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THE FOURIER EXPANSION OF $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$

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Abstract. We compute the Fourier coefficients of the weight one modular form $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ in terms of the number of representations of an integer as a sum of two squares. We deduce a relation between this modular form and translates of the modular form $\eta(z)^4/\eta(2z)^2$. In the last section we use our main result to give an elementary proof of an identity by Victor Kac.

1. Introduction

In this note we consider the η-product

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \prod_{n\geqslant 1} \frac{(1-q^n)^2}{1-q^n+q^{2n}}. \tag{1.1}$$

where $q = e^{2\pi iz}$. Recall that $\eta(z)$ is Dedekind's eta function

$$\eta(z) = e^{\pi iz/12} \prod_{n\geqslant 1} (1-q^n).$$

The η-product $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ is a modular form of weight 1 and level 6. Since it is invariant under the transformation $z \mapsto z+1$, it has a Fourier expansion of the form

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \sum_{n\geqslant 0} a_6(n)\, q^n, \tag{1.2}$$

where the Fourier coefficients $a_6(n)$ are integers. For general information on η-products, see [6, Sect. 2.1].

Our main result expresses $a_6(n)$ in terms of the number $r(n)$ of representations of n as the sum of two squares, i.e the number of elements $(x,y) \in \mathbb{Z}^2$ such that $x^2+y^2 = n$. Observe that $r(n)$ is divisible by 4 for all $n \geqslant 1$ (for $n = 0$ we have $r(0) = 1$). The sequence $r(n)$ appears as Sequence A004018 in [7].

Theorem 1.1. *For all non-negative integers m we have*

$$\begin{aligned} a_6(3m) &= (-1)^m\, r(3m), \\ a_6(3m+1) &= (-1)^{m+1}\, \frac{r(3m+1)}{4}, \\ a_6(3m+2) &= (-1)^{m+1}\, \frac{r(3m+2)}{2}. \end{aligned}$$

We next relate $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ to the weight one modular form $\eta(z)^4/\eta(2z)^2$ and two of its translates.

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Theorem 1.2. *Set $j = e^{2\pi i/3}$. We have the following linear relation between weight one modular forms:*

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \frac{1}{4}\frac{\eta(z)^4}{\eta(2z)^2} + \frac{1-j}{4}\frac{\eta(z+1/3)^4}{\eta(2z+2/3)^2} + \frac{1-j^2}{4}\frac{\eta(z+2/3)^4}{\eta(2z+4/3)^2}.$$

Both modular forms $\eta(z)\eta(2z)\eta(3z)/\eta(6z)$ and $\eta(z)^4/\eta(2z)^2$ came up naturally in [5], where we computed the number $C_n(q)$ of ideals of codimension n of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials in two variables over a finite field $\mathbb{F}_q$ of cardinality q. Equivalently, $C_n(q)$ is the number of $\mathbb{F}_q$-points of the Hilbert scheme of n points on a two-dimensional torus. We proved that $C_n(q)$ is the value at q of a palindromic one-variable polynomial $C_n(x) \in \mathbb{Z}[x]$ with integer coefficients, which we computed completely (see [5, Th. 1.3]).

We also showed (see [5, Cor. 6.2]) that the generating function of the polynomials $C_n(x)$ can be expressed as the following infinite product:

$$1 + \sum_{n\geqslant 1} \frac{C_n(x)}{x^n}\, q^n = \prod_{n\geqslant 1} \frac{(1-q^n)^2}{1-(x+x^{-1})q^n + q^{2n}}\,. \tag{1.3}$$

It follows from the previous equality that $C_n(1) = 0$. Actually, we proved (see [5, Th. 1.3 and 1.4]) that there exists a polynomial $P_n(x) \in \mathbb{Z}[x]$ such that $C_n(x) = (x-1)^2 P_n(x)$. Moreover, $P_n(x)$ is palindromic, has non-negative coefficients and its value at $x = 1$ is equal to the sum of divisors of n: $P_n(1) = \sum_{d|n} d$.

