# Some results on the $\xi(\mathbf{s})$ and $\boldsymbol{\Xi}(\mathbf{t})$ functions associated with Riemann's $\zeta(\mathbf{s})$ function. * 

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#### Abstract

We report on some properties of the $\xi(s)$ function and its value on the critical line, $\Xi(t)=\xi\left(\frac{1}{2}+i t\right)$. First, we present some identities that hold for the log derivatives of a holomorphic function. We then reexamine Hadamard's product-form representation of the $\xi(s)$ function, and present a simple proof of the horizontal monotonicity of the modulus of $\xi(s)$. We then show that the $\Xi(t)$ function can be interpreted as the autocorrelation function of a weakly stationary random process, whose power spectral function $S(\omega)$ and $\Xi(t)$ form a Fourier transform pair. We then show that $\xi(s)$ can be formally written as the Fourier transform of $S(\omega)$ into the complex domain $\tau=t-i \lambda$, where $s=\sigma+i t=\frac{1}{2}+\lambda+i t$. We then show that the function $S_{1}(\omega)$ studied by Pólya has $g(s)$ as its Fourier transform, where $\xi(s)=g(s) \zeta(s)$. Finally we discuss the properties of the function $g(s)$, including its relationships to Riemann-Siegel's $\vartheta(t)$ function, Hardy's Z-function, Gram's law and the Riemann-Siegel asymptotic formula.


Key words: Riemann's $\zeta(s)$ function, $\xi(s)$ and $\Xi(t)$ functions, Riemann hypothesis, Monotonicity of the modulus $\xi(t)$, Hadamard's product formula, Pólya's Fourier transform representation, Fourier transform to the complex domain, Riemann-Siegel's asymptotic formula, Hardy's Z-function.

## 1 Definition of $\xi(\mathbf{s})$ and its properties

The Riemann zeta function is defined by

$$
\begin{equation*}
\zeta(s)=\sum_{n=1}^{\infty} n^{-s}, \text { for } \Re(s)>1 \tag{1}
\end{equation*}
$$

which is then defined for the entire $s$-domain by analytic continuation (See Riemann [15] and Edwards [3]). In this article we investigate some properties of the function $\xi(s){ }^{1}$ defined by (see Appendix A)

$$
\begin{equation*}
\xi(s)=\frac{s(s-1)}{2} \pi^{-s / 2} \Gamma(s / 2) \zeta(s) \tag{2}
\end{equation*}
$$

The function $\xi(s)$ is an entire function with the following "reflective" property:

$$
\begin{equation*}
\xi(1-s)=\xi(s) \tag{3}
\end{equation*}
$$

If we write

$$
s=\sigma+i t=\frac{1}{2}+\lambda+i t
$$

[^0]the property (3) is paraphrased as
\[

$$
\begin{align*}
& \Re\left\{\xi\left(\frac{1}{2}+\lambda+i t\right)\right\}=\Re\left\{\xi\left(\frac{1}{2}-\lambda+i t\right)\right\},  \tag{4}\\
& \Im\left\{\xi\left(\frac{1}{2}+\lambda+i t\right)\right\}=-\Im\left\{\xi\left(\frac{1}{2}-\lambda+i t\right)\right\}, \tag{5}
\end{align*}
$$
\]

By setting $\lambda=0$ in (5), we find

$$
\begin{equation*}
\Im\left\{\xi\left(\frac{1}{2}+i t\right)\right\}=0, \quad \text { for all } t \tag{6}
\end{equation*}
$$

which implies that $\xi(s)$ is real on the "critical line." Thus, if we define a real-valued function

$$
\begin{equation*}
\Xi(t)=\xi\left(\frac{1}{2}+i t\right)=\Re\left\{\xi\left(\frac{1}{2}+i t\right)\right\}, \tag{7}
\end{equation*}
$$

the Riemann hypothesis can be paraphrased as "The zeros of $\Xi(t)$ are all real," which is indeed the way Riemann stated his conjecture, now known as the Riemann hypothesis or RH for short.

By applying Laplace's equation to $\Im\{\xi(s)\}$ and using (6), we readily find

$$
\begin{equation*}
\left.\frac{\partial^{2} \Im\{\xi(s)\}}{\partial \lambda^{2}}\right|_{\lambda=0}=0 \tag{8}
\end{equation*}
$$

Thus, it follows that $\Im\{\xi(s)\}$ must be a polynomial in $\lambda$ of degree 1 in the vicinity of $\lambda=0$, viz.,

$$
\begin{equation*}
\Im\{\xi(s)\} \approx b(t) \lambda, \text { for } \lambda \approx 0 \tag{9}
\end{equation*}
$$

where $b(t)$ is a function of $t$ only, independent of $\lambda$.
Similarly, by applying Laplace's equation to $\Re\{\xi(s)\}$ and using the Cauchy-Riemann equation:

$$
\begin{equation*}
\frac{\partial \Re\{\xi(s)\}}{\partial t}=-\frac{\partial \Im\{\xi(s)\}}{\partial \lambda} \tag{10}
\end{equation*}
$$

and using (9), we find that the real part of $\xi(s)$ is a polynomial in $\lambda$ of degree 2 :

$$
\begin{equation*}
\Re\{\xi(s)\} \approx \frac{b^{\prime}(t)}{2} \lambda^{2}, \quad \text { for } \lambda \approx 0 \tag{11}
\end{equation*}
$$

where $b^{\prime}(t)=\frac{d b(t)}{d t}$.

## 2 Preliminaries

### 2.1 Logarithmic Differentials of Holomorphic Functions

We begin with the following lemma that is applicable to any holomorphic function.
Lemma 2.1. For a holomorphic function $f(s)$ we have

$$
\begin{align*}
& \frac{1}{|f(s)|} \cdot \frac{\partial|f(s)|}{\partial \sigma}=\Re\left\{\frac{f^{\prime}(s)}{f(s)}\right\}  \tag{12}\\
& \frac{1}{|f(s)|} \cdot \frac{\partial|f(s)|}{\partial t}=-\Im\left\{\frac{f^{\prime}(s)}{f(s)}\right\} \tag{13}
\end{align*}
$$

wherever $f(s) \neq 0$, where $f^{\prime}(s)=\frac{d f(s)}{d s}$.
Proof. See Kobayashi [8].
By differentiating the logarithm of $f(s)$ further, we obtain

Corollary 2.1. For the holomorphic function $f(s)$ of Lemma 2.1 the following identities also hold:

$$
\begin{align*}
& \frac{1}{|f(s)|} \frac{\partial^{2}|f(s)|}{\partial \sigma^{2}}-\left(\frac{1}{|f(s)|} \frac{\partial|f(s)|}{\partial \sigma}\right)^{2}=\Re\left\{\frac{f^{\prime \prime}(s)}{f(s)}-\left(\frac{f^{\prime}(s)}{f(s)}\right)^{2}\right\}  \tag{14}\\
& \frac{1}{|f(s)|} \frac{\partial^{2}|f(s)|}{\partial t^{2}}-\left(\frac{1}{|f(s)|} \frac{\partial|f(s)|}{\partial t}\right)^{2}=-\Re\left\{\frac{f^{\prime \prime}(s)}{f(s)}-\left(\frac{f^{\prime}(s)}{f(s)}\right)^{2}\right\} . \tag{15}
\end{align*}
$$

wherever $f(s) \neq 0$, where $f^{\prime \prime}(s)=\frac{d^{2} f(s)}{d s^{2}}$.
Proof. See Kobayashi [8].

