# NEW IDENTITIES FOR BINARY KRAWTCHOUK POLYNOMIALS, BINOMIAL COEFFICIENTS AND CATALAN NUMBERS 

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#### Abstract

Аbstract. We obtain new combinatorial identities for integral values of binary Krawtchouk polynomials $K_{p}^{2 m}(x), 0 \leq p \leq 2 m$, by computing the characters of the $p$-exterior representations on certain elements of order 2 of $\mathrm{SO}(2 m)$. From this identities, we deduce several new relations for binomial coefficients and Catalan numbers.


## 1. Introduction

For each $0 \leq p \leq n$, the binary Krawtchouk polynomial (BKP for short) of order $n$ and degree $p$ is defined by

$$
\begin{equation*}
K_{p}^{n}(x)=\sum_{j=0}^{p}(-1)^{j}\binom{x}{j}\binom{n-x}{p-j} \tag{1.1}
\end{equation*}
$$

where $\binom{x}{j}=x(x-1) \cdots(x-j+1) / j$ ! for $j \geq 1$ and $\binom{x}{0}=1$. Thus, by definition, $K_{p}^{n}(j) \in \mathbb{Z}$ for every integer $j$. One can easily check that

$$
\begin{equation*}
K_{p}^{n}(0)=\binom{n}{p}, \quad K_{p}^{n}(1)=\left(1-\frac{2 p}{n}\right)\binom{n}{p}, \quad K_{p}^{n}(n)=(-1)^{p}\binom{n}{p} \tag{1.2}
\end{equation*}
$$

and that we also have $K_{0}^{n}(x)=1$ and $K_{1}^{n}(x)=n-2 x$.
These polynomials form a discrete family $\left\{K_{p}^{n}(x)\right\}_{p=0}^{n}$ of orthogonal polynomials with respect to the binomial distribution. They satisfy several identities such as orthogonality, 3 -term recursions in the 3 variables, modularity properties, integral formulae, relations with other families of orthogonal polynomials, etc. See [14] for a survey on binary Krawtchouk polynomials and its properties (also [4] and [13]).

Binary Krawtchouk polynomials appear in several problems related with the abelian group $\mathbb{Z}_{2}^{k}$, for some $k$. The most commonly known examples of this are applications to combinatorial problems or to coding theory (see [13] for a survey). In combinatorics, BKP's appear in: (a) the existence or not of the inverse of the Radon transform on $\mathbb{Z}_{2}^{k}$ ([5]), (b) reconstruction problems on graphs (switching, reorientation, sign [25]) and (c) multiple perfect coverings of $\mathbb{Z}_{2}^{n}$ ([10], [27]). Also, in the context of binary codes, BKP's play a role in: (a) the existence or not of binary perfect codes ([26]), (b) alternative expressions for the MacWilliams identities relating the weight enumerator of the code with the corresponding enumerator of its dual ([11]) and (c) in some universal bounds for codes ([15]). Notably, in all of these

[^0]problems, the relevant question is the existence or not of integral zeros of the BKP's involved (see [4], [7], [8], [9], [13] for results related to integral zeros of BKP's).

Less known is the ubiquity of Krawtchouk polynomials in spectral geometry. In this setting, BKP's were used to study isospectrality problems for elliptic differential operators $D$ acting on $\mathbb{Z}_{2}^{k}$-manifolds (i.e. compact flat Riemannian manifolds having holonomy group isomorphic to $\mathbb{Z}_{2}^{k}$ ). Here, $D$ is either a Dirac-type operator (spin Dirac or signature operator, see [16], [17], [18], [19]) or a Laplacian (Hodge Laplacian, $p$-Laplacian, full Laplacian, see [19], [20], [21], [22]). Again, the existence of integral zeros play a key role in the results.

In brief, in the first two sections after the Introduction, we obtain new identities for integral values of binary Krawtchouk polynomials and in the subsequent sections we give applications of them to binomial coefficients and Catalan numbers.

An outline of the paper is as follows. In Section 2, we first recall some facts on the $p$-exterior representations of $\mathrm{SO}(2 m), 0 \leq p \leq 2 m$, and give an explicit expression for $\chi_{p}(x)$, the character values of these representations at elements $x$ of the maximal torus $T_{2 m}$ of $\mathrm{SO}(2 m)$. Then, by relating the $p$-characters $\chi_{p}(x)$ at elements $x$ of order 2 with BKP's, we obtain a new identity for binary Krawtchouk polynomials (see Theorem[2.2) of even order at even values and degree $p$, i.e. $K_{p}^{2 m}(2 j)$, in terms of the integral values $K_{k}^{m}(j)$ of polynomials of smaller degree $k$ (see also Corollary [2.4).

