical systems form an active field of research in the scope of resurgence [19, 25]. A detailed exposition of the application of resurgence theory to differential equations can be found in [13]. Unfortunately, the exact calculation of an overall factor of the asymptotic expansion of a solution of an ODE, called *Stokes constant*, turns out to be difficult for many problems. This fact severely limits the utility of the method for enumeration problems, as the dominant factor of the asymptotic expansion is of most interest and the detailed structure of the asymptotic expansion is secondary.

In this section it will be sketched, for the sake of completeness, how the presented combinatorial framework fits into the realm of differential equations. The given elementary properties each have their counterpart in resurgence's alien calculus [23, II.6].

Theorem 4.1 serves as a good starting point to analyze differential equations with power series solutions in $\mathbb{R}[[x]]^\alpha_\beta$. Given an analytic function $F \in \mathbb{R}\{x, y_0, \ldots, y_L\}$, the $\mathcal{A}^\alpha_\beta$-derivation can be applied on the ordinary differential equation

$$0 = F(x, f(x), f'(x), f''(x), \ldots, f^{(L)}(x)).$$

The chain rule for analytic functions (Theorem 4.1) gives

$$0 = \sum_{l=0}^{L} \frac{\partial F}{\partial y_l}(x, y_0, \ldots, y_L)\Big|_{\substack{y_m = f^{(m)}(x)\\ m\in\{0,\ldots,L\}}} (\mathcal{A}^\alpha_\beta f^{(l)})(x). \tag{A.1}$$

The differential equation becomes a linear equation for the asymptotic expansions of the derivatives $f^{(l)}$. This raises the question how these different asymptotic expansions relate to each other.

Proposition A.1. *If $f \in \mathbb{R}[[x]]^\alpha_\beta$, then $f'(x) \in \mathbb{R}[[x]]^\alpha_{\beta+2}$ and*

$$(\mathcal{A}^\alpha_\beta x^2 f'(x))(x) = \left(\alpha^{-1} - x\beta + x^2 \frac{\partial}{\partial x}\right)(\mathcal{A}^\alpha_\beta f)(x). \tag{A.2}$$

Proof. The statements can be verified by using $f'(x) = \sum_{n=0}^\infty n f_n x^{n-1}$ and substituting an asymptotic expansion up to order $R-1$ from eq. (2.2). Set $h(x) = x^2 f'(x)$ such that, for $n \geq 1$

$$h_n = (n-1)f_{n-1} =$$
$$= \sum_{k=0}^{R-1} \alpha^{n-1+\beta-k}(n-1)\Gamma(n-1+\beta-k)[x^k](\mathcal{A}^\alpha_\beta f)(x) + \mathcal{O}\left(\alpha^n \Gamma(n+\beta-R)\right).$$

An elementary calculation using $x\Gamma(x) = \Gamma(x+1)$ and $k[x^k] = [x^k]x\partial_x$ reveals,

$$h_n = \sum_{k=0}^{R-1} \alpha^{n+\beta-k}\Gamma(n+\beta-k)[x^k](\alpha^{-1} - x\beta + x^2\partial_x)(\mathcal{A}^\alpha_\beta f)(x) + \mathcal{O}\left(\alpha^n\Gamma(n+\beta-R)\right),$$

which verifies eq. (A.2) and shows that $h \in \mathbb{R}[[x]]^\alpha_\beta$. Since $x^2 f'(x) = h(x)$, it follows from Proposition 2.13 that $f'(x) \in \mathbb{R}[[x]]^\alpha_{\beta+2}$. ☐

Corollary A.2. *If $F \in \mathbb{R}\{x, y_0, \ldots, y_L\}$ and $f \in \mathbb{R}[[x]]^\alpha_\beta$ is a solution of the differential equation*

$$0 = F(x, f(x), f'(x), f''(x), \ldots, f^{(L)}(x)), \tag{A.3}$$

then $(\mathcal{A}^\alpha_\beta f)(x)$ is a solution of the linear differential equation

$$0 = \sum_{l=0}^{L} x^{2L-2l} \frac{\partial F}{\partial y_l}(x, y_0, \ldots, y_L)\Big|_{\substack{y_m = f^{(m)}(x)\\ m\in\{0,\ldots,L\}}} (\alpha^{-1} - x\beta + x^2\partial_x)^l (\mathcal{A}^\alpha_\beta f)(x). \tag{A.4}$$

Remark A.3. Even if it is known that the solution to a differential equation has a well-behaved asymptotic expansion, Corollary A.2 provides this asymptotic expansion only up to the initial values for the linear differential equation (A.4). Note that the form of the asymptotic expansion can still depend non-trivially on the initial values of the solution f of the nonlinear differential equation.