### 2.2 The Product Formula for $\xi(\mathrm{s})$

Hadamard [5] obtained in 1893 the following product-form representation

$$
\begin{equation*}
\xi(s)=\frac{1}{2} e^{B s} \prod_{n}\left[\left(1-\frac{s}{\rho_{n}}\right) e^{s / \rho_{n}}\right] \tag{16}
\end{equation*}
$$

using Weirstrass's factorization theorem, which asserts that any entire function can be represented by a product involving its zeroes. In (16), the product is taken over all (infinitely many) zeros $\rho_{n}$ 's of the function $\xi(s)$, and $B$ is a real constant. Detailed accounts of this formula are found in many books (see e.g., Edwards [3], Iwaniec [7] Patterson [13] and Titchmarsh [18]. Sondow and Dumitrescu [16] and Matiyasevich et al. [11] explored the use of the above product form, hoping to find a possible proof of the Riemann hypothesis.

By taking the logarithm of (16) and differentiating it, we obtain

$$
\begin{equation*}
\frac{\xi^{\prime}(s)}{\xi(s)}=B+\sum_{n}\left(\frac{1}{s-\rho_{n}}+\frac{1}{\rho_{n}}\right) \tag{17}
\end{equation*}
$$

From the definition of $\xi(s)$ in (2), we have

$$
\begin{equation*}
\frac{\xi^{\prime}(s)}{\xi(s)}=\frac{1}{s}+\frac{1}{s-1}-\frac{\log \pi}{2}+\Psi\left(\frac{s}{2}\right)+\frac{\zeta^{\prime}(s)}{\zeta(s)} \tag{18}
\end{equation*}
$$

where

$$
\Psi(s)=\frac{\Gamma^{\prime}(s)}{\Gamma(s)}
$$

is the digamma function.
We equate (17) to (18), use the identity $\Psi\left(\frac{s}{2}+1\right)=\frac{1}{s}+\frac{1}{2} \Psi\left(\frac{s}{2}\right)$, and set $s=0$, obtaining

$$
\begin{equation*}
B+\sum_{n}\left(-\frac{1}{\rho_{n}}+\frac{1}{\rho_{n}}\right)=-1-\frac{1}{2}+\frac{1}{2} \Psi(1)+\frac{\zeta^{\prime}(0)}{\zeta(0)} \tag{19}
\end{equation*}
$$

By using $\zeta^{\prime}(0) / \zeta(0)=\log (2 \pi)$, and $\Psi(1)=\Gamma^{\prime}(1)=-\gamma($ where $\gamma=0.5772218 \ldots$ is the Euler constant $)$, we determine the constant $B$ as

$$
\begin{equation*}
B=\log (2 \pi)-1-\frac{1}{2} \log \pi-\gamma / 2=\frac{1}{2} \log (4 \pi)-1-\gamma / 2=-0.0230957 \ldots \tag{20}
\end{equation*}
$$

Davenport (1] pp. 81-82) derives an alternative expression for $B$. The reflective property of $\xi(s)$ gives the identity

$$
\begin{equation*}
\frac{\xi^{\prime}(s)}{\xi(s)}=-\frac{\xi^{\prime}(1-s)}{\xi(1-s)} \tag{21}
\end{equation*}
$$

which, together with (17), yields

$$
\begin{equation*}
B+\sum_{n}\left(\frac{1}{s-\rho_{n}}+\frac{1}{\rho_{n}}\right)=-B-\sum_{n}\left(\frac{1}{1-s-\rho_{n}}+\frac{1}{\rho_{n}}\right) \tag{22}
\end{equation*}
$$

Thus,

$$
\begin{align*}
B & =-\sum_{n} \frac{1}{\rho_{n}}-\frac{1}{2}\left(\sum_{n} \frac{1}{s-\rho_{n}}-\sum_{n} \frac{1}{s-\left(1-\rho_{n}\right)}\right) \\
& =-\sum_{n} \frac{1}{\rho_{n}}=-2 \sum_{n=1}^{\infty} \frac{\sigma_{n}}{\sigma_{n}^{2}+t_{n}^{2}}, \tag{23}
\end{align*}
$$

Note that the two summed terms in the parenthesis in the first line of the above cancel to each other, because if $\rho_{n}$ is a zero, so is $1-\rho_{n}$. To obtain the final expression in the above, we use the property that when $\rho_{n}=\sigma_{n}+i t_{n}$ is a zero, so is its complex conjugate $\rho_{n}^{*}=\sigma_{n}-i t_{n}$, thus we enumerate zeros in such a way that $\rho_{n}^{*}=\rho_{-n}$.

By substituting (23) back into (16), we obtain

$$
\begin{align*}
\xi(s) & =\frac{1}{2} \exp \left(-s \sum_{n} \frac{1}{\rho_{n}}\right) \prod_{n}\left(1-\frac{s}{\rho_{n}}\right) e^{s / \rho_{n}}=\frac{1}{2} \prod_{n} e^{-s / \rho_{n}}\left(1-\frac{s}{\rho_{n}}\right) e^{s / \rho_{n}} \\
& =\frac{1}{2} \prod_{n}\left(1-\frac{s}{\rho_{n}}\right) \tag{24}
\end{align*}
$$

This is nothing but the product form

$$
\xi(s)=\xi(0) \prod_{n}\left(1-\frac{s}{\rho_{n}}\right)
$$

which Edwards (see 3 p. 18 and pp. 46-47) attributes to Riemann.
Then, Eqn.(17) is simplified to

$$
\begin{equation*}
\frac{\xi^{\prime}(s)}{\xi(s)}=\sum_{n} \frac{1}{s-\rho_{n}} \tag{25}
\end{equation*}
$$

From this and Lemma 2.1 we have

$$
\begin{equation*}
\frac{1}{|\xi(s)|} \frac{\partial|\xi(s)|}{\partial \sigma}=\Re\left(\sum_{n} \frac{1}{s-\rho_{n}}\right)=\sum_{n} \frac{\sigma-\sigma_{n}}{\left(\sigma-\sigma_{n}\right)^{2}+\left(t-t_{n}\right)^{2}} \tag{26}
\end{equation*}
$$

Thus, we arrive at the following theorem concerning the monotonicity of the $|\xi(s)|$ function, which Sondow and Dumitrescu [16] proved in a little more complicated way based on (16) instead of (24). Matiyaesevich et al. 11 also discuss the monotonicity of the $\xi(s)$ and other functions.

