SYMMETRIES RELATED TO DOMINO TILINGS ON A CHESSBOARD

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Abstract: *In this paper we study different kinds of symmetries related to the domino tilings of chessboards.*

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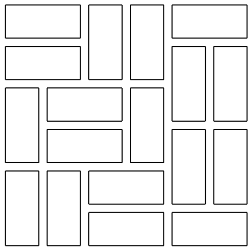
1. PARTITIONING THE DOMINOS

In 1961 Kasteleyn and independently Temperley and Fischer proved that the number of different domino tilings of a $(2r) \times (2n)$ chessboard is exactly

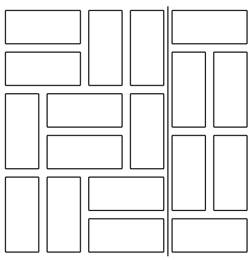
$$\prod_{m=1}^{r} \prod_{k=1}^{n} \left(4\cos^2 \frac{m\pi}{2r+1} + 4\cos^2 \frac{k\pi}{2n+1}\right) .$$

This formula is well-known as the *Kasteleyn* formula. For example, for the 6×6 chessboard we have 6728 different tilings. As an example, we show one of them here.

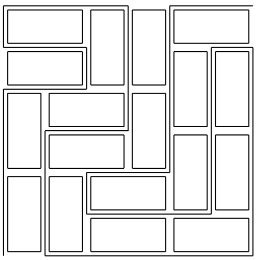


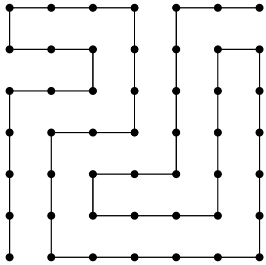
Note that $6728 = 2^3 29^2$ is really a nice number; however, none of Kasteleyn's eighteen $\cos^2$ terms is constructible with ruler and compass. Interestingly enough, in case of a 16×16 chessboard, all terms of the Kasteleyn formula is constructible. (The interested reader can find the details in our recent paper listed in the references.)

It is a well-known math competition problem that for any domino tiling out of the 6728 different variations, we can always find a straight line completely separating the dominos into two nonempty groups. (No domino is bisectible, of course.) Our next figure shows the solution for our example above.



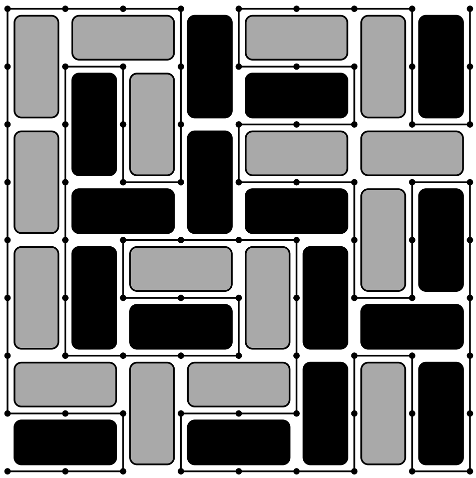
In the recent paper we ask a similar question. As in graph theory, by a Hamiltonian path, we mean such a path which starts at a vertex, ends at another vertex, and goes through all other vertices. In summary, a Hamiltonian path visits each vertex exactly once. (Here we consider each vertex of any elementary square of the chessboard.) On a 6×6 chessboard the number of vertices is 49. Therefore, if there exists any Hamiltonian path, the length of each Hamiltonian path is 48. Our next figure shows a Hamiltonian path connecting two opposite corners of the chessboard.





