Tuenter polynomials and a Catalan triangle

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Abstract

We consider Tuenter polynomials as linear combinations of descending factorials and show that coefficients of these linear combinations are expressed via a Catalan triangle of numbers. We also describe a triangle of coefficients in terms of some polynomials.

1 Preliminaries. Tuenter polynomials

The polynomials we are going to study in this brief note are defined by a recursion [7]

$$P_{k+1}(n) = n^2 \left(P_k(n) - P_k(n-1)\right) + nP_k(n-1), \quad n \in \mathbb{N} \tag{1.1}$$

with initial condition $P_0(n) = 1$. The first few polynomials yielded by (1.1) are as follows.

$$P_1(n) = n,$$

$$P_2(n) = n(2n-1),$$
$$P_3(n) = n(6n^2 - 8n + 3),$$
$$P_4(n) = n(24n^3 - 60n^2 + 54n - 17),$$
$$P_5(n) = n(120n^4 - 480n^3 + 762n^2 - 556n + 155),$$
$$P_6(n) = n(720n^5 - 4200n^4 + 10248n^3 - 12840n^2 + 8146n - 2073).$$

Let us refer to these polynomials as Tuenter ones. Introducing a recursion operator $R := n^2 (1 - \Lambda^{-1}) + n\Lambda^{-1}$, where Λ is a shift operator acting as $\Lambda(f(n)) = f(n+1)$, one can write $P_k(n) = R^k(1)$. The sense of these polynomials is that they help to count the sum

$$S_r(n) = \sum_{j=0}^{2n} \binom{2n}{j} |n-j|^r$$

for odd r.

Bruckman in [2] asked to prove that $S_3(n) = n^2 \left(\begin{array}{c} 2n \\ n \end{array} \right)$. Strazdins in [6] solved this problem and conjectured that $S_{2k+1}(n) = \tilde{P}_k(n) \left(\begin{array}{c} 2n \\ n \end{array} \right)$ with some monic polynomial $\tilde{P}_k(n)$ for any $k \geq 0$. Tuenter showed in [7] that it is almost true. More exactly, he proved that

$$S_{2k+1}(n) = P_k(n) n \left(\begin{array}{c} 2n \\ n \end{array} \right) = P_k(n) \frac{(2n)!}{(n-1)!n!}.$$

One can see that polynomial $\tilde{P}_k(n)$ is monic only for $k = 0, 1$. The recursion (1.1) follows from [7]

$$S_{r+2}(n) = n^2 S_r(n) - 2n(2n-1) S_r(n-1).$$

Also, as was noticed in [7], polynomials $P_k(n)$ can be obtained as a special case of Dumont-Foata polynomials of three variables [3].

2 The Tuenter polynomials as linear combinations of descending factorials

Consider descending factorials

$$(n)_k := n(n-1)(n-2)\cdots(n-k+1).$$

It can be easily seen that

$$R((n)_k) = k^2 (n)_k + (k+1)(n)_{k+1}. \tag{2.1}$$

Let us consider $P_k(n)$ as linear combinations of descending factorials

$$P_k(n) = \sum_{j=1}^{k} c_{j,k} (n)_j,$$

with some coefficients $c_{j,k}$ to be calculated. For example, for the first few $P_k(n)$ we get

$$P_1(n) = (n)_1,$$

$$P_2(n) = (n)_1 + 2(n)_2,$$

$$P_3(n) = (n)_1 + 10(n)_2 + 6(n)_3,$$

$$P_4(n) = (n)_1 + 42(n)_2 + 84(n)_3 + 24(n)_4,$$

$$P_5(n) = (n)_1 + 170(n)_2 + 882(n)_3 + 720(n)_4 + 120(n)_5,$$

$$P_6(n) = (n)_1 + 682(n)_2 + 8448(n)_3 + 15048(n)_4 + 6600(n)_5 + 720(n)_6.$$

With (2.1) we can easily derive recurrence relations for the coefficients $c_{j,k}$. Indeed, from

$$\begin{aligned} P_{k+1}(n) &= \sum_{j=1}^{k+1} c_{j,k+1}(n)_j \\ &= R(P_k(n)) \\ &= \sum_{j=1}^{k} c_{j,k}\left(j^2(n)_j + (j+1)(n)_{j+1}\right) \end{aligned}$$

we get

$$c_{j,k+1} = j^2 c_{j,k} + j c_{j-1,k}, \quad j \geq 1, \quad k \geq j. \tag{2.2}$$

To use (2.2), one must agree that $c_{0,k} = c_{k+1,k} = 0$ for $k \geq 1$. Then, starting from $c_{1,1} = 1$ we obtain the whole set $\{c_{j,k} : j \geq 1, k \geq j\}$. For example, $c_{1,k} = 1$ for all $k \geq 1$, while for $j = 2$ we obviously get a recursion

$$c_{2,k+1} = 4c_{2,k} + 2, \quad c_{2,1} = 0.$$

As can be easily seen, a solution of this equation is given by

$$c_{2,k} = \frac{1}{3}\left(2^{2k-1} - 2\right), \quad k \geq 2. \tag{2.3}$$

Remark 2.1. It is interesting to note that integer sequence (2.3), known as A020988 in [5] gives n-values of local maxima for $s(n) := \sum_{j=1}^{n} a(j)$, where $\{a(n)\}$ is the Golay-Rudin-Shapiro sequence [1].