Theorem 2.1 (Monotonicity of Modulus Function $|\xi(s)|$ ). Let $\sigma_{\text {sup }}$ be the supremum of the real parts of all zeros:

$$
\sigma_{\text {sup }}=\sup _{n}\left\{\sigma_{n}\right\}
$$

Then the modulus $|\xi(\sigma+i t)|$ is a monotone increasing function of $\sigma$ in the region $\sigma>\sigma_{\text {sup }}$ for all real $t$. Likewise, the modulus is a monotone decreasing function of $\sigma$ in the region $\sigma<\sigma_{\mathrm{inf}}$, where

$$
\sigma_{\mathrm{inf}}=\inf _{n}\left\{\sigma_{n}\right\}=1-\sigma_{\text {sup }}
$$

Proof. It is apparent from (26) that $|\xi(s)|$ is a monotone increasing function of $\sigma$ in the range $\sigma>\sigma_{\text {sup }} \geq \frac{1}{2}$ for all $t$. Because of the reflective property (3) it then readily follows that $|\xi(s)|$ is a monotone decreasing function of $\sigma$ in the range $\sigma<1-\sigma_{\text {sup }} \leq \frac{1}{2}$.

Thus, if all zeta zeros are located on the critical line, i.e., if $\sigma_{\text {sup }}=\sigma_{\mathrm{inf}}=\frac{1}{2}$, the derivative of the modulus $|\xi(s)|$ is positive for $\sigma>\frac{1}{2}$, and negative for $\sigma<\frac{1}{2}$. Thus, we have shown the necessity of monotonicity of the modulus function $|\xi(s)|$, which has been one of major concerns towards a proof of the Riemann hypothesis.
Corollary 2.2 (Monotonicity of Modulus Function $|\xi(s)|$, if the Riemann hypothesis is true). If all zeta zeros are on the critical line, the modulus $|\xi(\sigma+i t)|$ is a monotone increasing function of $\sigma$ in the right half plane, $\sigma>\frac{1}{2}$. Likewise, the modulus is a monotone decreasing function of $\sigma$ in the left half plane, $\sigma<\frac{1}{2}$.
Proof. The above discussion that has led to this corollary should suffice as a proof.

### 2.3 Functions $\mathbf{a}(\lambda, \mathbf{t}), \mathbf{b}(\lambda, \mathbf{t}), \alpha(\lambda, \mathbf{t}), \beta(\lambda, \mathbf{t})$ and Their Properties

Take the imaginary part of both sides of (25) and set $s=\frac{1}{2}+i t$. By noting that $\xi(s)$ is real for $\sigma=\frac{1}{2}$, we obtain

$$
\begin{equation*}
\left.\frac{1}{\xi(s)} \frac{\partial \Im(\xi(s))}{\partial \sigma}\right|_{\sigma=\frac{1}{2}}=\sum_{n} \frac{t-t_{n}}{\left(t-t_{n}\right)^{2}+\left(\frac{1}{2}-\sigma_{n}\right)^{2}} \tag{27}
\end{equation*}
$$

Recall the function $b(t)$ defined in (9). Then, the LHS of the above is $\frac{b(t)}{\Xi(t)}$, where

$$
\begin{align*}
& \Xi(t)=\xi\left(\frac{1}{2}+i t\right)=\frac{1}{2} \prod_{n}\left(1-\frac{\frac{1}{2}+i t}{\sigma_{n}+i t_{n}}\right)  \tag{28}\\
& b(t)=\left.\frac{\partial \Im\{\xi(s)\}}{\partial \sigma}\right|_{\sigma=\frac{1}{2}}=\Xi(t) \cdot \sum_{n} \frac{t-t_{n}}{\left(t-t_{n}\right)^{2}+\left(\frac{1}{2}-\sigma_{n}\right)^{2}} . \tag{29}
\end{align*}
$$

Differentiate (25) once more, and we obtain

$$
\frac{\xi^{\prime \prime}(s) \xi(s)-\xi^{\prime 2}(s)}{\xi^{2}(s)}=-\sum_{n} \frac{1}{\left(s-\rho_{n}\right)^{2}}
$$

which can be rearranged to yield

$$
\begin{equation*}
\frac{\xi^{\prime \prime}(s)}{\xi(s)}=\left(\frac{\xi^{\prime}(s)}{\xi(s)}\right)^{2}-\sum_{n} \frac{1}{\left(s-\rho_{n}\right)^{2}} \tag{30}
\end{equation*}
$$

Taking the real part of both sides, and evaluating them at $s=\frac{1}{2}+i t$, we find

$$
\begin{align*}
\frac{2 a(t)}{\Xi(t)} & =-\left(\frac{b(t)}{\Xi(t)}\right)^{2}+\sum_{n} \frac{\left(t-t_{n}\right)^{2}-\left(\frac{1}{2}-\sigma_{n}\right)^{2}}{\left[\left(\frac{1}{2}-\sigma_{n}\right)^{2}+\left(t-t_{n}\right)^{2}\right]^{2}} \\
& =\frac{-b^{2}(t)^{2}+b^{\prime}(t) \Xi(t)-b(t) \Xi^{\prime}(t)}{\Xi^{2}(t)} \tag{31}
\end{align*}
$$

where

$$
\begin{equation*}
2 a(t)=\left.\frac{\partial^{2} \xi(s)}{\partial \sigma^{2}}\right|_{\sigma=\frac{1}{2}} \tag{32}
\end{equation*}
$$

From the Cauchy-Riemann equation we find

$$
\begin{equation*}
\Xi^{\prime}(t)=-\left.\frac{\partial \Im\{\xi(s)\}}{\partial \sigma}\right|_{\sigma=\frac{1}{2}}=-b(t) \tag{33}
\end{equation*}
$$

By substituting this into (31), we obtain a surprisingly simple result:

$$
\begin{equation*}
a(t)=\frac{1}{2} b^{\prime}(t)=-\frac{1}{2} \Xi^{\prime \prime}(t) \tag{34}
\end{equation*}
$$

which can be alternatively obtained by applying the Laplace equation to (32).
The above formulae carry over to any point $s=\frac{1}{2}+\lambda+i t$ :

Lemma 2.2. Let us define

$$
\begin{align*}
2 a(\lambda, t) & =\frac{\partial^{2} \Re\{\xi(s)\}}{\partial \lambda^{2}}=-\Re\left\{\xi^{\prime \prime}(t)\right\}  \tag{35}\\
b(\lambda, t) & =\frac{\partial \Im\{\xi(s)\}}{\partial \lambda} \tag{36}
\end{align*}
$$

where $\xi^{\prime \prime}(s)$ is the second partial derivative of $\xi(s)$ with respect to $t$. Then, the following relations hold:

$$
\begin{align*}
a(\lambda, t) & =\frac{1}{2} b^{\prime}(\lambda, t)  \tag{37}\\
b(\lambda, t) & =-\Re\left\{\xi^{\prime}(t)\right\} \tag{38}
\end{align*}
$$

Proof. By applying the Cauchy-Riemann equations and Laplace's equation, the above relations can be easily derived.