In Section 3, we generalize the reduction formula obtained in Theorem [2.2(see Theorem 3.1) giving rise to a whole new family of identities for BKP's of the form $K_{p}^{2^{r} m}\left(2^{s} j\right)$. We then exhibit several explicit computations illustrating the results.

In Section 4, by evaluating the expressions previously obtained for BKP's, we present recursive relations between binomial coefficients (see 4.1)-(4.3)). In particular, expressions for $\binom{2 m}{2 q},\binom{2 m}{2 q+1},\binom{2 m+1}{2 q}$ and $\binom{2 m+1}{2 q+1}$ in terms of $\binom{m}{q}$ and falling factorials $(q)_{0},(q)_{1}, \ldots,(q)_{q}$ are given in Theorem4.3. Also, for any $r, m, q \in \mathbb{N}_{0}$, we obtain the values for $\binom{2^{r} m}{2^{r} q}$ and $\binom{2^{r} m}{2^{r} q+1}$ modulo $2,4,8$ and 16 (see Proposition 4.7) and modulo some higher powers of 2 (see Propositions 4.11 and 4.12).

In the last two sections, we apply the results of Section 4 to study central binomial coefficients $c_{m}=\binom{2 m}{m}$ and Catalan numbers $C_{m}$. Explicit expressions and recursions for $c_{m}$ can be found in (5.1) - (5.5) and Proposition 5.1) while mixed expressions between $c_{m}$ 's and integral values of BKP's are given in Proposition 5.5, In Proposition6.1 we obtain new recursion formulas for Catalan numbers $C_{2 n}$ and $C_{2 n+1}$ in terms of $C_{0}, \ldots, C_{n}$. Finally, we give some congruence relations for $C_{m}$ modulo 2, 4, 8 and 16 .

## 2. A reduction formula for binary Krawtchouk polynomials

As we mentioned in the Introduction, certain properties of binary Krawtchouk polynomials lead to (spectral) geometrical results on $\mathbb{Z}_{2}^{k}$-manifolds. Here, in contrast, we will use a geometric result (characters of exterior representations) to obtain a relation between Krawtchouk polynomials.

Characters of p-exterior representations. Let $n=2 m$ with $m \in \mathbb{N}$. Consider the special orthogonal group $\mathrm{SO}(2 m)=\left\{A \in M_{2 m}(\mathbb{R}): A A^{t}=A^{t} A=I\right.$, $\left.\operatorname{det} A=1\right\}$ of $\mathbb{R}^{2 m}$. The maximal torus of $\mathrm{SO}(2 m)$ is $T_{2 m}=\left\{x\left(t_{1}, \ldots, t_{m}\right): t_{1}, \ldots, t_{m} \in \mathbb{R}\right\}$ where

$$
\begin{equation*}
x\left(t_{1}, \ldots, t_{m}\right)=\operatorname{diag}\left(B_{1}, \ldots, B_{m}\right), \tag{2.1}
\end{equation*}
$$

is the block diagonal matrix with blocks $B_{i}=\left(\begin{array}{c}\cos t_{i}-\sin t_{i} \\ \sin t_{i} \\ \cos t_{i}\end{array}\right)$ for $i=1, \ldots, m$.
For $0 \leq p \leq m$, let $\left(\tau_{p}, \Lambda^{p}\left(\mathbb{R}^{2 m}\right)_{\mathbb{C}}\right)$ be the $p$-exterior representation of $\mathrm{SO}(2 m)$. Each $\tau_{p}$ is irreducible for $0 \leq p \leq m-1$ and $\tau_{m}$ is the sum of 2 irreducible representations $\tau_{m}^{+}$and $\tau_{m}^{-}$given by the splitting $\bigwedge^{m}\left(\mathbb{R}^{2 m}\right)_{\mathbb{C}}=\Lambda_{+}^{m}\left(\mathbb{R}^{2 m}\right)_{\mathbb{C}} \oplus \bigwedge_{-}^{m}\left(\mathbb{R}^{2 m}\right)_{\mathbb{C}}$. Let $\chi_{p}$ and $\chi_{m}^{ \pm}$denote the character of $\tau_{p}$ and $\tau_{m}^{ \pm}$, respectively.

