The log-convexity of the poly-Cauchy numbers

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Abstract

In 2013, Komatsu introduced the poly-Cauchy numbers, which generalize Cauchy numbers. Several generalizations of poly-Cauchy numbers have been considered since then. One particular type of generalizations is that of multiparameter-poly-Cauchy numbers. In this paper, we study the log-convexity of the multiparameter-poly-Cauchy numbers of the first kind and of the second kind. In addition, we also discuss the log-behavior of multiparameter-poly-Cauchy numbers.

Key words: Cauchy numbers, Poly-Cauchy numbers, multiparameter-poly-Cauchy numbers, log-convexity, log-concavity.

2000 Mathematics Subject Classification: 05A19, 05A20, 11B83.

1 Introduction

Komatsu [10], introduced two kinds of poly-Cauchy numbers $c_n^{(k)}$ and $\widehat{c}_n^{(k)}$. The first kind $c_n^{(k)}$ is given by

$$c_n^{(k)} = \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 x_2 \cdots x_k)_n dx_1 dx_2 \cdots dx_k,$$

and the second kind $\widehat{c}_n^{(k)}$ is given by

$$\widehat{c}_n^{(k)} = (-1)^n \underbrace{\int_0^1 \cdots \int_0^1}_{k} \langle x_1 x_2 \cdots x_k \rangle_n dx_1 dx_2 \cdots dx_k,$$

*The first author was supported in part by the grant of Wuhan University and by the grant of Hubei Provincial Experts Program.

†The second author was supported by a grant of "The First-class Discipline of Universities in Shanghai".

where k is a positive integer, and $(x)_n = x(x-1)\cdots(x-n+1$ $(n \ge 1)$ with $(x)_0 = 1$ and $\langle x\rangle_n = x(x+1)\cdots(x+n-1)$ with $\langle x\rangle_0 = 1$. The first few values of $c_n^{(k)}$ and $\widehat{c}_n^{(k)}$ are

$$c_0^{(k)} = 1, c_1^{(k)} = \frac{1}{2^k}, c_2^{(k)} = -\frac{1}{2^k} + \frac{1}{3^k}, c_3^{(k)} = \frac{2}{2^k} - \frac{3}{3^k} + \frac{1}{4^k}, c_4^{(k)} = -\frac{6}{2^k} + \frac{11}{3^k} - \frac{6}{4^k} + \frac{1}{5^k}, \cdots$$
$$\widehat{c}_0^{(k)} = 1, \widehat{c}_1^{(k)} = -\frac{1}{2^k}, \widehat{c}_2^{(k)} = \frac{1}{2^k} + \frac{1}{3^k}, \widehat{c}_3^{(k)} = -\frac{2}{2^k} - \frac{3}{3^k} - \frac{1}{4^k}, \widehat{c}_4^{(k)} = \frac{6}{2^k} + \frac{11}{3^k} + \frac{6}{4^k} + \frac{1}{5^k}, \cdots$$

Some sequences derived from denominators and numerators of poly-Cauchy numbers of the first kind and of the second kind can be found in [16, A224094–A224101,A219247,A224102–A224107,A224109]. When $k = 1$, $c_n = c_n^{(1)}$ and $\widehat{c}_n = \widehat{c}_n^{(1)}$ are the Cauchy numbers of the first kind and the second kind, respectively (see [3]). The basic properties of the two kinds of the poly-Cauchy numbers are studied in [9, 10]. Several generalizations of poly-Cauchy numbers have been considered since then. One particular type of generalizations is that of the multiparameter-poly-Cauchy numbers ([12]). For a k-tuple of real numbers $L = (l_1, \dots, l_k)$ and a n-tuple of real numbers $A = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$, define the q-multiparameter-poly-Cauchy polynomials of the first kind $c_{n,L,A,q}^{(k)}(z)$ by

$$c_{n,L,A,q}^{(k)}(z) = \int_0^{l_1} \cdots \int_0^{l_k} (x_1 \cdots x_k - \alpha_0 - z) \cdots (x_1 \cdots x_k - \alpha_{n-1} - z) d_q x_1 \cdots d_q x_k\,.$$