We now derive similar functions and their relations by interchanging $\Re\{\xi(s)\}$ and $\Im\{\xi(s)\}$.
Corollary 2.3. Let us define

$$
\begin{align*}
2 \alpha(\lambda, t) & =\frac{\partial^{2} \Im\{\xi(s)\}}{\partial \lambda^{2}}=-\Im\left\{\xi^{\prime \prime}(s)\right\}  \tag{39}\\
\beta(\lambda, t) & =\frac{\partial \Re\{\xi(s)\}}{\partial \lambda} \tag{40}
\end{align*}
$$

Then the following relations hold:

$$
\begin{align*}
\alpha(\lambda, t) & =-\frac{1}{2} \beta^{\prime}(\lambda, t)  \tag{41}\\
\beta(\lambda, t) & =\Im\left\{\xi^{\prime}(s)\right\}  \tag{42}\\
\frac{\partial a(\lambda, t)}{\partial \lambda} & =\alpha^{\prime}(\lambda, t), \quad \frac{\partial \alpha(\lambda, t)}{\partial \lambda}=-a^{\prime}(\lambda, t)  \tag{43}\\
\frac{\partial \beta(\lambda, t)}{\partial \lambda} & =b^{\prime}(\lambda, t), \quad \frac{\partial b(\lambda, t)}{\partial \lambda}=-\beta^{\prime}(\lambda, t) \tag{44}
\end{align*}
$$

Proof. By applying the Cauchy-Riemann equations and Laplace's equation, the above relations can be easily derived.

## 3 The Fourier transform representation of $\xi(s)$

### 3.1 Integral representation of $\xi(\mathbf{s})$

We begin with the following integral representation of $\xi(s)$ (see Appendix A) found in Edwards [3], p.16.

$$
\begin{equation*}
\xi(s)=\frac{1}{2}-\frac{s(1-s)}{2} \int_{1}^{\infty} \psi(x)\left(x^{s / 2}+x^{(1-s) / 2}\right) \frac{d x}{x} \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi(x)=\sum_{n=1}^{\infty} e^{-n^{2} \pi x} \tag{46}
\end{equation*}
$$

is called the theta function. By applying integration by parts to (45) and Jacobi's identity for the theta function ${ }^{2}$ Edwards ([3], p. 17) gives the following expression by generalizing Riemann's result, which holds for any complex number $s$ :

$$
\begin{equation*}
\xi(s)=4 \int_{1}^{\infty} \frac{d\left[x^{3 / 2} \psi^{\prime}(x)\right]}{d x} x^{-1 / 4} \cosh \left[\frac{1}{2}\left(s-\frac{1}{2}\right) \log x\right] d x \tag{48}
\end{equation*}
$$

[^1]By writing

$$
\begin{equation*}
\frac{d\left[x^{3 / 2} \psi^{\prime}(x)\right]}{d x} x^{-1 / 4}=\pi x^{1 / 4} D(x) \tag{49}
\end{equation*}
$$

with $D(x)$ defined by

$$
\begin{equation*}
D(x)=\sum_{n=1}^{\infty} n^{2}\left(n^{2} \pi x-\frac{3}{2}\right) e^{-n^{2} \pi x}>0, \text { for } x \geq 1 \tag{50}
\end{equation*}
$$

we can write (48) as

$$
\begin{equation*}
\xi(s)=4 \pi \int_{1}^{\infty} x^{1 / 4} D(x) \cos \left(\frac{\tau \log x}{2}\right) d x \tag{51}
\end{equation*}
$$

where $\tau$ is a complex number defined by

$$
\begin{equation*}
\tau=t-i \lambda=-i\left(s-\frac{1}{2}\right) \tag{52}
\end{equation*}
$$

and we used the identity $\cosh (i y)=\cos y$. By changing the variable from $x$ to $\omega$ by

$$
\begin{equation*}
\omega=\frac{\log x}{2}, \quad x \geq 1 \tag{53}
\end{equation*}
$$

and defining

$$
\begin{equation*}
S(\omega)=8 \pi e^{5 \omega / 2} D\left(e^{2 \omega}\right), \quad \omega \geq 0 \tag{54}
\end{equation*}
$$

we can write (51) as

$$
\begin{equation*}
\xi(s)=\int_{0}^{\infty} S(\omega) \cos (\omega \tau) d \omega \tag{55}
\end{equation*}
$$

which is a compact expression for

$$
\begin{equation*}
\xi\left(\frac{1}{2}+\lambda+i t\right)=\int_{0}^{\infty} S(\omega)(\cos \omega t \cosh (\omega \lambda)+i \sin \omega t \sinh (\omega \lambda)) d \omega \tag{56}
\end{equation*}
$$

On the critical line $s=\frac{1}{2}+i t$ (i.e., when $\lambda=0$ ), the above reduces to a more familiar formula

$$
\begin{equation*}
\Xi(t)=\int_{0}^{\infty} S(\omega) \cos (\omega t) d \omega \tag{57}
\end{equation*}
$$

### 3.2 The kernel function $S(\omega)$ as a power spectral function.

The kernel $S(\omega)$ defined by (54) is positive for all $\omega \geq 0$, because $D(x)$ is positive for $x \geq 1$. Therefore, $S(\omega)$ can qualify as a spectral density function of a certain wide-sense stationary (a.k.a. weakly stationary) process, and we can interpret $\Xi(t)$ as its autocorrelation function (see e.g., 10 p. 349). In this context, the Fourier transforms between the spectrum $S(\omega)$ and the function $\Xi(t)$ are what is known as the WienerKhinchin theorem (a.k.a. the Wiener-Khinchin-Einstein theorem). The inverse transform to (57), given below by (61), exists when $\Xi(t)$ is absolutely integrable.

The Fourier transform representation (57) has been studied by George Pólya 14 and others (see e.g., Titchmarsh [18, Chapter 10). Dimitrov and Rusev [2] give a comprehensive review of the past work on "zeros of entire Fourier transforms," including Pólya's work.