For $x \in T_{2 m}$ there are combinatorial expressions for $\chi_{p}(x), 0 \leq p \leq m$, that we now present (see Proposition 3.7 in [16]). If $I_{m}=\{1, \ldots, m\}$ and $x=x\left(t_{1}, \ldots, t_{m}\right)$ then we have

$$
\begin{equation*}
\chi_{p}(x)=\sum_{\substack{\ell=0 \\(-1)^{\ell+p}=1}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}} \sum_{\left\{j_{1}, \ldots, j_{\ell}\right\} \subset I_{m}}\left(\prod_{h=1}^{\ell} \cos t_{j_{h}}\right) \tag{2.2}
\end{equation*}
$$

for $0 \leq p \leq m-1$. By duality, $\chi_{2 m-p}(x)=\chi_{p}(x)$, we know the characters also for the values $m+1 \leq p \leq 2 m$. Furthermore,

$$
\begin{equation*}
\chi_{m}^{ \pm}(x)=\left(\sum_{\substack{\ell=1 \\ \ell \text { odd }}}^{m} 2^{\ell-1}\binom{m-\ell}{\frac{m-\ell}{2}} \sum_{\left\{j_{1}, \ldots, j_{\ell}\right\} \subset I_{m}}\left(\prod_{h=1}^{\ell} \cos t_{j_{h}}\right)\right) \pm 2^{m-1} i^{m}\left(\prod_{j=1}^{m} \sin t_{j}\right) \tag{2.3}
\end{equation*}
$$

Note that by (2.3), since $\chi_{m}^{+}+\chi_{m}^{-}=\chi_{m}$, (2.2) also holds for $p=m$.
Remark 2.1. Clearly, $\chi_{n}(i d)=2^{n}$. If $x \in T_{2 m}$ is of order 2 then $\chi_{n}(x)=0$. In fact, $\Lambda=\Lambda^{0} \oplus \cdots \oplus \Lambda_{+}^{m} \oplus \bigwedge_{-}^{m} \oplus \cdots \oplus \Lambda^{n}$. Thus, by using (2.5) and (2.11) below, we have $\chi_{n}(x)=\sum_{p=0}^{n} \chi_{p}(x)=\sum_{p=0}^{n} K_{p}^{n}(2 j)=0$ for any $1 \leq j \leq m$.

The reduction formula. By using (2.2), we will now express $K_{p}^{2 m}(2 j)$ as certain integral linear combination of $K_{\ell}^{m}(j)$ for some alternating indices $\ell$ in $\{0,1, \ldots, p\}$.

Theorem 2.2. Let $m \in \mathbb{N}$ and $j \in \mathbb{N}_{0}$. For $0 \leq p \leq 2 m$ we have

$$
\begin{equation*}
K_{p}^{2 m}(2 j)=\sum_{\substack{\ell=0 \\ \ell=p(2)}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}} K_{\ell}^{m}(j) . \tag{2.4}
\end{equation*}
$$

Proof. If $C \in \mathbb{R}^{n \times n}$, let $n_{C}=\operatorname{dim}\left(\mathbb{R}^{n}\right)^{C}$, i.e. the dimension of the space fixed by $C$. Let $B$ be a diagonal matrix in $\mathrm{SO}(n)$ of order 1 or 2 , that is

$$
B=\operatorname{diag}(\underbrace{-1, \ldots,-1}_{e}, \underbrace{1, \ldots, 1}_{f})
$$

where $\varepsilon_{i} \in\{ \pm 1\}, 1 \leq i \leq n$, with an even number of -1 's (since $\operatorname{det}(B)=1$ ). If $e_{1}, \ldots, e_{n}$ is the canonical basis of $\mathbb{R}^{n}$, put $I_{B}=\left\{1 \leq i \leq n: B e_{i}=e_{i}\right\}$, hence $n_{B}=\left|I_{B}\right|$, and $I_{B}^{\prime}=\left\{1 \leq j \leq n: j \notin I_{B}\right\}$. We have that

$$
\begin{equation*}
\chi_{p}(B)=K_{p}^{n}\left(n-n_{B}\right)=\sum_{\substack{J \subseteq I_{n} \\|J|=p}}(-1)^{\left|J \cap I_{B}^{\prime}\right|} \tag{2.5}
\end{equation*}
$$

(see (3.2) and Remark 3.6 in in [22]). Actually, (2.5) holds for every $B \in \operatorname{SO}(n)$ of order 2, not necessarily diagonal (see the proof of Theorem 2.1 in [19]).