Here, Jackson's q-integral is defined by

$$\int_0^x f(t) d_q t = (1-q)x \sum_{n=0}^\infty f(q^n x) q^n\,,$$

and

$$[x]_q = \frac{1-q^x}{1-q} \to x \quad (q \to 1)\,.$$

The q-multiparameter-poly-Cauchy polynomials of the first kind can be expressed explicitly in terms of the multiparameter Stirling numbers of the first kind $S_1(n, m, A)$, defined by

$$(t-\alpha_0)(t-\alpha_1)\cdots(t-\alpha_{n-1}) = \sum_{m=0}^n S_1(n,m,A) t^m\,.$$

Namely, we have

$$c_{n,L,A,q}^{(k)}(z) = \sum_{m=0}^n S_1(n,m,A) \sum_{i=0}^m \binom{m}{i} \frac{(-z)^i (l_1 \cdots l_k)^{m-i+1}}{[m-i+1]_q^k}$$

([12, Theorem 1]).

When $q \to 1$, $l_1 = \cdots = l_k = 1$ and $z = 0$, we have

$$c_{n,A}^{(k)} = \lim_{q \to 1} c_{n,(1,\dots,1),A,q}^{(k)}(0) = \sum_{m=0}^n \frac{S_1(n,m,A)}{(m+1)^k}\,.$$

Furthermore, if $A=(0,1,\dots,n-1)$, then $S_1(n,m)=(-1)^{n-m}S_1(n,m,A)$ are the unsigned Stirling numbers of the first kind and $c_n^{(k)}=c_{n,(0,1,\dots,n-1)}^{(k)}$ are poly-Cauchy numbers of the first kind.

Similarly, define the q-multiparameter-poly-Cauchy polynomials of the second kind $\widehat{c}_{n,L,A,q}^{(k)}(z)$ by

$$\widehat{c}_{n,L,A,q}^{(k)}(z)=\int_0^{l_1}\cdots\int_0^{l_k}(-x_1\cdots x_k-\alpha_0+z)\cdots(-x_1\cdots x_k-\alpha_{n-1}+z)d_qx_1\cdots d_qx_k\,.$$

When $q\to 1$, $l_1=\cdots=l_k=1$, $\alpha_i=i\rho$ $(i=0,1,\dots,n-1)$ and $z=0$, the number $\widehat{c}_{n,A}^{(k)}$ is reduced to the poly-Cauchy numbers of the second kind with a parameter ρ ([11]). Furthermore, if $\rho=1$, then the number $\widehat{c}_{n,A}^{(k)}$ is reduced to the poly-Cauchy numbers of the second kind $\widehat{c}_n^{(k)}$ ([10]). If $k=1$, then $\widehat{c}_n^{(1)}=\widehat{c}_n$ is the classical Cauchy number ([3]). The q-multiparameter-poly-Cauchy polynomials of the second kind can be also expressed explicitly in terms of the multiparameter Stirling numbers of the first kind as follows.

$$\widehat{c}_{n,L,A,q}^{(k)}(z)=\sum_{m=0}^n(-1)^mS_1(n,m,A)\sum_{i=0}^m\binom{m}{i}\frac{(-z)^i(l_1\cdots l_k)^{m-i+1}}{[m-i+1]_q^k}$$

([12, Theorem 2]). When $q\to 1$, $l_1=\cdots=l_k=1$ and $z=0$, we have

$$\widehat{c}_{n,A}^{(k)}=\lim_{q\to 1}\widehat{c}_{n,(1,\dots,1),A,q}^{(k)}(0)=\sum_{m=0}^n\frac{(-1)^mS_1(n,m,A)}{(m+1)^k}\,.$$

In [18], the log-convexity of Cauchy numbers of the first kind and of the second kind has been studied. We recall some other definitions and notations used in this paper.