From the above observation that $S(\omega)$ is positive for $\omega \geq 0$, we can readily establish the following proposition:

Theorem 3.1. The modulus $\mid \Xi(t)) \mid$ is maximum at $t=0$, i.e.,

$$
\begin{equation*}
|\Xi(t)| \leq \Xi(0)=0.4971 \ldots, \quad \text { for all } t \tag{58}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
\int_{0}^{\infty} \Xi(t) d t=3 \pi\left(\frac{\pi^{1 / 4}}{\Gamma(3 / 4)}-1\right)=2.8067 \ldots \tag{59}
\end{equation*}
$$

Proof. From (55), it readily follows that

$$
\begin{equation*}
|\Xi(t)| \leq \int_{0}^{\infty}|S(\omega)| d \omega=\int_{0}^{\infty} S(\omega) d \omega=\Xi(0) \tag{60}
\end{equation*}
$$

Since $\zeta\left(\frac{1}{2}\right)=-1.46035 \ldots 3^{3}$, and $g\left(\frac{1}{2}\right)=-\frac{1}{8} \pi^{-1 / 4} \Gamma\left(\frac{1}{4}\right)=-0.3404 \ldots$, we have $\Xi(0)=\xi\left(\frac{1}{2}\right)=g\left(\frac{1}{2}\right) \zeta\left(\frac{1}{2}\right)=$ 0.4971 ....

From the Wiener-Khinchin inverse formula, which holds when $\Xi(t)$ is absolutely integrable, we have

$$
\begin{equation*}
S(\omega)=\frac{2}{\pi} \int_{0}^{\infty} \Xi(t) \cos (\omega t) d t \tag{61}
\end{equation*}
$$

By setting $\omega=0$, we readily find

$$
\begin{equation*}
S(0)=\frac{2}{\pi} \int_{0}^{\infty} \Xi(t) d t \tag{62}
\end{equation*}
$$

By setting $\omega=0$ in (54), we have

$$
\begin{equation*}
S(0)=8 \pi D(1)=8\left(\frac{3}{2} \psi^{\prime}(1)+\psi^{\prime \prime}(1)\right) \tag{63}
\end{equation*}
$$

The function $\psi(x)$ satisfies the aforementioned Jacobi's identity (47). By differentiating the identity equation, we find

$$
\begin{equation*}
2 \psi^{\prime}(x)=-\frac{1}{2} x^{-3 / 2}-x^{-3 / 2} \psi(1 / x)-2 x^{-5 / 2} \psi^{\prime}(1 / x) \tag{64}
\end{equation*}
$$

By setting $x=1$ in (64) we obtain

$$
\begin{equation*}
\psi^{\prime}(1)=-\frac{1}{8}(1+2 \psi(1)) \tag{65}
\end{equation*}
$$

The value of $\psi(1)$ is known (see e.g., Yi [19], Theorem 5.5 in p. 398)

$$
\begin{equation*}
\psi(1)=\frac{1}{2}\left(\frac{\pi^{1 / 4}}{\Gamma(3 / 4)}-1\right)=\frac{1}{2}\left(\frac{1.3313}{1.2254}-1\right)=0.0432 \ldots \tag{67}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\psi^{\prime}(1)=-\frac{1}{8} \frac{\pi^{1 / 4}}{\Gamma(3 / 4)}=-0.1358 \ldots \tag{68}
\end{equation*}
$$

The numerical evaluation of $\psi^{\prime \prime}(1)$ is straightforward, since its series representation converges rapidly:

$$
\begin{equation*}
\psi^{\prime \prime}(1)=\pi^{2} \sum_{n=1}^{\infty} n^{4} e^{-\pi n^{2}} \approx \pi^{2} \sum_{n=1}^{2} n^{4} e^{-\pi n^{2}}=0.4271 \ldots \tag{69}
\end{equation*}
$$

Thus, we finally evaluate

$$
\begin{equation*}
\int_{0}^{\infty} \Xi(t) d t=\frac{\pi}{2} S(0)=4 \pi\left(\frac{3}{2} \psi^{\prime}(1)+\psi^{\prime \prime}(1)\right)=2.8067 \ldots \tag{70}
\end{equation*}
$$

[^2]The variable $t$ of the complex variable $s=\sigma+i t=\frac{1}{2}+\lambda+i t$ is often called the height in the zeta function related literature. In view of the Wiener-Khinchin theorem (57) and (61), it may be appropriate to interpret $t$ as "time" and the variable $\omega$ of $S(\omega)$ as the "(angular) frequency." Then, we may refer to the complex number $\tau$ defined by (52) as "complex-time." Use of the complex-time $\tau$ allow the compact representation (55) given earlier, viz.

$$
\begin{equation*}
\xi(s)=\int_{0}^{\infty} S(\omega) \cos (\omega \tau) d \omega \tag{71}
\end{equation*}
$$

This interpretation of Riemann's result (48) will shed some new light to the Fourier transform representation of the $\xi(s)$ function. We will further discuss this in a later section.

## 4 Further results on the Fourier transform representation

### 4.1 Decomposition of $\mathbf{S}(\omega)$

In the Fourier transform representation (155) the kernel function $S(\omega)$ can be expressed as

$$
\begin{equation*}
S(\omega)=\sum_{n=1}^{\infty} S_{n}(\omega) \tag{72}
\end{equation*}
$$

with

$$
\begin{equation*}
S_{n}(\omega)=8 \pi e^{5 \omega / 2} D_{n}\left(e^{2 \omega}\right) \tag{73}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{n}(x)=n^{2}\left(n^{2} \pi x-\frac{3}{2}\right) e^{-n^{2} \pi x} \tag{74}
\end{equation*}
$$

The Fourier transform can therefore be written as a summation of infinite components, i.e.,

$$
\begin{equation*}
\xi(s)=\sum_{n=1}^{\infty} f_{n}(s) \tag{75}
\end{equation*}
$$

with

$$
\begin{align*}
f_{n}(s) & =\int_{0}^{\infty} S_{n}(\omega) \cos (\omega \tau) d \omega \\
& =8 \pi \int_{0}^{\infty} e^{5 \omega / 2} D_{n}\left(e^{2 \omega}\right) \cos (\omega \tau) d \omega \tag{76}
\end{align*}
$$

The switching in the order between the summation over $n$ and the integration over $\omega$, as used in (76) and (75), can be justified, because the series $\sum_{n=1}^{N} S_{n}(\omega)$ uniformly converges to $S(\omega)$ as $N \rightarrow \infty$ in the entire range $\omega \geq 0$. Note also that in the range $\omega \geq 0, S(\omega)$ is predominantly determined by its first components $S_{1}(\omega)$, leaving $S_{n}(\omega), n \geq 2$ negligibly smaller. However, any attempt to replace $S(\omega)$ by $S_{1}(\omega)$ in an effort to prove the Riemann hypothesis would fail, as argued by Titchmarsh (see [18, Chapter 10, p. 256).