Now, let $n=2 m$ and let $B$ be any matrix in $\mathrm{SO}(2 m)$ of order $\leq 2$. Such $B$ is conjugate in $\mathrm{SO}(2 m)$ to an element $x_{B} \in T_{2 m}$ as in (2.1), we denote this by $B \sim x_{B}$. Without loss of generality we can assume that

$$
B=\operatorname{diag}(\underbrace{-1, \ldots,-1}_{e}, \underbrace{1, \ldots, 1}_{f})
$$

with $e+f=2 m$ ( $e=0$ if and only if $B=I d$ ). Clearly $B$ is conjugate to $x_{B}$, where

$$
x_{B}=x(\underbrace{\pi, \ldots, \pi}_{j}, \underbrace{0, \ldots, 0}_{m-j}),
$$

with $e=2 j$ and $f=2(m-j)$. By evaluating (2.2) at $x_{B}$, we have

$$
\begin{equation*}
\chi_{p}\left(x_{B}\right)=\sum_{\substack{\ell=0 \\(-1)^{\ell+p}=1}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}} \sum_{\substack{J \subset I_{m} \\|J|=\ell}}(-1)^{\left|J \cap I_{j}\right|} . \tag{2.6}
\end{equation*}
$$

Since $B \sim x_{B}$, the characters of $B$ and $x_{B}$ coincide and we can equate (2.5) to (2.6). Thus, since $n-n_{B}=e=2 j$, we have

$$
K_{p}^{2 m}(2 j)=\chi_{p}(B)=\chi_{p}\left(x_{B}\right)=\sum_{\substack{\ell=0 \\(-1)^{\ell+p}=1}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}} K_{\ell}^{m}(j)
$$

for every $0 \leq p \leq n=2 m$ and $1 \leq j \leq m$.
Note that expression (2.4) trivially holds for $j \in \mathbb{Z} \backslash\{0,1, \ldots, m\}$ since $K_{p}^{n}(j)=0$ for every $j<0$ or $j>m$.

Remark 2.3. Since $0 \leq \ell \leq p \leq 2 m$, we have $\binom{m-\ell}{(p-\ell) / 2}=0$ for $\ell>2 m-p$ and $K_{\ell}^{m}(j)=0$ for $\ell>m$. Thus, the upper limit in the sum in Theorem 2.2, say $\rho$, is actually the minimum between $p, m$ and $2 m-p$, if $p$ and $m$ have the same parity; or the minimum between $p, m-1$ and $2 m-p$, otherwise. Thus, putting $\mu_{p}(m)=m$ if $p \equiv m(\bmod 2)$ and $\mu_{p}(m)=m-1$ if $p \not \equiv m(\bmod 2)$, we have that

$$
\rho=\rho(p, m)=\min \left\{p, \mu_{p}(m), 2 m-p\right\}
$$

is the true upper limit in the summation in (2.4).
By considering the cases $p$ even or odd separately, we get simpler expressions for (2.4) as follows

$$
\begin{align*}
K_{2 q}^{2 m}(2 j) & =\sum_{k=0}^{q} 4^{k}\binom{m-2 k}{q-k} K_{2 k}^{m}(j)  \tag{2.7}\\
K_{2 q+1}^{2 m}(2 j) & =2 \sum_{k=0}^{q} 4^{k}\binom{m-2 k-1}{q-k} K_{2 k+1}^{m}(j)
\end{align*}
$$

with $q \leq m$ in the even case and $q \leq m-1$ in the odd case.
There are 3 basic symmetry relations between binary Krawtchouk polynomials; namely, $K_{k}^{n}(n-k)=K_{n-k}^{n}(k), K_{k}^{n}(j)=(-1)^{j} K_{n-k}^{n}(j)$ and

$$
\begin{equation*}
\binom{n}{j} K_{k}^{n}(j)=\binom{n}{k} K_{j}^{n}(k) . \tag{2.8}
\end{equation*}
$$

By using (2.8) we can get an expression for $K_{2 j}^{2 m}(p)$ similar to (2.4).