When $x = e^{2i\pi/k}$ with $k = 2, 3, 4$, or 6, then $x + x^{-1} = 2\cos(2\pi/k)$ is an integer. For such an integer k, we define the sequence $a_k(n)$ by

$$\sum_{n\geqslant 0} a_k(n)\, q^n = \prod_{n\geqslant 1} \frac{(1-q^n)^2}{1 - 2\cos(2\pi/k)\, q^n + q^{2n}}. \tag{1.4}$$

Since $2\cos(2\pi/k)$ is an integer, so is each $a_k(n)$. It follows from (1.3) that these integers are related to the polynomials $C_n(x)$ by

$$C_n(e^{2i\pi/k}) = a_k(n)\, e^{2ni\pi/k}.$$

In [5] we computed $a_2(n)$, $a_3(n)$, and $a_4(n)$ explicitly in terms of well-known arithmetical functions. In particular, we established the equality

$$a_2(n) = (-1)^n\, r(n), \tag{1.5}$$

where $r(n)$ is the number of representations of n as the sum of two squares.

We also observed in [5, (1.8)] that

$$\sum_{n\geqslant 0} a_2(n)\, q^n = \frac{\eta(z)^4}{\eta(2z)^2} \quad \text{and} \quad \sum_{n\geqslant 0} a_6(n)\, q^n = \frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)}. \tag{1.6}$$

The question of finding an explicit expression for $a_6(n)$ had been left open in [5]. This is now solved with Theorem 1.1 of this note. In view of this theorem, of (1.5), and of (1.6), for all $m \geqslant 0$ we obtain

$$\begin{cases} a_6(3m) = a_2(3m), \\ a_6(3m+1) = \dfrac{a_2(3m+1)}{4}, \\ a_6(3m+2) = -\dfrac{a_2(3m+2)}{2}. \end{cases} \tag{1.7}$$

We had experimentally observed (see [5, Footnote 7]) that $a_6(n) = 0$ whenever $a_2(n) = 0$. As a consequence of (1.7) we can now state that $a_6(n) = 0$ if and only if $a_2(n) = 0$, i.e. if and only n is not the sum of two squares.

Remarks 1.3. (a) The sequence $a_6(n)$ is Sequence A258210 in [7]. The sequence $a_6(3n+1)$ is probably the opposite of Sequence A258277 in *loc. cit.*

(b) It can be seen from Table 1 that $a_6(n)$ is not a multiplicative function. Indeed, $a_6(10) \neq a_6(2)a_6(5)$ or $a_6(18) \neq a_6(2)a_6(9)$ or $a_6(20) \neq a_6(4)a_6(5)$.

TABLE 1. *First values of* $a_6(n)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$a_6(n)$	-1	-2	0	1	4	0	0	-2	-4	2	0	0	-2	0	0	1	4	4	0	-4

Theorems 1.1 and 1.2 will be proved in the next two sections. In Section 4 we explain how to obtain an elementary proof of an identity which Victor Kac [4] obtained using his theory of contragredient Lie superalgebras.

2. PROOF OF THEOREM 1.1

2.1. For any odd integer m we set $\xi(m) = -2\sin(m\pi/6)$. Because of the well-known properties of the sine function, $\xi(m)$ depends only on the class of m modulo 12 and we have the following equalities for all odd m:

$$\xi(-m) = -\xi(m) \quad \text{and} \quad \xi(m+6) = -\xi(m), \tag{2.1}$$

which is equivalent to $\xi(-m) = -\xi(m)$ and $\xi(6-m) = \xi(m)$.

We have

$$\xi(m) = \begin{cases} -1 & \text{if } m \equiv 1 \text{ or } 5 \pmod{12}, \\ -2 & \text{if } m \equiv 3 \pmod{12}, \\ 1 & \text{if } m \equiv 7 \text{ or } 11 \pmod{12}, \\ 2 & \text{if } m \equiv 9 \pmod{12}. \end{cases} \tag{2.2}$$

Next consider the excess function $E_1(n;4)$ defined by

$$E_1(n;4) = \sum_{d|n,\, d\equiv 1 \,(\mathrm{mod}\ 4)} 1 - \sum_{d|n,\, d\equiv -1 \,(\mathrm{mod}\ 4)} 1.$$

It is a multiplicative function, i.e. $E_1(mn;4) = E_1(m;4)\,E_1(n;4)$ whenever m and n are coprime. It is well known that the excess function can be computed in terms of the prime decomposition of n. Write $n = 2^c p_1^{a_1} p_2^{a_2} \cdots q_1^{b_1} q_2^{b_2} \cdots$, where all p_i, q_i are distinct prime numbers such that $p_i \equiv 1 \pmod 4$ and $q_i \equiv 3 \pmod 4$ for all $i \geqslant 1$. Then $E_1(n;4) = 0$ if and only if one of the exponents b_i is odd. If all b_i are even, then

$$E_1(n;4) = (1+a_1)(1+a_2)\cdots\,. \tag{2.3}$$

In the sequel we will need the following result.