### 4.2 The Fourier transform of $\mathbf{S}(\omega)$ in $-\infty<\omega<\omega$.

Now let us consider the Fourier transform of $S(\omega)$ defined over the entire real line $-\infty<\omega<\infty$, instead of the positive line $\omega \geq 0$. Note that the kernel $S(\omega)$ of (54) extended to the range $-\infty<\omega<\infty$ is symmetric, i.e.,

$$
\begin{equation*}
S(-\omega)=S(\omega), \quad-\infty<\omega<\infty \tag{77}
\end{equation*}
$$

which can be shown using Jacobi's identity (47). See 9 for a derivation of (77).
The Fourier transform representation (55) can then be rewritten as

$$
\begin{equation*}
\xi(s)=\frac{1}{2} \int_{-\infty}^{\infty} S(\omega) e^{i \omega \tau} d \omega=\frac{1}{2} \int_{-\infty}^{\infty} S(\omega) e^{\omega\left(s-\frac{1}{2}\right)} d \omega \tag{78}
\end{equation*}
$$

Since the kernel $S(\omega)$ is a symmetric real function, we can readily derive the reflective property $\xi(1-s)=\xi(s)$ and thus $\xi(s)$ is real on the critical line.

The kernel $S_{n}(\omega)$ of (73) can be written as

$$
\begin{equation*}
S_{n}(\omega)=8 \pi n^{2} e^{\frac{5 \omega}{2}} D_{1}\left(n^{2} e^{2 \omega}\right) \tag{79}
\end{equation*}
$$

with

$$
\begin{equation*}
D_{1}(x)=\left(\pi x-\frac{3}{2}\right) e^{-\pi x} \tag{80}
\end{equation*}
$$

Furthermore, we can write $S_{n}(\omega)$ in terms of $S_{1}(\omega)$ as follows:

$$
\begin{equation*}
S_{n}(\omega)=\frac{1}{\sqrt{n}} S_{1}(\omega+\log n), \quad n=1,2,3, \ldots \tag{81}
\end{equation*}
$$

By substituting (72) and (81) into the above, we obtain

$$
\begin{align*}
\xi(s) & =\frac{1}{2} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} S_{n}(\omega) e^{i \omega \tau} d \omega=\frac{1}{2} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} S_{1}(\omega+\log n) e^{i \omega \tau} d \omega \\
& =\frac{1}{2} \int_{-\infty}^{\infty} S_{1}\left(\omega^{\prime}\right) e^{i \omega^{\prime} \tau} d \omega^{\prime} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{-i \tau \log n} \tag{82}
\end{align*}
$$

where we set $\omega+\log n=\omega^{\prime}$ in the above derivation. The summed term is nothing but the zeta function $\zeta(s)$, i.e.,

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{-i \tau \log n}=\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}+i \tau}}=\zeta\left(\frac{1}{2}+i \tau\right)=\zeta(s) \tag{83}
\end{equation*}
$$

The result (82) can be compactly expressed as

$$
\begin{equation*}
\xi(s)=\xi_{1}(s) \zeta(s) \tag{84}
\end{equation*}
$$

By writing

$$
\begin{equation*}
g(s)=\frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \tag{85}
\end{equation*}
$$

we can state the following proposition by referring to (2):
Theorem 4.1. (The Fourier transform of $\mathbf{S}_{\mathbf{1}}(\omega)$ )
The function $g(s)$ (85) that transforms $\zeta(s)$ into $\xi(s)$ by multiplication is the Fourier transform of $S_{1}(\omega)$ to the domain $\tau$, i.e.,

$$
\begin{equation*}
g(s)=\frac{1}{2} \int_{-\infty}^{\infty} S_{1}(\omega) e^{i \omega \tau} d \omega=\xi_{1}(s) \tag{86}
\end{equation*}
$$

where $\tau=t-i \lambda=t-i\left(\sigma-\frac{1}{2}\right)=-i\left(s-\frac{1}{2}\right)$.
Proof. See 9 .

Let us denote the Fourier transform of $S_{n}(\omega)$ as $\xi_{n}(s)$ :

$$
\begin{equation*}
\xi_{n}(s)=\frac{1}{2} \int_{-\infty}^{\infty} S_{n}(\omega) e^{i \omega \tau} d \omega=\xi_{1}(s) n^{-s} \tag{87}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi(s)=\sum_{n=1}^{\infty} \xi_{n}(s) \tag{88}
\end{equation*}
$$

Note that the functions $\xi_{n}(s)$ are individually complex functions even on the critical line, since $S_{n}(\omega)$ are not symmetric functions, thus $\xi_{n}(s)$ 's do not enjoy the reflective property that their sum $\xi(s)$ does. If we define

$$
\begin{equation*}
\bar{\xi}_{n}(s)=\frac{1}{2}\left[\xi_{n}(s)+\xi_{n}(1-s)\right]=\frac{1}{2}\left[g_{n}(s) n^{-s}+g_{n}(1-s) n^{s-1}\right] \tag{89}
\end{equation*}
$$

this function is reflective and

$$
\begin{equation*}
\xi(s)=\sum_{n=1}^{\infty} \bar{\xi}_{n}(s) \tag{90}
\end{equation*}
$$

### 4.3 Properties of the $\mathrm{g}(\mathrm{s})$ function

In this section we discuss some properties of $g(s)$ defined by (85), and its relations to the Riemann-Siegel function and Hardy's Z-function.

We set $s=\frac{1}{2}+i t$ in $g(s)$ and define real functions $a(t)$ and $b(t)$ :

$$
\begin{align*}
a(t) & =\Re\left\{\log \Gamma\left(\frac{1}{4}+\frac{i t}{2}\right)\right\}, \\
b(t) & =\Im\left\{\log \Gamma\left(\frac{1}{4}+\frac{i t}{2}\right)\right\} . \tag{91}
\end{align*}
$$

Then, we can write

$$
\begin{equation*}
g\left(\frac{1}{2}+i t\right)=-\frac{1}{2}\left(t^{2}+\frac{1}{4}\right) \pi^{-1 / 4} e^{-i \frac{t}{2} \log \pi} e^{a(t)+i b(t)} . \tag{92}
\end{equation*}
$$

