A GEOMETRY WHERE EVERYTHING IS BETTER THAN NICE

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Abstract. We present a geometry in the disk whose metric truth is curiously arithmetic.

1. The better-than-nice metric

Consider the metric

$$g = \frac{4}{1-r^2}(dx^2+dy^2)$$

in the unit disk. Here $r^2 = x^2+y^2$ as usual. Everything of interest can be computed explicitly, and with surprising results.

1.1. **Hypocycloids in the disk.** Consider the curve

$$c(t) = (1-a)e^{i\theta(t)} + ae^{-i\phi(t)}, \qquad 0 < a < 1,$$

thought of as a point on a circle of radius a turning at a rate $\dot{\phi}$ in the clockwise direction as the centre of the circle rotates on a circle of radius $1-a$ rotating at a rate $\dot{\theta}$ in the counterclockwise direction.

For the small circle of radius a to roll without slipping on the inside of the circle of radius 1 requires the point $c(t)$ to have velocity 0 when $|c(t)| = 1$, which is the relation

$$a\dot{\phi} = (1-a)\dot{\theta}.$$

Because g is rotationally invariant, without loss of generality we may take

$$\theta(t) = \frac{at}{2\sqrt{a(1-a)}}, \qquad \phi(t) = \frac{(1-a)t}{2\sqrt{a(1-a)}}.$$

Theorem 1.1. *The curve $c(t)$ is a geodesic for the metric g, parameterized by arclength.*

Proof. For our g, the equations $\ddot{u}^i + \Gamma^i_{jk}\dot{u}^j\dot{u}^k = 0$ determining geodesics $u(t) = (x(t), y(t))$ parameterized proportional to arclength are

$$(1-x^2-y^2)\ddot{x} + x(\dot{x}^2-\dot{y}^2) + 2y\dot{x}\dot{y} = 0 \quad \text{and}$$
$$(1-x^2-y^2)\ddot{y} - y(\dot{x}^2-\dot{y}^2) + 2x\dot{x}\dot{y} = 0,$$

which can be expressed in terms of $z = x+iy$, as the single equation

$$(1-z\bar{z})\ddot{z} + \bar{z}\dot{z}^2 = 0. \tag{1}$$

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Given the formula for $c(t)$ it is straightforward to verify that $c(t)$ satisfies (1) and moreover that $|\dot{c}| = \sqrt{1-|c|^2}/2$, from which it follows that $\|\dot{c}\|_g = 1$, meaning that the parameterization is by arclength. q.e.d.

Theorem 1.2. *The closed geodesics (i.e. keep rolling the generating circle of the hypocycloid until it closes up) have length $4\pi\sqrt{n}$, and the number of geometrically distinct geodesics of length $4\pi\sqrt{n}$ is given by the arithmetic function $\psi(n)$.*

The function $\psi(n)$ counts the number of different ways that the integer n may be written as a product $n = pq$, with $p \leq q$, $(p, q) = 1$. Values of this function are tabulated in sequence $A007875$ in the online encyclopedia of integer sequences [1].

Proof. Beginning at $c(0) = 1$, the geodesic $c(t)$ first returns to the boundary circle at

$$c\left(4\pi\sqrt{a(1-a)}\right) = e^{2\pi i(1-a)},$$

returning again at points of the form $e^{2\pi i m(1-a)}$ $(m \in \mathbf{Z}_+)$. The corresponding succession of cycloidal geodesic arcs winds clockwise around the origin if $0 < a < 1/2$ and counterclockwise if $1/2 < a < 1$; when $a = 1/2$, $c(t)$ traverses back and forth along the x-axis. Thus $c(t)$ forms a once-covered closed geodesic precisely when $2\pi m(1-a) = q2\pi$ for some relatively prime pair of positive integers $q < m$, in which case the geodesic has length

$$4\pi m\sqrt{a(1-a)} = 4\pi\sqrt{(m-q)q}.$$

To count geometrically distinct closed geodesics we restrict to $0 < a \leq 1/2$, in which case $p := m - q = ma \leq m(1-a) = q$. Given any relatively prime positive integers $p \leq q$ the geodesic of length $4\pi\sqrt{pq}$ occurs when $a = p/(p+q)$. q.e.d.

1.2. **Eigenfunctions and eigenvalues of the Laplacian.** Set the Laplacian Δ to be

$$\Delta = -g^{ab}\nabla_a\nabla_b$$

where ∇_a is the covariant derivative operator associated to the metric g via the Levi-Civita connection. Consider the eigenvalue problem

$$\Delta u = \lambda u$$

for functions u with the boundary value $u(r = 1) = 0$. This problem has a number of remarkable features.