Corollary 2.4. For $0 \leq j, p \leq m \in \mathbb{N}$ we have

$$
\begin{equation*}
K_{2 j}^{2 m}(p)=\frac{\binom{2 m}{2 j}}{\binom{2 m}{p}\binom{m}{j}} \sum_{\substack{\ell=0 \\ \ell \equiv p(2)}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}}\binom{m}{\ell} K_{j}^{m}(\ell) . \tag{2.9}
\end{equation*}
$$

Proof. The result follows directly by first applying (2.8) to $K_{2 j}^{2 m}(p)$ and then using (2.4) and (2.8) again.

Remark 2.5. With expressions (2.4) and (2.9), we can recursively compute $K_{p}^{2 m}(j)$, by using BKP's of order $2 m$, for $j$ even and any $p$ or for $p$ even and any $j$. It would remain to cover the cases $K_{\text {odd }}^{2 m}$ (odd), which are only $m^{2}$ cases out of $(2 m+1)^{2}$. Hence, asymptotically, we cover $75 \%$ of the cases, as one could expect a priori. The other symmetry relations are of no help for this matter.

As a direct consequence of the previous result we have the following cancellation property for Krawtchouk polynomials.
Corollary 2.6. Let $m, j \in \mathbb{N}$. For $0 \leq p \leq m$ we have

$$
\begin{equation*}
\sum_{p=0}^{2 m} \sum_{\substack{\ell=0 \\ \ell \equiv p(2)}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}} K_{\ell}^{m}(j)=2^{m} \sum_{\ell=0}^{m} K_{\ell}^{m}(j)=0 . \tag{2.10}
\end{equation*}
$$

Proof. In [19] we have shown that

$$
\begin{equation*}
K_{0}^{n}(j)+K_{1}^{n}(j)+\cdots+K_{n}^{n}(j)=0 \tag{2.11}
\end{equation*}
$$

for every $1 \leq j \leq n$, and hence the second identity in $(2.10)$ follows. To get the first identity in (2.10) we apply (2.11) to (2.4). Note that $\sum_{p=0}^{2 m} \sum_{\ell=0}^{p}=\sum_{\ell=0}^{2 m} \sum_{p=\ell}^{2 m}$ and, since $K_{k}^{n}(j)=0$ for $j>n$ and $\binom{m-\ell}{\frac{p-\ell}{2}}=0$ for $\frac{p-\ell}{2}>m-\ell$, we have

$$
\sum_{p=0}^{2 m} \sum_{\substack{\ell=0 \\ \ell \equiv p(2)}}^{p} 2^{\ell}\binom{m-\ell}{\frac{p-\ell}{2}} K_{\ell}^{m}(j)=\sum_{\ell=0}^{m} 2^{\ell}\left(\sum_{\substack{p=\ell \\ \ell \equiv p(2)}}^{2 m-\ell}\binom{m-\ell}{\frac{p-\ell}{2}}\right) K_{\ell}^{m}(j) .
$$

Finally, since

$$
\sum_{\substack{p=\ell \\ \ell \equiv p(2)}}^{2 m-\ell}\binom{m-\ell}{\frac{p-\ell}{2}}=\sum_{k=0}^{m-\ell}\binom{m-\ell}{k}=2^{m-\ell}
$$

the left hand side of (2.10) equals $2^{m} \sum_{\ell=0}^{m} K_{\ell}^{m}(j)$, as desired.
Example 2.7. We now illustrate the big cancellations present in the corollary. Let $m=4$ and $j=3$. The terms of the inner sum of the left hand side of (2.10) are

$$
\begin{array}{ll}
p=0,1,7,8 & 2^{0}\binom{4}{0} K_{0}^{4}(3), \quad 2^{1}\binom{3}{0} K_{1}^{4}(3), \\
p=2,6 & 2^{1}\binom{3}{3} K_{1}^{4}(3), \quad 2^{0}\binom{4}{4} K_{0}^{4}(3) \\
p=3,5 & 2^{0}\left(\begin{array}{l}
4 \\
1
\end{array} K_{0}^{4}(3)+2^{2}\binom{2}{0} K_{2}^{4}(3), \quad 2^{0}\binom{4}{3} K_{0}^{4}(3)+2^{2}\binom{2}{2} K_{2}^{4}(3)\right. \\
p=4 & 2^{1}\binom{3}{1} K_{1}^{4}(3)+2^{3}\binom{1}{0} K_{3}^{4}(3), \quad 2^{1}\binom{3}{2} K_{1}^{4}(3)+2^{3}\binom{1}{1} K_{3}^{4}(3) \\
p & 2^{0}\binom{4}{2} K_{0}^{4}(3)+2^{2}\binom{2}{1} K_{2}^{4}(3)+2^{4}\binom{0}{0} K_{4}^{4}(3)
\end{array}
$$