Lemma 2.1. *Let n be a positive integer which is not divisible by* 3. *We have*

$$\sum_{d|n,\, d \text{ odd}} \xi(d) = -E_1(n;4) \quad \textit{and} \quad \sum_{d|n,\, d \text{ odd}} \xi(3d) = -2E_1(n;4).$$

Proof. Let d be an odd divisor of n; it is not divisible by 3 since n is not. Therefore, $d \equiv 1, 5, 7$ or $11 \pmod{12}$. Observe that $d \equiv 1$ or $5 \pmod{12}$ if and only if $d \equiv 1 \pmod 4$ since $d \equiv 3 \pmod{12}$ is excluded. Similarly, $d \equiv 7$ or $11 \pmod{12}$ if and only if $d \equiv 3 \pmod 4$. Now, $\xi(d) = -1$ if $d \equiv 1$ or 5, and $\xi(d) = 1$ if $d \equiv 7$ or $11 \pmod{12}$. Consequently,

$$\sum_{d|n\,,\, d \text{ odd}} \xi(d) = \sum_{d|n\,,\, d\equiv 3 \pmod 4} 1 - \sum_{d|n\,,\, d\equiv 1 \pmod 4} 1 = -E_1(n;4).$$

Similarly, $\xi(3d) = \xi(3) = -2$ if $d \equiv 1$ or 5, and $\xi(3d) = \xi(9) = 2$ if $d \equiv 7$ or $11 \pmod{12}$. Therefore,

$$\sum_{d|n\,,\, d \text{ odd}} \xi(3d) = \sum_{d|n\,,\, d\equiv 3 \pmod 4} 2 - \sum_{d|n\,,\, d\equiv 1 \pmod 4} -2 = -2E_1(n;4).$$

□

2.2. We now express $a_6(n)$ in terms of the function ξ introduced above.

Proposition 2.2. *We have*

$$a_6(n) = \sum_{d|n\,,\, d \text{ odd}} \xi\left(\frac{2n}{d} - d\right). \tag{2.4}$$

Note that $2n/d - d$ is an odd integer since d is an odd divisor of n.

Proof. Set $u = \pi/k$ and $\omega = d$ in Formula (9.3) of [2, p. 10]. It becomes

$$\sum_{n\geqslant 0} a_k(n)\, q^n = 1 - 4\sin(\pi/k) \sum_{n\geqslant 1} \left(\sum_{d|n\,,\, d \text{ odd}} \sin\left(\left(\frac{2n}{d} - d\right)\frac{\pi}{k}\right)\right) q^n. \tag{2.5}$$

Consider the special case $k = 6$ of (2.5). Since $\sin(\pi/6) = 1/2$, Equality (2.5) becomes

$$\begin{aligned}\sum_{n\geqslant 0} a_6(n)\, q^n &= 1 - 2\sum_{n\geqslant 1}\left(\sum_{d|n\,,\, d \text{ odd}} \sin\left(\left(\frac{2n}{d} - d\right)\frac{\pi}{6}\right)\right) q^n \\ &= 1 + \sum_{n\geqslant 1}\left(\sum_{d|n\,,\, d \text{ odd}} \xi\left(\frac{2n}{d} - d\right)\right) q^n\end{aligned}$$

in view of the definition of ξ. The formula for $a_6(n)$ follows. □

Proof of Theorem 1.1. Let us first mention the following well-known fact (see [1, § 51, Th. 65]): the number $r(n)$ of representations of n as a sum of two squares is related to the excess function $E_1(n;4)$ by

$$r(n) = 4\,E_1(n;4) \tag{2.6}$$

for all $n \geqslant 0$. It follows from this fact and from (1.5) that

$$a_2(n) = (-1)^n 4\,E_1(n;4). \tag{2.7}$$

We now distinguish three cases according to the residue of n modulo 3.