By defining two real functions $r(t)$ and $\vartheta(t)$

$$
\begin{align*}
& r(t)=-\frac{1}{2}\left(t^{2}+\frac{1}{4}\right) \pi^{-1 / 4} e^{a(t)} \\
& \vartheta(t)=b(t)-\frac{t}{2} \log \pi=\Im\left\{\log \Gamma\left(\frac{1}{4}+\frac{i t}{2}\right)\right\}-\frac{t}{2} \log \pi \tag{93}
\end{align*}
$$

we can rewrite (92) as

$$
\begin{equation*}
g\left(\frac{1}{2}+i t\right)=r(t) e^{i \vartheta(t)} \tag{94}
\end{equation*}
$$

The function $\vartheta(t)$ of (93) is called the Riemann-Siegel theta function, and the function $Z(t)$ defined by

$$
\begin{equation*}
Z(t)=\zeta\left(\frac{1}{2}+i t\right) e^{i \vartheta(t)} \tag{95}
\end{equation*}
$$

is often referred to as Hardy's $Z$-function [6, which is real for real $t$ and has the same zeros as $\zeta(s)$ at $s=\frac{1}{2}+i t$, with $t$ real. Thus, locating the Riemann zeros on the critical line reduces to locating zeros on the real line of $Z(t)$. Furthermore,

$$
|Z(t)|=\left|\zeta\left(\frac{1}{2}+i t\right)\right| .
$$

Consider the following Stirling approximation formula for $\Gamma(s)$ :

$$
\begin{equation*}
\log \Gamma(s) \approx \frac{1}{2} \log \frac{2 \pi}{s}+s(\log s-1) \tag{96}
\end{equation*}
$$

Then

$$
\begin{equation*}
\log \Gamma(s / 2) \approx\left(1-\frac{s}{2}\right) \log 2+\frac{1}{2} \log \pi+\left(\frac{s-1}{2}\right) \log s-\frac{s}{2} \tag{97}
\end{equation*}
$$

By evaluating the above at $s=\frac{1}{2}+i t$, we have

$$
\begin{align*}
\log \Gamma\left(\frac{1}{4}+\frac{i t}{2}\right) & =a(t)+i b(t) \\
& \approx \frac{3}{4} \log 2+\frac{1}{2} \log \pi-\left(\frac{1}{4}+\frac{t \theta(t)}{2}\right)-\frac{1}{8} \log \left(t^{2}+\frac{1}{4}\right)+i\left[\frac{t}{4} \log \left(t^{2}+\frac{1}{4}\right)-\frac{t}{2}-\frac{t}{2} \log 2-\frac{\theta(t)}{4}\right] \tag{98}
\end{align*}
$$

where

$$
\begin{equation*}
\theta(t)=\tan ^{-1} 2 t \tag{99}
\end{equation*}
$$

Thus, we obtain

$$
\begin{align*}
& r(t) \approx-2^{-\frac{1}{4}} \pi^{\frac{1}{4}}\left(t^{2}+\frac{1}{4}\right)^{\frac{7}{8}} e^{-\frac{1}{4}-\frac{\theta(t) t}{2}} \\
& \vartheta(t) \approx \frac{t}{2} \log \frac{t}{2 \pi e}-\frac{\theta(t)}{4}+\frac{t}{4} \log \left(1+\frac{1}{4 t^{2}}\right) \tag{100}
\end{align*}
$$

If we set

$$
\begin{equation*}
A(t)=-r(t), \quad \text { and } \quad \varphi(t)=\vartheta(t)+\pi \tag{101}
\end{equation*}
$$

then,

$$
\begin{equation*}
g\left(\frac{1}{2}+i t\right)=A(t) e^{i \varphi(t)} \tag{102}
\end{equation*}
$$

We denote the real and imaginary parts of $g\left(\frac{1}{2}+i t\right)$ by $G(t)$ and $\hat{G}(t)$, respectively, viz:

$$
\begin{equation*}
g\left(\frac{1}{2}+i t\right)=G(t)+i \hat{G}(t) \tag{103}
\end{equation*}
$$

Then it is apparent that

$$
\begin{equation*}
G(t)=A(t) \cos \varphi(t), \quad \text { and } \quad \hat{G}(t)=A(t) \sin \varphi(t) \tag{104}
\end{equation*}
$$

For sufficiently large $t \gg 1, \theta(t) \approx \frac{\pi}{2}$. Thus, $A(t)$ and $\varphi(t)$ can be approximated by

$$
\begin{align*}
& A(t) \approx(2 e \pi)^{-\frac{1}{4}} e^{-\frac{\pi t}{4}} t^{\frac{7}{4}}, \text { for } t \gg 1 \\
& \varphi(t) \approx \frac{t}{2} \log \frac{t}{2 e \pi}+\frac{7 \pi}{8}, \text { for } t \gg 1 \tag{105}
\end{align*}
$$

The function $A(t)$ is strictly positive for all $t$, hence $G(t)$ becomes zero only when $\varphi(t)=n \pi+\frac{\pi}{2}$ for some integer $n$. Similarly, $\hat{G}(t)$ crosses zero only when $\varphi(t)=n \pi$ for integer $n$. Thus, the number of zeros $N(T)$ of $G(t)$ in $(0, T)$ is given by

$$
\begin{equation*}
N(T)=\frac{\varphi(T)}{\pi} \approx \frac{T}{2 \pi} \log \frac{T}{2 \pi e}+\frac{7}{8}, \quad T>T(\epsilon) \tag{106}
\end{equation*}
$$

The same result should hold for the number of zeros $N(T)$ of $\hat{G}(T)$ in $(0, T)$. The above $N(T)$ agrees to the asymptotic "Riemann-von Mangoldt formula" for the number of zeros of $\zeta\left(\frac{1}{2}+i t\right)$ (and hence the number of zeros of $\xi\left(\frac{1}{2}+i t\right)$, as well), which Riemann conjectured in his 1859 lecture and proved by von Mangoldt in 1905 (see e.g., [3, 12]).

Gram 44 observed in 1909 that zeros of $Z(t)$ and zeros of $\sin \vartheta(t)$ alternate on the $t$ axis, with some few exception (see Edwards [3] p. 125). His observation is consistent with our analysis given above that the
number of zeros $\hat{G}(t)=A(t) \sin \varphi(t)=-A(t) \sin \vartheta(t)$ in the interval $[0, t]$ is asymptotically equivalent to that of $\zeta\left(\frac{1}{2}+i t\right)$ (and hence that of $\Xi(t)$ as well). If we define the complex function

$$
\begin{equation*}
z(s)=\frac{\xi(s)}{r(t)} \tag{107}
\end{equation*}
$$

then $z(s)$ is reflective. Furthermore $z\left(\frac{1}{2}+i t\right)=Z(t)$, because (94) and (95) imply

$$
\begin{equation*}
Z(t)=\frac{\Xi(t)}{r(t)} \tag{108}
\end{equation*}
$$

Let $G_{n}(t)$ denote the value on the critical line of $\bar{\xi}_{n}(s)$ defined in (89), i.e.,

$$
\begin{align*}
G_{n}(t) & =\bar{\xi}_{n}\left(\frac{1}{2}+i t\right)=\left.\frac{1}{2}\left[g(s) n^{-s}+g(1-s) n^{s-1}\right]\right|_{s=\frac{1}{2}+i t}=\frac{1}{2}\left[(G(t)+i \hat{G}(t)) n^{-\frac{1}{2}-i t}+(G(t)-i \hat{G}(t)) n^{-\frac{1}{2}+i t}\right] \\
& =G(t) n^{-\frac{1}{2}} \cos (t \log n)+\hat{G}(t) n^{-\frac{1}{2}} \sin (t \log n)=A(t) n^{-\frac{1}{2}} \cos (\varphi(t)-t \log n) \tag{109}
\end{align*}
$$