respectively. The sum of all these terms is

$$
2^{0} S_{4} K_{0}^{4}(3)+2^{1} S_{3} K_{1}^{4}(3)+2^{2} S_{2} K_{2}^{4}(3)+2^{3} S_{1} K_{3}^{4}(3)+2^{4} S_{0} K_{4}^{4}(3)
$$

where $S_{j}=\binom{j}{0}+\cdots+\binom{j}{j}$ for $0 \leq j \leq 4$. Thus, the left hand side of (2.10) is

$$
\sum_{p=0}^{8} \sum_{\substack{\ell=0 \\ \ell \equiv p(2)}}^{p} 2^{\ell}\binom{4-\ell}{\frac{p-\ell}{2}} K_{\ell}^{4}(3)=2^{4}\left(K_{0}^{4}(3)+K_{1}^{4}(3)+K_{2}^{4}(3)+K_{3}^{4}(3)+K_{4}^{4}(3)\right)=0
$$

since the values for $K_{i}^{4}(3)$ with $i=0, \ldots, 4$ are $1,-2,0,2$ and -1 respectively.

## 3. A family of identities for BKP's

Formula (2.4) can be iterated to obtain (more involved) expressions for $K_{p}^{2^{r} m}\left(2^{s} j\right)$ in terms of $K_{p}^{m}(j)$ 's for different $p$ 's. For instance, for $r=s=2$ and $j \leq m$ odd, we have

$$
K_{p}^{4 m}(4 j)=\sum_{\substack{\ell=0 \\ \ell \equiv p(2)}}^{p} 2^{\ell}\binom{2 m-\ell}{\frac{p-\ell}{2}} K_{\ell}^{2 m}(2 j)=\sum_{\substack{\ell=0 \\ \ell \equiv p(2)}}^{p} 2^{\ell}\binom{2 m-\ell}{\frac{p-\ell}{2}} \sum_{\substack{k=0 \\ k \equiv \ell(2)}}^{\ell} 2^{k}\binom{m-k}{\frac{\ell-k}{2}} K_{k}^{m}(j),
$$

where we have applied (2.4) twice. That is,

$$
\begin{equation*}
K_{p}^{4 m}(4 j)=\sum_{\substack{0 \leq k \leq \ell \leq p \\ k \equiv \ell \equiv p(2)}} 2^{k+\ell}\binom{2 m-\ell}{\frac{p-\ell}{2}}\binom{m-k}{\frac{\ell-k}{2}} K_{k}^{m}(j) . \tag{3.1}
\end{equation*}
$$

However, for $K_{p}^{4 m}(2 j)$ we can apply (2.4) only once.
A generalized reduction formula. By continuing with the previous process, we can express the values of $K_{p}^{2^{r} m}(x)$ for $x$ even as linear combinations of values of some BKP's of much smaller orders, thus generalizing Theorem 2.2. We will need the following notation. For $r, s \in \mathbb{N}$, with $r$ fixed, put

$$
f(s, r):=(r-s) \chi_{[1, r]}(s)= \begin{cases}r-s & \text { if } s \leq r,  \tag{3.2}\\ 0 & \text { if } s>r,\end{cases}
$$

where $\chi_{[1, r]}$ is the characteristic function of the interval $[1, r]$. Note that $f(t, t)=0$ for every $t$.