(a) We start with the case $n \equiv 1 \pmod 3$. We have $n = 3\ell + 1$ for some non-negative integer ℓ. Since the odd divisors d of n are not divisible by 3, they must satisfy $d \equiv 1, 5, 7$ or $11 \pmod{12}$. Such divisors are invertible $\pmod{12}$ and we have $d^2 \equiv 1 \pmod{12}$. Consequently,

$$\frac{2n}{d} - d \equiv \frac{2nd^2}{d} - d \equiv 2nd - d \pmod{12}.$$

Hence,

$$\xi\left(\frac{2n}{d} - d\right) = \xi(2nd - d) = \xi(6d\ell + d) = ((-1)^d)^\ell \xi(d) = (-1)^\ell \xi(d)$$

in view of (2.1). Therefore, by Proposition 2.2,

$$a_6(n) = (-1)^\ell \sum_{d|n,\, d \text{ odd}} \xi(d).$$

Together with Lemma 2.1 and (2.7), this implies

$$a_6(n) = (-1)^{\ell+1} E_1(n; 4) = (-1)^{n+\ell+1} a_2(n)/4.$$

Finally observe that n is odd (resp. even) if ℓ is even (resp. odd). Therefore, $a_6(n) = a_2(n)/4$.

(b) Now consider the case $n \equiv 2 \pmod{3}$. We have $n = 3\ell + 2$ for some non-negative integer ℓ. Again the odd divisors d of n must satisfy $d \equiv 1, 5, 7$ or $11 \pmod{12}$ since they are not divisible by 3. Consequently, as above,

$$\xi\left(\frac{2n}{d} - d\right) = \xi(2nd - d) = \xi(6d\ell + 3d) = (-1)^\ell \xi(3d).$$

By Lemma 2.1 and (2.7), we obtain

$$\begin{aligned} a_6(n) &= (-1)^\ell \sum_{d|n,\, d \text{ odd}} \xi(3d) \\ &= (-1)^{\ell+1} 2E_1(n; 4) = (-1)^{n+\ell+1} a_2(n)/2. \end{aligned}$$

Since n and ℓ are of the same parity, we have $a_6(n) = -a_2(n)/2$.

(c) Finally we consider the case when n is divisible by 3. We write $n = 3^N t$, where $N \geqslant 1$ and t is not divisible by 3. Any odd divisor d of n is of the form $d = 3^r s$ for some odd divisor s of t and $0 \leqslant r \leqslant N$. Since t and its divisors s are not divisible by 3 and since s is odd, we again have $s \equiv 1, 5, 7$ or $11 \pmod{12}$. Recall that for such s we have $s^2 \equiv 1 \pmod{12}$. Thus, for $d = 3^r s$, we obtain

$$\frac{2n}{d} - d \equiv \left(2 \cdot 3^{N-r} t - 3^r\right) s \pmod{12}.$$

If $r = 0$, then $2n/d - d \equiv (6 \cdot 3^{N-1} t - 1)s \pmod{12}$. Therefore,

$$\xi\left(\frac{2n}{d} - d\right) = \xi\left((6 \cdot 3^{N-1} t - 1)s\right) = (-1)^t \xi(-s) = (-1)^{t-1} \xi(s).$$

in view of (2.1).

If $0 < r < N$, then $2n/d - d \equiv (6 \cdot 3^{N-r-1} t - 3^r)s \pmod{12}$. Therefore,

$$\xi\left(\frac{2n}{d} - d\right) = \xi\left((6 \cdot 3^{N-r-1} t - 3^r)s\right) = (-1)^t \xi(-3^r s) = (-1)^{t-1} \xi(3^r s).$$

Now, $3^r \equiv 3 \pmod{12}$ if r is odd, and $3^r \equiv -3$ if $r > 0$ is even. Then by (2.1),

$$\xi\left(\frac{2n}{d} - d\right) = (-1)^{t-r} \xi(3s).$$

Now consider the case $r = N$. If N is odd, then $3^N \equiv 3 \pmod{12}$ and

$$\xi\left(\frac{2n}{d} - d\right) = \xi\left((2t - 3^N)s\right) = \xi((2t-3)s).$$

Now, if t is odd, then $t \equiv 1, 5, 7$ or $11 \pmod{12}$. We have $2t - 3 \equiv 7$ or $11 \pmod{12}$ and the multiplication by 7 or by 11 exchanges the sets $\{1, 5\}$ and $\{7, 11\}$. Since by (2.2) the function ξ takes opposite values on such sets, we have $\xi((2t-3)s) = -\xi(s)$. Consequently, $\xi(2n/d - d) = -\xi(s)$ when t is odd.