Remark A.4. The linear differential equation (A.4) only has a non-trivial solution if α^{-1} is the root of a certain polynomial. If this root is not real or if two roots have the same modulus, the present formalism has to be generalized to complex and multiple α to express the asymptotic expansion of a general solution. This generalization is straightforward. We merely need to generalize Definition 2.1 of suitable sequences to:

Definition A.5. For given $\beta \in \mathbb{R}_{>0}$ and $\alpha_1, \ldots, \alpha_L \in \mathbb{C}$ with $|\alpha_1| = |\alpha_2| = \ldots = |\alpha_L| =: \alpha > 0$ let $\mathbb{C}[[x]]^{\alpha_1,\ldots,\alpha_L}_\beta \subset \mathbb{C}[[x]]$ be the subspace of complex power series, such that $f \in \mathbb{C}[[x]]^{\alpha_1,\ldots,\alpha_L}_\beta$ if and only if there exist sequences of complex numbers $(c^f_{k,l})_{k\in\mathbb{N}_0, l\in[1,L]}$, which fulfill

$$f_n = \sum_{k=0}^{R-1}\sum_{l=1}^{L} c^f_{k,l}\alpha_l^{n+\beta-k}\Gamma(n+\beta-k) + \mathcal{O}\left(\alpha^n\Gamma(n+\beta-R)\right) \qquad \forall R \in \mathbb{N}_0. \tag{A.5}$$

B Technical inequalities and identities

Lemma B.1. *If $C \in \mathbb{R}_{>0}$, $u \in \mathbb{R}_{\geq 1}$, and $R, s \in \mathbb{N}_0$ such that $s \geq 4\left(R\log u + C\right) \geq 1$, then*

$$\sum_{m=s}^{\infty} \frac{C^m}{m!} \leq \frac{1}{u^R}. \tag{B.1}$$

Proof. It follows from $\binom{s+m}{m} \geq 1 \Rightarrow s!m! \leq (s+m)!$ that

$$\sum_{m=s}^{\infty}\frac{C^m}{m!} = C^s\sum_{m=0}^{\infty}\frac{C^m}{(m+s)!} \leq \frac{C^s}{s!}\sum_{m=0}^{\infty}\frac{C^m}{m!} = e^C\frac{C^s}{s!}. \tag{B.2}$$

Observe that $\frac{1}{s!} \leq \left(\frac{e}{s}\right)^s$, because $e^s \geq \frac{s^s}{s!}$ and that $\left(\frac{eC}{s}\right)^s$ is monotonically decreasing, seen as a function in s, since $\frac{eC}{s} \leq 1$. Accordingly,

$$\leq e^C \left(\frac{eC}{s}\right)^s \leq \left(\frac{eC}{4\left(R\log u + C\right)}\right)^{4(R\log u + C)} = e^C \left(\frac{e^{1-\log 4}C}{R\log u + C}\right)^{4(R\log u + C)}. \tag{B.3}$$

Finally, because $R\log u \geq 0$ and $(1 - \log 4) \approx -0.39 \leq -\frac{1}{4}$

$$\leq e^C \left(e^{1-\log 4}\right)^{4(R\log u + C)} \leq e^{-R\log u} = \frac{1}{u^R}. \tag{B.4}$$

□

Lemma B.2. *If $a \in \mathbb{R}_{>0}$ and $m, k \in \mathbb{N}_0$ with $a > m + k$, then*

$$\binom{a-1}{m}\Gamma(a-m-k) = \sum_{l=0}^{m} \binom{k+l-1}{l} \frac{\Gamma(a-k-l)}{(m-l)!}. \tag{B.5}$$

Proof. This is an exercise in binomial identities. Negating upper indices, using the Chu-Vandermonde identity and negating upper indices again gives,

$$\binom{a-1}{m} = (-1)^m \binom{m-a}{m} = (-1)^m \sum_{l=0}^{m} \binom{-k}{l}\binom{m-a+k}{m-l} \tag{B.6}$$

$$= \sum_{l=0}^{m} \binom{k+l-1}{l}\binom{a-k-l-1}{m-l}. \tag{B.7}$$

The statement follows by writing the second binomial coefficient on the right hand side as a product of Γ-functions and requiring that $a > m + k$. □