Theorem 3.1. Let $m, p, r, s \in \mathbb{N}, j \in \mathbb{N}_{0}$ such that $2(\nu-1) \leq p \leq 2^{r} m$ and $2^{s} j \leq 2^{r} m$ where $\nu=\min \{r, s\}$. Then, we have

$$
\begin{equation*}
K_{p}^{2^{r} m}\left(2^{s} j\right)=\sum_{\substack{0 \leq p_{\nu} \leq \cdots \leq p_{1} \leq p \\ p_{\nu} \equiv \cdots \equiv p_{1} \equiv p(2)}} 2^{p_{1}+\cdots+p_{\nu}}\left(\prod_{k=1}^{\nu}\left(\frac{2^{r-k} m-p_{k}}{p_{k-1} p_{k}}\right)\right) K_{p_{\nu}}^{2^{f(s, r)} m}\left(2^{f(r, s)} j\right) \tag{3.3}
\end{equation*}
$$

where we use the convention $p_{0}=p$.
Note. If $r=s$ there is a big simplification in (3.3), since in this case $K_{p_{\nu}}^{2^{f(s, r)} m}\left(2^{f(r, s)} j\right)$ equals $K_{p_{r}}^{m}(j)$ for $m, j$ odd. By taking $r=s=1$ we get Theorem 2.2

Proof. We will apply Theorem 2.2 the maximum possible number of times, which is $\nu$. So, we will consider the cases $s \leq r$ and $s>r$ separately.
(i) Assume first that $s \leq r$. Hence we can apply (2.4) just $s$ times. We proceed by induction on $r$ using (2.4). The first step in the induction, i.e. $r=2(r=1$ is just Theorem [2.2], is done in the observation before the statement. For the inductive step, suppose first that $s=r$. By considering $m^{\prime}=2^{r-1} m$ and $j^{\prime}=2^{r-1} j$, we can apply Theorem 2.2 and get

$$
K_{p}^{2^{r} m}\left(2^{r} j\right)=K_{p}^{2 m^{\prime}}\left(2 j^{\prime}\right)=\sum_{\substack{0 \leq \ell \leq p \\ \ell \equiv p(2)}} 2^{\ell}\binom{m^{\prime}-p}{\frac{p-\ell}{2}} K_{\ell}^{2 r-1} m\left(2^{r-1} j\right) .
$$

Thus, by the inductive hypothesis,

$$
K_{p}^{2^{r} m}\left(2^{r} j\right)=\sum_{\substack{0 \leq \ell \leq p \\ \ell \equiv p(2)}} 2^{\ell}\binom{m^{\prime}-p}{\frac{p-\ell}{2}} \sum_{\substack{0 \leq q_{r-1} \leq \cdots \leq q_{1} \leq \ell \\ q_{r-1} \equiv \cdots \equiv q_{1} \equiv p(2)}} 2^{q_{1}+\cdots+q_{r-1}} \prod_{k=1}^{r-1}\binom{2^{r-1-k} m-q_{k}}{\frac{q_{k-1}-q_{k}}{2}} K_{q_{r-1}}^{m}(j) .
$$

By renaming the indices as follows $p_{1}=\ell, p_{2}=q_{1}, \ldots, p_{r}=q_{r-1}$, we get

$$
\begin{equation*}
K_{p}^{2^{r} m}\left(2^{r} j\right)=\sum_{\substack{0 \leq p_{r} \leq \cdots \leq p_{1} \leq p \\ p_{r} \equiv \cdots \equiv p_{1} \equiv p(2)}} 2^{p_{1}+\cdots+p_{r}}\left(\prod_{k=1}^{r}\binom{2^{r-k} m-p_{k}}{\frac{p_{k-1}-p_{k}}{2}}\right) K_{p_{r}}^{m}(j), \tag{3.4}
\end{equation*}
$$

which equals expression (3.3) with $\nu=s=r$ and $f(r, r)=0$.
For the case when $s<r$, we proceed similarly as before. By applying (2.4) a number $s$ of times, we get

$$
\begin{equation*}
K_{p}^{2^{r} m}\left(2^{s} j\right)=\sum_{\substack{0 \leq p_{s} \leq \cdots \leq p_{2} \leq p_{1} \leq p \\ p_{s} \equiv \cdots \equiv p_{1} \equiv p(\bmod 2)}} 2^{p_{1}+\cdots+p_{s}}\left(\prod_{k=1}^{s}\binom{2^{r-k_{m} m-p_{k}}}{\frac{p_{k-1}-p_{k}}{2}}\right) K_{p_{s}}^{2^{r-s} m}(j), \tag{3.5}
\end{equation*}
$$

which equals (3.3) since $\nu=s, f(s, r)=r-s$ and $f(r, s)=0$, in this case.
(ii) When $s>r$, we can apply (2.4) a number $r$ of times. By proceeding similarly as before, we get

$$
\begin{equation*}
K_{p}^{2^{r} m}\left(2^{s} j\right)=\sum_{\substack{0 \leq p_{r} \leq \cdots \leq p_{2} \leq p_{1} \leq p \\ p_{r} \equiv \cdots \equiv p_{1} \equiv p(\bmod 2)}} 2^{p_{1}+\cdots+p_{r}}\left(\prod_{k=1}^{r}\left(\frac{2^{r-k} m-p_{k}}{2_{k-1} p_{k}}\right)\right) K_{p_{r}}^{m}\left(2^{s-r} j\right) . \tag{3.6}
\end{equation*}
$$