If t is even, then $t \equiv 2, 4, 8$ or $10 \pmod{12}$. Then $2t - 3 \equiv 1$ or $5 \pmod{12}$. The multiplication by 1 or by 5 preserves each set $\{1, 5\}$ and $\{7, 11\}$, so that by (2.2) we have $\xi((2t-3)s) = \xi(s)$. In conclusion,

$$\xi(2n/d - d) = (-1)^t\, \xi(s)$$

when $r = N$ is odd.

If $r = N$ is even, then $3^N \equiv -3 \pmod{12}$ and $\xi(2n/d - d) = \xi((2t - 3^N)s) = \xi((2t+3)s)$. A reasoning as in the odd N case shows that when N is even we have

$$\xi(2n/d - d) = (-1)^{t-1}\, \xi(s).$$

We can now compute $a_6(n)$. We start with the case of odd N. Collecting the above information, we obtain

$$\begin{aligned}
a_6(n) &= \sum_{d|n,\, d\,\mathrm{odd}} \xi\left(\frac{2n}{d} - d\right) = \sum_{s|t,\, s\,\mathrm{odd}} \sum_{r=0}^{N} \xi\left(\frac{2\cdot 3^N t}{3^r s} - d\right) \\
&= \sum_{s|t,\, s\,\mathrm{odd}} \left((-1)^{t-1}\xi(s) + \left(\sum_{r=1}^{N-1} (-1)^{t-r}\right) \xi(3s) + (-1)^t\, \xi(s) \right) \\
&= \left((-1)^{t-1} + (-1)^t\right) \sum_{s|t,\, s\,\mathrm{odd}} \xi(s) = 0.
\end{aligned}$$

On the other hand, since the power of 3 in n is odd, then by (2.3) we have $a_2(n) = (-1)^n 4\, E_1(n, 4) = 0$. Therefore, $a_6(n) = a_2(n)$ in this case.

If N is even, then

$$\begin{aligned}
a_6(n) &= \sum_{d|n,\, d\,\mathrm{odd}} \xi\left(\frac{2n}{d} - d\right) = \sum_{s|t,\, s\,\mathrm{odd}} \sum_{r=0}^{N} \xi\left(\frac{2\cdot 3^N t}{3^r s} - d\right) \\
&= \sum_{s|t,\, s\,\mathrm{odd}} \left((-1)^{t-1}\xi(s) + \left(\sum_{r=1}^{N-1} (-1)^{t-r}\right) \xi(3s) + (-1)^{t-1}\xi(s) \right) \\
&= \sum_{s|t,\, s\,\mathrm{odd}} \left(2(-1)^{t-1}\xi(s) + (-1)^{t-1}\xi(3s)\right) \\
&= (-1)^{t-1} \left(2 \sum_{s|t,\, s\,\mathrm{odd}} \xi(s) + \sum_{s|t,\, s\,\mathrm{odd}} \xi(3s) \right) \\
&= (-1)^t 4\, E_1(t; 4)
\end{aligned}$$

by Lemma 2.1. Now, by multiplicativity of the excess fonction,

$$E_1(n; 4) = E_1(3^N; 4)\, E_1(t; 4) = E_1(t; 4)$$

since $E_1(3^N;4) = 1$ for even N. Finally, t and n being of the same parity, we have

$$a_6(n) = (-1)^t\, 4\, E_1(t;4)) = (-1)^n\, 4\, E_1(n;4)) = a_2(n).$$

Q.e.d. □

3. Proof of Theorem 1.2

Set $f(q) = \eta(z)\eta(2z)\eta(3z)/\eta(6z) = \sum_{n\geqslant 0} a_6(n)\, q^n$ and $g(q) = \eta(z)^4/\eta(2z)^2 = \sum_{n\geqslant 0} a_2(n)\, q^n$; see (1.6). To prove Theorem 1.2 it suffices to check that

$$f(q) = ag(q) + bg(jq) + cg(j^2q),$$

where $a = 1/4$, $b = (1-j)/4$, and $c = (1-j^2)/4$. Now,

$$\begin{aligned} ag(q) + bg(jq) + cg(j^2q) &= a \sum_{n\geqslant 0} a_2(n)\, q^n + b \sum_{n\geqslant 0} a_2(n)\, j^n q^n \\ &\quad + c \sum_{n\geqslant 0} a_2(n)\, j^{2n} q^n \\ &= (a+b+c) \sum_{m\geqslant 0} a_2(3m)\, q^{3m} \\ &\quad + (a + jb + j^2c) \sum_{m\geqslant 0} a_2(3m+1)\, q^{3m+1} \\ &\quad + (a + j^2b + jc) \sum_{m\geqslant 0} a_2(3m+2)\, q^{3m+2}. \end{aligned}$$