Theorem 1.3. *The eigenfunctions and eigenvalues satisfy*

(1) *The eigenvalues λ_n are precisely the positive integers $n = 1, 2, 3, \ldots$.*
(2) *The eigenfunctions are polynomials.*
(3) *The dimension of the eigenspace for eigenvalue n is the number of divisors of n. (The number of divisors function is denoted by $\tau(n)$.)*

Proof. Since the operator

$$-\Delta + \lambda = \frac{1-r^2}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \lambda$$

is analytic hypoelliptic, distributional solutions to the eigenvalue equation $\Delta u = \lambda u$ are necessarily real analytic, and representable near $(x, y) = (0, 0)$ by absolutely convergent Fourier series

$$u(r, \theta) = \sum_{n\in\mathbf{Z}} a_n(r)e^{in\theta}.$$

Observing that $(-\Delta + \lambda)(a_n(r)e^{in\theta})$ has the form $A_n(r)e^{in\theta}$ for some A_n, it follows that u is an eigenfunction of Δ only if each summand $a_n(r)e^{in\theta}$ is, so it suffices to consider just products of the form

$$u(r, \theta) = f(r)e^{in\theta}.$$

Expressing $-\Delta + \lambda$ in polar coordinates yields

$$(-\Delta + \lambda)(f(r)e^{in\theta}) = \frac{1 - r^2}{4}\left(f''(r) + \frac{1}{r}f'(r) + \left(\frac{4\lambda}{1 - r^2} - \frac{n^2}{r^2}\right)f(r)\right)e^{in\theta}.$$

Therefore $u(r, \theta) = f(r)e^{in\theta}$ is an eigenfunction of Δ only if f satisfies

$$r^2(1 - r^2)f'' + r(1 - r^2)f' + (4\lambda r^2 - n^2(1 - r^2))f = 0.$$

Assume for definiteness that $n \geq 0$ and write $f(r) = r^n g(r^2)$, so that g satisfies the hypergeometric equation

$$r(1 - r)g'' + (c - (a + b + 1)r)g' - ab\, g = 0,$$

where

$$a = \left(n + \sqrt{n^2 + 4\lambda}\right)/2, \quad b = \left(n - \sqrt{n^2 + 4\lambda}\right)/2, \text{ and } c = n + 1.$$

This has a unique non-singular solution (up to scalar multiplication), the hypergeometric function

$$g(r) = {}_2F_1(a, b; n + 1; r) \quad \text{where} \quad g(1) = \frac{n!}{\Gamma(1 + a)\Gamma(1 + b)},$$

which is zero at $r = 1$ if and only if $b = -m$ for some integer $m \geq 1$. This implies $\lambda = m(m + n)$ is a positive integer, proving part (1) of the theorem. If $n > 0$ there are two corresponding eigenfunctions $u(r, \theta) = r^n g(r^2)e^{\pm in\theta}$, making a total of $\tau(\lambda)$ eigenfunctions for each positive integer eigenvalue λ. That these are linearly independent (part (3)), and that the eigenfunctions are polynomials (part (2)) can be verified using an explicit formula for the eigenfunctions, as follows.

Using the formulation $\Delta = -(1-z\bar{z})\frac{\partial^2}{\partial\bar{z}\partial z}$, where $z = x+iy$, one can check directly that the Rodrigues-type formula

$$u^{(p,q)}(z) = \frac{(-1)^p}{q(p + q - 1)!}(1 - z\bar{z})\frac{\partial^{p+q}}{\partial\bar{z}^p\partial z^q}(1 - z\bar{z})^{p+q-1} \tag{2}$$

represents eigenfunctions corresponding to $\lambda = pq$, i.e., $\Delta u^{(p,q)} = pq\, u^{(p,q)}$. q.e.d.

1.3. **Two corollaries.**

Corollary 1.4. *The spectral function is precisely the square of the Riemann zeta function*

$$\sum_n \frac{1}{(\lambda_n)^s} = \sum_n \frac{\tau(n)}{n^s} = (\zeta(s))^2.$$

In a more applied vein, consider a unit radius circular membrane fixed at the boundary (i.e. a drumhead) having constant tensile force per unit length S and radially varying density $\rho(r) = 4S/(1-r^2)$. Small transverse displacements of the membrane $w(x, y, t)$ are governed by the equation

$$w_{tt} + \Delta w = 0, \tag{3}$$

so squared eigenfrequencies of the membrane correspond to eigenvalues of Δ.