Let $\{z_n\}_{n\ge 0}$ be a sequence of positive numbers. If for all $j\ge 1$, $z_j^2\le z_{j-1}z_{j+1}$ (respectively $z_j^2\ge z_{j-1}z_{j+1}$), the sequence $\{z_n\}_{n\ge 0}$ is called log-convex (respectively log-concave).

The log-behavior (log-convexity and log-concavity) are important properties of combinatorial sequences, and they play an important role in many subjects such as quantum physics, white noise theory, probability, economics and mathematical biology. See for instance [1, 2, 4, 5, 6, 8, 14, 15, 17].

If $z_0\le z_1\le\cdots\le z_{m-1}\le z_m\ge z_{m+1}\ge\cdots$ for some m, then $\{z_n\}_{n\ge 0}$ is called unimodal, and m is called a mode of the sequence.

In this paper, we study the log-convexity of multiparameter-poly-Cauchy numbers of the first kind and of the second kind. In addition, we also discuss the log-behavior of multiparameter-poly-Cauchy numbers.

2 The log-convexity of poly-Cauchy numbers

In this section, we mainly discuss the log-behavior of $\{c_n^{(k)}\}_{n\ge 2}$, $\{\widehat{c}_n^{(k)}\}_{n\ge 0}$, $\{c_{n,A}^{(k)}\}_{n\ge 2}$, and $\{\widehat{c}_{n,A}^{(k)}\}_{n\ge 0}$. For convenience, put

$$\sigma_n^{(k)}=(-1)^{n-1}c_n^{(k)},\quad \sigma_{n,A}^{(k)}=(-1)^{n-1}c_{n,A}^{(k)}\quad (n\ge 1),$$

and

$$\omega_n^{(k)} = (-1)^n \widehat{c}_n^{(k)}, \quad \omega_{n,A}^{(k)} = (-1)^n \widehat{c}_{n,A}^{(k)} \quad (n \ge 0).$$

In [18], the log-convexity of Cauchy numbers was discussed. First, we shall investigate the log-convexity of the poly-Cauchy numbers of the two kinds.

Lemma 2.1 *[13] If $\{y_n\}_{n\ge 0}$ is log-convex, then the Stirling transformation of the first kind $z_n = \sum_{m=0}^{n} \left[{n \atop m}\right] y_m$ preserves the log-convexity.*

Theorem 2.1 *The sequences $\{c_n^{(k)}\}_{n\ge 2}$ and $\{\widehat{c}_n^{(k)}\}_{n\ge 0}$ are log-convex.*

Proof. We first prove the log-convexity of $\{\sigma_n^{(k)}\}_{n\ge 2}$. For $n \ge 1$,

$$\begin{aligned}
&\left(c_n^{(k)}\right)^2 - c_{n-1}^{(k)} c_{n+1}^{(k)}\\
&= \Bigg[\underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-1-x_1 \cdots x_k) dx_1 \cdots dx_k\Bigg]^2\\
&\quad - \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-2-x_1 \cdots x_k) dx_1 d \cdots dx_k\\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-x_1 \cdots x_k) dx_1 \cdots dx_k.
\end{aligned}$$

For $0 \le x_j \le 1$ $(1 \le j \le k)$, $n-1 \le n - x_1 x_2 \cdots x_k \le n$. Then for $n \ge 3$,

$$\begin{aligned}
&\left(c_n^{(k)}\right)^2 - c_{n-1}^{(k)} c_{n+1}^{(k)}\\
&\le \Bigg[\underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-1-x_1 \cdots x_k) dx_1 \cdots dx_k\Bigg]^2\\
&\quad - \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-2-x_1 \cdots x_k) dx_1 d \cdots dx_k\\
&\quad \times (n-1) \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-1-x_1 \cdots x_k) dx_1 \cdots dx_k\\
&= - \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (1 - x_1 \cdots x_k) \cdots (n-1-x_1 \cdots x_k) dx_1 \cdots dx_k\\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k)^2 (1 - x_1 \cdots x_k) \cdots (n-2-x_1 \cdots x_k) dx_1 \cdots dx_k\\
&\le 0.
\end{aligned}$$

Hence, sequence $\{c_n^{(k)}\}_{n\geq 2}$ is log-convex.