It follows from (1.7) that

$$\begin{aligned} ag(q) + bg(jq) + cg(j^2q) &= (a+b+c) \sum_{m\geqslant 0} a_6(3m)\, q^{3m} \\ &\quad + 4(a + jb + j^2c) \sum_{m\geqslant 0} a_6(3m+1)\, q^{3m+1} \\ &\quad - 2(a + j^2b + jc) \sum_{m\geqslant 0} a_6(3m+2)\, q^{3m+2}. \end{aligned}$$

The right-hand side is equal to $f(q)$ since $a+b+c = 1$, $a + jb + j^2c = 1/4$, and $a + j^2b + jc = -1/2$. Q.e.d.

4. An elementary proof of an identity by Victor Kac

In [4, p. 122] Victor Kac derived four identities for η-products from his theory of contragredient Lie superalgebras. One of these identities, labelled (new$_4$) in *loc. cit.*, can be rephrased in the following form:

$$\frac{\eta^2(2z)\eta(3z)}{\eta(z)\eta(6z)} = \sum_{n\in\mathbb{Z}} (-1)^n f(n)\, q^{n^2},$$

where $f(3m) = 1$, $f(3m+1) = -1$, and $f(3m+2) = 0$. This immediately implies

$$\frac{\eta^2(2z)\eta(3z)}{\eta(z)\eta(6z)} = \sum_{n\geqslant 0} b(n)\, q^{n^2}, \tag{4.1}$$

where $b(0) = 1$, $b(3m) = 2\,(-1)^m = 2\,(-1)^{3m}$ for $m > 0$, and $b(n) = (-1)^{n-1}$ if n is not a multiple of 3. See also [6, Th. 8.2].

In a mail dated March 22, 2016 Günter Köhler observed that our η-product (1.1) can be written as the product of Kac's η-product (4.1) and the η-product $\eta(z)^2/\eta(2z)$ (the latter two being modular forms of weight $1/2$). Indeed,

$$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)} = \frac{\eta^2(2z)\eta(3z)}{\eta(z)\eta(6z)} \cdot \frac{\eta(z)^2}{\eta(2z)}. \tag{4.2}$$

Now Gauss proved (see [2, (7.324)] or [3, 19.9 (i)]) the following:

$$\frac{\eta(z)^2}{\eta(2z)} = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = \sum_{n\geqslant 0} a(n)\, q^{n^2}, \tag{4.3}$$

where $a(0) = 1$ and $a(n) = 2(-1)^n$ for all $n > 0$. Therefore, the expansion of the right-hand side of (4.2) is given by

$$\frac{\eta^2(2z)\eta(3z)}{\eta(z)\eta(6z)} \cdot \frac{\eta(z)^2}{\eta(2z)} = \sum_{n\geqslant 0} a'_6(n)\, q^n,$$

where

$$a'_6(n) = \sum_{\substack{x,y\geqslant 0 \\ x^2+y^2=n}} a(x)b(y). \tag{4.4}$$

Consequently, an alternative way to prove Theorem 1.1 is to establish the following lemma.

Lemma 4.1. *For all* $m \geqslant 0$,

$$\begin{aligned} a'_6(3m) &= (-1)^m\, r(3m), \\ a'_6(3m+1) &= (-1)^{m+1}\, \frac{r(3m+1)}{4}, \\ a'_6(3m+2) &= (-1)^{m+1}\, \frac{r(3m+2)}{2}. \end{aligned}$$

Conversely, since the right-hand side of (4.3) is an invertible formal power series, the lemma combined with Gauss's identity (4.3) and with our elementary proof of Theorem 1.1 yields an elementary proof of Kac's identity (4.1).