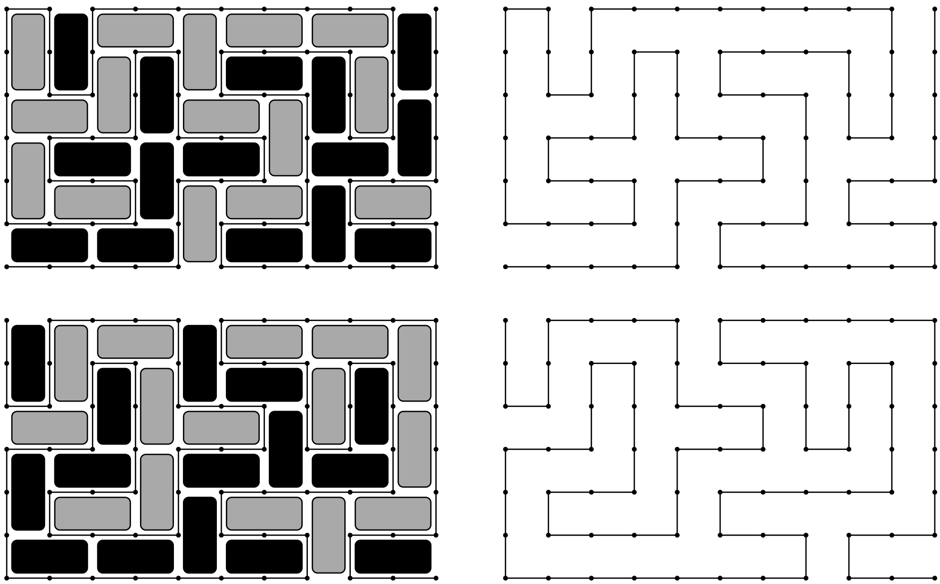
We give the reader an exercise: Find a Hamiltonian path which connects the top-left and the bottom-right corners.

Or next figure gives a larger chessboard tiling with a corresponding Hamiltonian path. On both sides of the Hamiltonian path we find 16 dominos.



The chessboard can be any rectangle with sides of even lengths. The next figures show a chessboard with two Hamiltonian paths. The domino tilings are the same in both cases. Note that the Hamiltonian path connecting the first and the third corners is very different from the Hamiltonian path connecting the second and the fourth corners of the chessboard.

Given an arbitrary domino tiling, in our recent paper listed in the references we showed the existence of a Hamiltonian path between the bottom-left and the top-right corners. Similarly, there is another Hamiltonian path between the top-left and the bottom-right corners. Furthermore, these two Hamiltonian paths both have the following nice property. Assume that we start from either the bottom-left or the bottom-right corner. Then both Hamiltonian paths keep the one-way traffic rule that the first (i.e. the left-most), the third, the fifth, etc. vertical lines all go to the north, and the second, the fourth, the sixth, etc. vertical lines all go to the south. Considering the horizontal lines, the first Hamiltonian path goes to the east on the bottom line, on the third line, on the fifth line, etc., and goes to west on the second line, on the fourth line, etc.; on the other hand, the second Hamiltonian path goes to the west on the bottom line, on the third line, on the fifth line, etc., and goes to east on the second line, on the fourth line, and so on.



2. GENERATING FUNCTION AND NEW RECURRENCE RELATION

Let c_n denote the number of domino tilings of a $6 \times (2n)$ chessboard.

As a recent result, from the OEIS Foundation Inc., we learn the generating function of the sequence $c_0, c_1, c_2, \ldots$:

$$\begin{aligned} g_6(x) &= \frac{1 - 27x + 177x^2 - 328x^3 + 177x^4 - 27x^5 + x^6}{(1-x)(1 - 39x + 377x^2 - 847x^3 + 377x^4 - 39x^5 + x^6)} \\ &= 1 + 13x + 281x^2 + 6728x^3 + 167089x^4 + 4213133x^5 + \ldots \end{aligned}$$

Observe that

$$(1-x)g_6(x) = \frac{u^3 - 27u^2 + 174u - 274}{u^3 - 39u^2 + 374u - 769}$$

where $u = x + x^{-1}$.

By using the above results we have the following values.

n	0	1	2	3	4	5	6	7
c_n	1	13	281	6728	167089	4213133	106912793	2720246633

In some sense this method of computing the numbers $c_0, c_1, c_2, \ldots$ is superior to the Kasteleyn formula because you do not have to worry about the numerical errors.

n	8	9	10
c_n	69289288909	1765722581057	45005025662792

n	11	12	13
c_n	1147185247901449	29242880940226381	745439797095329713

n	14	15
c_n	19002353776441540177	484398978524471931341

n	16	17
c_n	12348080425980866090537	314771823879840325570888

In general, we can get these values as the top-left entry of the matrix C^n where C is the following 20×20 symmetric matrix.