Recall the definition

$$\omega_n^{(k)} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k}. \tag{2.1}$$

It is easy to see that the sequence $\left\{\frac{1}{(m+1)^k}\right\}_{m\geq 0}$ is log-convex. By means of Lemma 2.1, we get that the sequence $\{\omega_n^{(k)}\}_{n\geq 0}$ is log-convex. ■

Theorem 2.2 *For the sequence $\{\omega_n^{(k)}\}_{n\geq 3}$, we have*

$$\omega_n^{(k)} < \begin{bmatrix} n \\ K_n \end{bmatrix} \sum_{m=1}^{n} \frac{1}{(m+1)^k}, \tag{2.2}$$

where K_n is the index of the maximal unsigned Stirling numbers of the first kind $\left[{n \atop m}\right]$ for all fixed $n \geq 3$.

Proof. For the Stirling numbers of the first kind, we know that

$$\begin{bmatrix} n \\ 1 \end{bmatrix} < \begin{bmatrix} n \\ 2 \end{bmatrix} < \cdots < \begin{bmatrix} n \\ K_n - 1 \end{bmatrix} < \begin{bmatrix} n \\ K_n \end{bmatrix} > \begin{bmatrix} n \\ K_n + 1 \end{bmatrix} > \cdots > \begin{bmatrix} n \\ n \end{bmatrix}.$$

where $K_n \sim \frac{n}{\ln n}(n \to \infty)$ (see [7]). We note that

$$\begin{aligned} \begin{bmatrix} n \\ 0 \end{bmatrix} &= 0, \quad n \geq 1, \\ \begin{bmatrix} n \\ m \end{bmatrix} \frac{1}{(m+1)^k} &< \begin{bmatrix} n \\ K_n \end{bmatrix} \frac{1}{(m+1)^k}, \quad 1 \leq m \leq n \quad (m \neq K_n), \end{aligned}$$

By applying (2.1), we obtain (2.2). ■

We now consider the log-convexity and unimodality of multiparameter-poly-Cauchy numbers of two kinds under some conditions.

Theorem 2.3 *Assume that the sequence $A = (0, \alpha_1, \ldots, \alpha_n, \ldots)$ satisfies that $\alpha_j \geq 1$, and $\alpha_j - \alpha_{j-1} \geq 1$ for $j \geq 1$. The sequences $\{c_{n,A}^{(k)}\}_{n\geq 2}$ and $\{\widehat{c}_{n,A}^{(k)}\}_{n\geq 0}$ are log-convex.*

Proof. Since $\alpha_j \geq 1$ and $\alpha_j - \alpha_{j-1} \geq 1$ $(j \geq 1)$, for $n \geq 3$ we have

$$\begin{aligned}
&\left(c_{n,A}^{(k)}\right)^2 - c_{n-1,A}^{(k)} c_{n+1,A}^{(k)} \\
&= \Bigg[\underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-1} - x_1 \cdots x_k) dx_1 \cdots dx_k \Bigg]^2 \\
&\quad - \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-2} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_n - x_1 \cdots x_k) dx_1 \cdots dx_k, \\
&= - \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k)^2 (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-2} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-1} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\quad - \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-2} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-1} - x_1 \cdots x_k) \\
&\qquad\qquad \times (\alpha_n - \alpha_{n-1} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\leq - \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1 \cdots x_k)^2 (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-2} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (\alpha_1 - x_1 \cdots x_k) \cdots (\alpha_{n-1} - x_1 \cdots x_k) dx_1 \cdots dx_k \\
&\leq 0,
\end{aligned}$$