Proof of Lemma 4.1 (provided by G. Köhler). (a) Suppose first $n = 3m + 1$ for some integer $m \geqslant 0$. We consider solutions $x \geqslant 0$, $y \geqslant 0$ of $x^2 + y^2 = n$. Since $n \equiv 1 \pmod 3$, exactly one of the integers x, y is a multiple of 3. Therefore the solutions can be coupled in pairs $((x, y), (y, x))$, where 3 divides x, but not y (hence $y > 0$). If $x > 0$, then the contribution of such a pair to $a'_6(n)$ is

$$\begin{aligned} &2(-1)^x \cdot (-1)^{y-1} + 2(-1)^y \cdot 2(-1)^x \\ &\qquad = 2(-1)^{x+y} = 2(-1)^{x^2+y^2} = 2(-1)^n = 2(-1)^{m+1} \end{aligned}$$

and it is 8 for $r(n)$, since one has to consider all pairs $(\pm x, \pm y)$ and $(\pm y, \pm x)$, which are distinct. If $x = 0$ (which occurs only if n is a square), then the contribution is

$$(-1)^{y-1} + 2(-1)^y = (-1)^y = (-1)^{y^2} = (-1)^n = (-1)^{m+1}$$

for $a'_6(n)$ and it is 4 for $r(n)$ (corresponding to the pairs $(0, \pm y), (\pm y, 0)$, which are distinct). Summing up and comparing the contributions, we obtain $a'_6(n) = (-1)^{m+1} r(n)/4$, which is the desired formula.

(b) Now let $n = 3m + 2$ for some integer $m \geqslant 0$. We again consider solutions $x \geqslant 0, y \geqslant 0$ of $x^2 + y^2 = n$. Since $n \equiv 2 \pmod 3$, none of x, y is divisible by 3, and in particular $x > 0$ and $y > 0$. The contribution of (x, y) is

$$2(-1)^x \cdot (-1)^{y-1} = 2(-1)^{x+y-1} = 2(-1)^{x^2+y^2-1} = 2(-1)^{n-1} = 2(-1)^{m+1}$$

for $a'_6(n)$ and it is 4 for $r(n)$. Summing up and comparing the contributions, we obtain $a'_6(n) = (-1)^{m+1} r(n)/2$.

(c) Finally let $n = 3m$. For $n = 0$, the result is clear, so we may assume that $n > 0$. Write $n = 3^N t$, where $N \geqslant 1$ and t is not divisible by 3. If N is odd, then by the remark preceding (2.3) and by (2.6) we have $r(n) = 4E(n; 4) = 0$. Hence the sum (4.4) defining $a'_6(n)$ is empty, which implies $a'_6(n) = 0 = (-1)^m r(3m)$.

So let $N = 2s > 0$ be even. It is easy to check that the solutions of $x^2 + y^2 = n = 3^{2s} t$ are of the form $x = 3^s u$ and $y = 3^s v$, where $u^2 + v^2 = t$. If t is not a square, then there is no solution where $u = 0$ or $v = 0$, and we obtain

$$\begin{aligned} a'_6(n) &= \sum_{\substack{u,v \geqslant 0 \\ u^2+v^2=t}} 2(-1)^{3^s u} \cdot 2(-1)^{3^s v} = 4 \sum_{\substack{u,v \geqslant 0 \\ u^2+v^2=t}} (-1)^u (-1)^v \\ &= 4 \sum_{\substack{u,v \geqslant 0 \\ u^2+v^2=t}} (-1)^{u+v} = 4 \sum_{\substack{u,v \geqslant 0 \\ u^2+v^2=t}} (-1)^{u^2+v^2} \\ &= 4(-1)^t \sum_{\substack{u,v \geqslant 0 \\ u^2+v^2=t}} 1 \\ &= (-1)^t r(t) = (-1)^m r(n). \end{aligned}$$

If $t = w^2$ is a square, then we have the additional solutions $(u, v) = (w, 0)$ and $(u, v) = (0, w)$, hence $(x, y) = (3^s w, 0)$ and $(x, y) = (0, 3^s w)$. This yields an additional contribution of $2(-1)^{3^s w} + 2(-1)^{3^s w} = 4(-1)^{(3^s w)^2} = 4(-1)^n$ for $a'_6(n)$, and for $r(n)$ it is 4; thus we have proved $a'_6(n) = (-1)^n r(n) = (-1)^m r(3m)$. □

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