Corollary 1.5. *Unlike the standard vibrating membrane whose eigenfrequencies are proportional to zeros of J_0, for each eigenfrequency $\omega_n = \sqrt{n}$ of (3), all the higher harmonics $m\omega_n$ $(2 < m \in \mathbf{Z}_+)$ are also eigenfrequencies.*

1.4. **Acoustic imaging and combinatorics.** Supplemented by the sequence of monomials $u^{(p,0)}(z) := z^p$ for $p \geq 0$, the eigenfunctions of Δ are the special functions suited to acoustic imaging of layered media. A layered medium consists of a stack of $n-1$ horizontal slabs between two semi-infinite half spaces, where each slab and half space has constant acoustic impedance. Thus impedance as a function of depth is a step function having jumps at the n interfaces. To image the layers, an impulsive horizontal plane wave is transmitted at time $t = 0$ from a reference plane in the upper half space down toward the stack of horizontal slabs, and the resulting echoes are recorded at the reference plane, producing a function $G(t)$ $(t > 0)$ (the boundary Green's function of the medium). Let L_1 denote the time required for acoustic waves to travel from the reference plane to the first interface and back, with $L_2, \ldots, L_n$ denoting two-way travel time within each successive layer. When a wave travels downward toward the jth interface, it is partly reflected back, with amplitude factor R_j, and partly transmitted, with transmission factor $1 - R_j$ $(1 \leq j \leq n)$.

Theorem 1.6.

$$G(t) = \sum_{k \in \{1\} \times \mathbf{Z}_+^{n-1}} \left(\prod_{j=1}^{n} u^{(k_j, k_{j+1})}(R_j) \right) \delta(t - \langle L, k \rangle).$$

Here $L = (L_1, \ldots, L_n)$, $u^{(0,q)} \equiv 0$ if $q \geq 1$ and $k_j = 0$ if $j > n$.

Proof. Expanding the binomial $(1 - z\bar{z})^{p+q-1}$ in the formula (2), and then applying the derivative $\partial^{p+q}/\partial\bar{z}^p\partial z^q$, yields

$$u^{(p,q)}(z) = \frac{(-1)^{q+\nu+1}}{q}(1 - z\bar{z})z^{m+\nu-q+1}\bar{z}^{m+\nu-p+1} \sum_{j=0}^{\nu} (-1)^j \frac{(j+\nu+m+1)!}{j!(j+m)!(\nu-j)!}(z\bar{z})^j,$$

where $m = |p-q|$ and $\nu = \min\{p,q\}-1$. Switching to polar form $z = re^{i\theta}$, it follows that

$$u^{(p,q)}(re^{i\theta}) = e^{i(p-q)\theta}\frac{(-1)^{q+\nu+1}}{q}(1-r^2)r^m\sum_{j=0}^{\nu}(-1)^j\frac{(j+\nu+m+1)!}{j!(j+m)!(\nu-j)!}r^{2j}.$$

For $\theta = 0, \pi$, the latter coincide with the functions $f^{(p,q)}$ occurring in [2, Thm. 2.4, 4.3]. q.e.d.

Each term $\left(\prod_{j=1}^n u^{(k_j,k_{j+1})}(R_j)\right)\delta(t-\langle L,k\rangle)$ corresponds to the set of all scattering sequences that have a common arrival time $t_i = \langle L,k\rangle$, with each individual scattering sequence weighted according to the corresponding succession of reflections and transmissions. A scattering sequence may be represented by a Dyck path as in Figure 1. The tensor products $\prod_{j=1}^n u^{(k_j,k_{j+1})}(R_j)$ thus have a combinatorial interpretation in that they count weighted Dyck paths having $2k_j$ edges at height j. See [2].

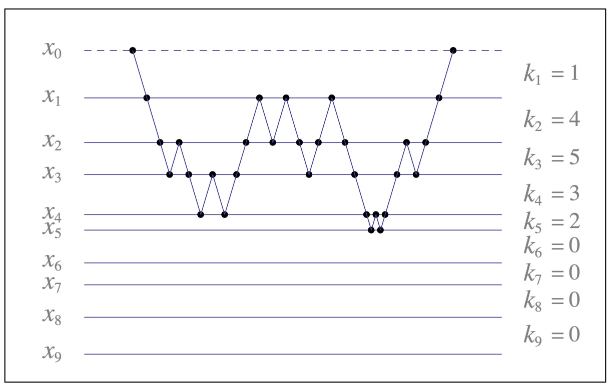

FIGURE 1. The Dyck path for a scattering sequence. Time increases to the right. Each node at depth x_j receives a weight according to the structure of the path at the node: weights $R_j, -R_j, 1-R_j, 1+R_j$ correspond respectively to down-up reflection, up-down reflection, downward transmission, upward transmission. The scattering sequence returns to the reference depth x_0 at time $t_i = \langle L,k\rangle$, where $k = (k_1,\dots,k_n)$. Its amplitude is the product of the weights.

REFERENCES

[1] *The On-Line Encyclopedia of Integer Sequences*, published electronically at https://oeis.org, 2016, Sequence A007875.

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