$$
C=\begin{bmatrix}
13&5&3&4&3&5&2&1&2&1&2&1&2&1&2&1&1&0&1&1\\
5&5&0&2&1&2&2&0&1&1&2&0&1&1&2&1&0&0&1&1\\
3&0&3&0&1&1&0&1&0&0&0&1&0&0&0&0&1&0&0&0\\
4&2&0&4&0&2&2&0&2&1&2&0&2&1&1&1&0&0&0&1\\
3&1&1&0&3&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0\\
5&2&1&2&0&5&1&0&2&1&1&0&2&1&2&1&1&0&0&1\\
2&2&0&2&0&1&4&0&1&2&2&0&1&2&1&1&0&0&0&1\\
1&0&1&0&1&0&0&2&0&0&0&1&0&0&0&0&0&0&0&0\\
2&1&0&2&0&2&1&0&4&2&1&0&2&1&1&2&0&0&0&1\\
1&1&0&1&0&1&2&0&2&5&1&0&1&2&1&2&0&1&0&1\\
2&2&0&2&0&1&2&0&1&1&2&0&1&1&1&1&0&0&0&1\\
1&0&1&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
2&1&0&2&0&2&1&0&2&1&1&0&2&1&1&1&0&0&0&1\\
1&1&0&1&0&1&2&0&1&2&1&0&1&2&1&1&0&0&0&1\\
2&2&0&1&0&2&1&0&1&1&1&0&1&1&2&1&0&0&0&1\\
1&1&0&1&0&1&1&0&2&2&1&0&1&1&1&2&0&0&0&1\\
1&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0\\
1&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&1&0&1&0&1&1&0&1&1&1&0&1&1&1&1&0&0&0&1
\end{bmatrix}
$$

We call our brand new method *the matrix power method for computing the Kasteleyn formula.* The proof of the correctness of this method is going to be published in a forthcoming paper. Note that the matrix power method enables us to compute the 4096th element, for example, much faster than the Kasteleyn formula or the generating function method. First you compute C^2, then C^4 as the square of C^2, then C^8 as the square of C^4, and so on.

Interestingly enough, the characteristic polynomial of both C and its inverse matrix C^{-1} can be written in the following form.

$$
\begin{aligned}
&\lambda^{10}(\left(\lambda^{10}+\lambda^{-10}\right)\\
&-63(\lambda^9+\lambda^{-9})+1561(\lambda^8+\lambda^{-8})-21023\left(\lambda^7+\lambda^{-7}\right)\\
&+176393\left(\lambda^6+\lambda^{-6}\right)-992383(\lambda^5+\lambda^{-5})+3912609(\lambda^4+\lambda^{-4})\\
&-11117\,602(\lambda^3+\lambda^{-3})+23182782\left(\lambda^2+\lambda^{-2}\right)-35879970(\lambda+\lambda^{-1})\\
&+41475390)
\end{aligned}
$$

A corollary of this fact is that the numbers c_n satisfy the following recurrence relation (for $n \geq 10$):

$$
\begin{aligned}
&c_{n+10} = -c_{n-10} \\
&+63(c_{n-9} + c_{n+9}) - 1561(c_{n-8} + c_{n+8}) + 21023\,(c_{n-7} + c_{n+7}) \\
&-176393\,(c_{n-6} + c_{n+6}) + 992383(c_{n-5} + c_{n+5}) - 3912609(c_{n-4} + c_{n+4}) \\
&+11117\,602(c_{n-3} + c_{n+3}) - 23182782\,(c_{n-2} + c_{n+2}) + 35879970(c_{n-1} + c_n) \\
&-41475390c_n
\end{aligned}
$$

This enables us to compute c_{n+10} if the previous twenty numbers $c_{n-10}, c_{n-9}, \ldots, c_n, \ldots, c_{n+9}$ are already known. Observe that this formula is symmetric with respect to c_{n-k} and c_{n+k} where $k = 1, 2, \ldots, 10$.

3. CONCLUSION

Each of the exponentially many different tilings of dominos show some symmetries. Here we gave some examples. The physical and chemical connections might also be interesting.

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