Similarly, for $n \geq 1$ we have

$$\begin{aligned}
&\left(\widehat{c}_{n,A}^{(k)}\right)^2 - \widehat{c}_{n-1,A}^{(k)}\widehat{c}_{n+1,A}^{(k)} \\
&= \Big[\underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1\cdots x_k(x_1\cdots x_k+\alpha_1)\cdots(x_1\cdots x_k+\alpha_{n-1})dx_1\cdots dx_k\Big]^2 \\
&\quad - \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1\cdots x_k(x_1\cdots x_k+\alpha_1)\cdots(x_1\cdots x_k+\alpha_{n-2})dx_1\cdots dx_k \\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1\cdots x_k(x_1\cdots x_k+\alpha_1)\cdots(x_1\cdots x_k+\alpha_{n})dx_1\cdots dx_k \\
&\leq - \underbrace{\int_0^1 \cdots \int_0^1}_{k} (x_1\cdots x_k)^2(x_1\cdots x_k+\alpha_1)\cdots(x_1\cdots x_k+\alpha_{n-1})dx_1\cdots dx_k \\
&\quad \times \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1\cdots x_k(x_1\cdots x_k+\alpha_1)\cdots(x_1\cdots x_k+\alpha_{n-2})dx_1\cdots dx_k, \\
&\leq 0.
\end{aligned}$$

Hence, the sequence $\{c_{n,A}^{(k)}\}_{n\geq 2}$ and $\{\widehat{c}_{n,A}^{(k)}\}_{n\geq 0}$ are log-convex. ■

Note that Theorem 2.3 generalizes Theorem 2.1. Clearly, Theorem 2.3 becomes Theorem 2.1 when $A = (0, 1, 2, \ldots, n, \ldots)$.

Theorem 2.4 *Suppose that the sequence $A = (0, \alpha_1, \ldots, \alpha_n, \ldots)$ satisfies that $\alpha_j \geq 0$ for $j \geq 1$. Then we have:*
(i) *if there exists $l \geq 3$ such that $\alpha_j \geq 2$ for $1 \leq j \leq l$ and $\alpha_j = 1$ for $j \geq l+1$, then $\{\sigma_{n,A}^{(k)}\}_{n\geq 1}$ is unimodal, and its single peak is at $l+1$;*
(ii) *if there exists $l \geq 3$ such that $\alpha_j \geq 1$ for $1 \leq j \leq l$ and $\alpha_j = 0$ for $j \geq l+1$, then $\{\omega_{n,A}^{(k)}\}_{n\geq 1}$ is unimodal, and its single peak is at $l+1$.*

Proof. (i) For $n \geq 1$,

$$\begin{aligned}
&\sigma_{n+1,A}^{(k)} - \sigma_{n,A}^{(k)} \\
&= \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1\cdots x_k(\alpha_1 - x_1\cdots x_k)\cdots(\alpha_{n-1} - x_1\cdots x_k)(\alpha_n - 1 - x_1\cdots x_k)dx_1\cdots dx_k.
\end{aligned}$$

We can verify that $\sigma_{n+1,A}^{(k)} - \sigma_{n,A}^{(k)} \geq 0$ for $1 \leq n \leq l$ and $\sigma_{n+1,A}^{(k)} - \sigma_{n,A}^{(k)} \leq 0$ for $n \geq l+1$. Hence, $\{\sigma_{n,A}^{(k)}\}_{n\geq 1}$ is unimodal, and its single peak is at $l+1$.

(ii) For $n \geq 1$,

$$\omega_{n+1,A}^{(k)} - \omega_{n,A}^{(k)}$$
$$= \underbrace{\int_0^1 \cdots \int_0^1}_{k} x_1 \cdots x_k (x_1 \cdots x_k + \alpha_1) \cdots (x_1 \cdots x_k + \alpha_{n-1})(x_1 \cdots x_k + \alpha_n - 1) dx_1 \cdots dx_k.$$

We can verify that $\omega_{n+1,A}^{(k)} - \omega_{n,A}^{(k)} \geq 0$ for $1 \leq n \leq l$ and $\omega_{n+1,A}^{(k)} - \omega_{n,A}^{(k)} \leq 0$ for $n \geq l+1$. Therefore, $\{\omega_{n,A}^{(k)}\}_{n\geq 1}$ is unimodal, and its single peak is at $l+1$. ∎

3 Acknowledgements

The authors would like to thank an anonymous referee whose helpful suggestions and comments have led to much improvement of the paper.

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