For the whole set of the coefficients $\{c_{j,k}\}$, we get the following.

Theorem 2.2. *A solution of equation (2.2) with $c_{0,k} = c_{k+1,k} = 0$ for $k \geq 1$ and $c_{1,1} = 1$ is given by*

$$c_{j,k} = \frac{j!}{(2j-1)!}\left(\sum_{q=1}^{j}(-1)^{q+j} B_{j,q} q^{2k-1}\right), \quad \forall j \geq 1, \quad k \geq j, \tag{2.4}$$

where the numbers

$$B_{j,q} := \frac{q}{j}\begin{pmatrix} 2j \\ j-q \end{pmatrix}$$

constitute a Catalan triangle [4].

Proof. Substituting (2.4) into (2.2) and collecting terms at q^{2k-1}, we obtain that sufficient condition for (2.4) to be a solution of (2.2) is that the numbers $B_{j,q}$ enjoy the relation

$$\frac{q^2 j!}{(2j-1)!} B_{j,q} = \frac{j^2 j!}{(2j-1)!} B_{j,q} - \frac{j!}{(2j-3)!} B_{j-1,q}, \quad \forall q = 1, \ldots, j-1.$$

Simplifying the latter we get the relation

$$(j-q)(j+q)B_{j,q} = (2j-1)(2j-2)B_{j-1,q}$$

which can be easily verified. Therefore the theorem is proved. □

The set $\{c_{j,k}\}$ can be presented as the number triangle

$$\begin{array}{ccccccc} & & & c_{1,1} & & & \\ & & c_{1,2} & & c_{2,2} & & \\ & c_{1,3} & & c_{2,3} & & c_{3,3} & \\ & & \ddots & & \ddots & & \ddots \end{array}$$

whose description is given by theorem 2.2.

Remark 2.3. From [4] one knows that the number $B_{j,q}$ can be interpreted as the number of pairs of non-intersecting paths of length j and distance q. The Catalan numbers itself (A000108) are

$$C_j := B_{j,1} = \frac{1}{j}\left(\begin{array}{c} 2j \\ j-1 \end{array}\right).$$

Therefore, we got an infinite number of integer sequences each of which is defined by numbers from the Catalan triangle and begins from $c_{j,j} = j!$. Let us list the first few ones. For example, one has,

$$c_{1,k} = 1,$$
$$c_{2,k} = \frac{1}{3}\left(2^{2k-1} - 2\right),$$
$$c_{3,k} = \frac{1}{20}\left(3^{2k-1} - 4\cdot 2^{2k-1} + 5\right),$$
$$c_{4,k} = \frac{1}{210}\left(4^{2k-1} - 6\cdot 3^{2k-1} + 14\cdot 2^{2k-1} - 14\right),$$
$$c_{5,k} = \frac{1}{3024}\left(5^{2k-1} - 8\cdot 4^{2k-1} + 27\cdot 3^{2k-1} - 48\cdot 2^{2k-1} + 42\right),$$
$$c_{6,k} = \frac{1}{55440}\left(6^{2k-1} - 10\cdot 5^{2k-1} + 44\cdot 4^{2k-1} - 110\cdot 3^{2k-1} + 165\cdot 2^{2k-1} - 132\right), \ldots$$

All these sequences are indeed integer because they are solutions of (2.2).

Let us replace $k \mapsto j+k$ in (2.2) and seek its solution in the form $c_{j,j+k} = F_k(j)j!$. Substituting the latter in (2.2) we come to the recurrence relation

$$F_k(j) - F_k(j-1) = j^2 F_{k-1}(j) \tag{2.5}$$

with conditions $F_0(j) = 1$ and $F_k(1) = 1$. A solution of (2.5) is

$$F_k(j) = 1 + \sum_{2\le\lambda_1\le j}\lambda_1^2 + \sum_{2\le\lambda_1\le\lambda_2\le j}\lambda_1^2\lambda_2^2 + \cdots + \sum_{2\le\lambda_1\le\cdots\le\lambda_k\le j}\lambda_1^2\lambda_2^2\cdots\lambda_k^2.$$

In particular,

$$F_1(j) = 1 + 2^2 + \cdots + j^2 = \frac{1}{6} j(j+1)(2j+1),$$

that is, $F_1(j)$ yields A000330 sequence of the square pyramidal numbers. The next two polynomials $F_k(j)$ are

$$F_2(j) = \frac{1}{360} j(j+1)(j+2)(2j+1)(2j+3)(5j-1)$$

and

$$F_3(j) = \frac{1}{45360} j(j+1)(j+2)(j+3)(2j+1)(2j+3)(2j+5)(35j^2 - 21j + 4).$$

Looking at these examples and others we can suppose that

$$F_k(j) = \prod_{q=0}^{k} (j+q) \prod_{q=0}^{k-1} (2j+2q+1) \tilde{F}_k(j),$$

where $\tilde{F}_k(j)$ is a polynomial of $(k-1)$ degree, which satisfy

$$\tilde{F}_k(1) = \frac{1}{(k+1)!(2k+1)!!}.$$

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