Thus, we find

$$
\begin{equation*}
\Xi(t)=\sum_{n=1}^{\infty} G_{n}(t)=A(t) \sum_{n=1}^{\infty} n^{-\frac{1}{2}} \cos (\varphi(t)-t \log n) \tag{110}
\end{equation*}
$$

where $A(t)=-r(t)$ and $\varphi(t)=\vartheta(t)+\pi$ are defined in (101), and

$$
\begin{equation*}
g\left(\frac{1}{2}+i t\right)=G(t)+i \hat{G}(t)=A(t) e^{i \varphi(t)}=-r(t) e^{i \vartheta(t)} \tag{111}
\end{equation*}
$$

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## Appendix A: Derivation of (2) and (45)

Although the essence of both equations is found in Riemann's original paper, we follow Edwards [3] and Matsumoto 12 . We begin with the integral representation of the gamma function

$$
\begin{equation*}
\Gamma(s)=\int_{0}^{\infty} u^{s-1} e^{-u} d u \tag{A.1}
\end{equation*}
$$

By setting $u=\pi n^{2} x$, we have

$$
\begin{equation*}
\Gamma(s)=\pi^{s} n^{2 s} \int_{0}^{\infty} x^{s-1} e^{-\pi n^{2} x} d x \tag{A.2}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\Gamma(s / 2)=\pi^{s / 2} n^{s} \int_{0}^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^{2} x} d x \tag{A.3}
\end{equation*}
$$

from which we obtain

$$
\begin{equation*}
\pi^{-s / 2} \Gamma(s / 2) n^{-s}=\int_{0}^{\infty} x^{\frac{s}{2}-1} e^{-\pi n^{2} x} d x \tag{A.4}
\end{equation*}
$$

By summing up over $n$ from 1 to infinity, we obtain

$$
\begin{equation*}
\pi^{-s / 2} \Gamma(s / 2) \zeta(s)=\int_{0}^{\infty} x^{\frac{s}{2}-1} \psi(x) d x \tag{A.5}
\end{equation*}
$$

where $\psi(x)$ is given defined in (46).

Let us write A.5) as $\nu(s)$, and the split the integration interval of the RHS into the two subintervals, $[0,1)$ and $[1, \infty)$, viz:

$$
\begin{align*}
\nu(s) & =\pi^{-s / 2} \Gamma(s / 2) \zeta(s)=\int_{0}^{\infty} x^{\frac{s}{2}-1} \psi(x) d x \\
& =\int_{0}^{1} x^{\frac{s}{2}-1} \psi(x) d x+\int_{1}^{\infty} x^{\frac{s}{2}-1} \psi(x) d x \tag{A.6}
\end{align*}
$$

By substituting Jacobi's identity for $\psi(x)$ given by (47) into the first integrand, we find

$$
\begin{align*}
\nu(s) & =\int_{0}^{1} x^{\frac{s}{2}-1}\left(x^{-1 / 2} \psi\left(x^{-1}\right)+\frac{1}{2} x^{-1 / 2}-\frac{1}{2}\right) d x+\int_{1}^{\infty} x^{\frac{s}{2}-1} \psi(x) d x \\
& =-\frac{1}{1-s}-\frac{1}{s}+\int_{1}^{\infty}\left(x^{\frac{s}{2}-1}+x^{\frac{1-s}{2}-1}\right) \psi(x) d x \tag{A.7}
\end{align*}
$$

It is apparent that $\nu(s)$ satisfies the reflective property, i.e.,

$$
\nu(1-s)=\nu(s)
$$

The function $\nu(s)$ is not an entire function since it has $s=0$ and $s=1$ as poles. By multiplying $\nu(s)$ by $-\frac{s(1-s)}{2}$, we define $\xi(s)$, viz.

$$
\begin{equation*}
\xi(s)=-\frac{1}{2} s(1-s) \nu(s)=\frac{1}{2} s(s-1) \pi^{-s / 2} \Gamma(s / 2) n^{-s} \zeta(s) \tag{A.8}
\end{equation*}
$$

which is (2).
The function $\xi(s)$ should satisfy the reflective property (3) since both $\nu(s)$ and $-\frac{s(1-s)}{2}$ are reflective. From A.7), we obtain

$$
\begin{equation*}
\xi(s)=\frac{1}{2}-\frac{1}{2} s(1-s) \int_{1}^{\infty}\left(x^{\frac{s}{2}}+x^{\frac{1-s}{2}}\right) \psi(x) \frac{d x}{x} \tag{A.9}
\end{equation*}
$$

which is (45). From the last expression, it is apparent that $\xi(0)=\xi(1)=\frac{1}{2}$.


[^0]:    *This paper was originally posted on http://hp.hisashikobayashi.com/\#5 January 22, 2016, under the title "No. 5. Some results on the $\xi(\mathbf{s})$ and $\boldsymbol{\Xi}(\mathbf{t})$ functions associated with Riemann's $\zeta(\mathbf{s})$ function: Towards a Proof of the Riemann Hypothesis."
    ${ }^{\dagger}$ Professor Emeritus, Department of Electrical Engineering, Princeton University, Princeton, NJ 08544.
    ${ }^{1}$ In Riemann's 1859 seminal paper [15] he was primarily concerned with the properties of this function evaluated on the critical line $s=\frac{1}{2}+i t$, which he denoted as $\xi(t)$. We write it as $\Xi(t)$ instead, as defined in (77). See, e.g., Titchmarsh [18] p. 16. Edwards [3 writes explicitly $\xi\left(\frac{1}{2}+i t\right)$ for $\Xi(t)$.

[^1]:    ${ }^{2}$ Jacobi's identity for the theta function $\psi(x)$ is

    $$
    \begin{equation*}
    2 \psi(x)+1=x^{-1 / 2}\left(2 \psi\left(x^{-1}\right)+1\right) \tag{47}
    \end{equation*}
    $$

[^2]:    ${ }^{3}$ See e.g. https://oeis.org/A059750