It is now clear that, by using (3.2), the expressions obtained in (3.4) - (3.6) take the single form in (3.3), and the result thus follows. Finally, note that we need to consider $p \geq 2(\nu-1)$, for if not the summation set would be empty.

Typically, one will apply (3.3) when $j$ and $m$ are odd, for if not we can keep absorbing the powers of 2 . However, we also want to consider the case $j=0$.

Remark 3.2. There are some redundancies in (3.3) because some terms can vanish. A more accurate, though complicated, expression can be obtained by taking into
account the exact limits of summation (see Remark (2.3). In this case we have

$$
\sum_{\substack{ \\0 \leq p_{\nu} \leq \cdots \leq p_{1} \leq p \\ p_{\nu} \equiv \cdots \equiv p_{1} \equiv p(2)}} \sum_{\substack{p_{\nu}=0 \\ p_{\nu} \equiv p(2)}}^{\rho_{\nu-1}} \cdots \sum_{\substack{p_{2}=0 \\ p_{2} \equiv p(2)}}^{\rho} \sum_{p_{1} \equiv p(2)}^{\rho}
$$

where the symbol $\rightsquigarrow$ means that the first summation (which is easier to write) should be replaced by the second summation (which looks more complicated but involves much less terms) and where

$$
\rho_{i}=\min \left\{p_{i-1}, \mu_{p_{i-1}}\left(2^{r-i+1} m\right), 2^{r-i} m-p_{i-1}\right\}
$$

for $i=0, \ldots, \nu-1$ with the conventions $\rho_{0}=\rho$ and $p_{0}=p$.
Explicit computations. We will now illustrate the formulas obtained in Theorems 2.2 and 3.1] with some examples. It will be helpful to have the values $K_{k}^{n}(j), 0 \leq i, j \leq n$, for $n=1,2, \ldots, 8$ at hand. Thus, we present them as matrices $K_{n}=\left(K_{i j}^{n}\right)$ with $K_{i j}^{n}=K_{i}^{n}(j)$ in Table 1 below.

Table 1. The values of $K_{p}^{n}(j)$ for $1 \leq n \leq 8$ and $0 \leq p, j \leq n$


Note that the sum of the even rows (i.e., the odd numbered ones) vanish; that is $\sum_{j=0}^{n} K_{i}^{n}(j)=0$ for $i$ odd. This is a general fact proved in [4], see (5.1) - (5.5). On the other hand, expression (2.11) accounts for the vanishing of the sum of the columns (even or odd, except for the first one).

Observe that in the even matrices $K_{2}, K_{4}, K_{6}, K_{8}$, there are several entries divisible by 2,4 and 8 . In the last two sections we will show that this is indeed a general phenomenon.

Example 3.3. Here we compute the single value $K_{2}^{8}(4)$ using 3 different expressions: the definition (1.1) and Theorems 2.2 and 3.1 We have

$$
\begin{aligned}
& K_{2}^{8}(4)=\binom{4}{0}\binom{4}{2}-\binom{4}{1}\binom{4}{1}+\binom{4}{2}\binom{4}{0}=6-16+6=-4, \\
& K_{2}^{8}(4)=2^{0}\binom{4}{1} K_{0}^{4}(2)+2^{2}\binom{2}{0} K_{2}^{4}(2)=4+4(-2)=-4, \\
& K_{2}^{8}(4)=\binom{4}{1}\binom{2}{0} K_{0}^{2}(1)+2^{2}\binom{2}{0}\binom{2}{1} K_{0}^{2}(1)+2^{4}\binom{2}{0}\binom{0}{0} K_{2}^{2}(1)=4+8-16=-4,
\end{aligned}
$$

by (1.1), (2.2) and (3.1), respectively.
Example 3.4. We will compute the values $K_{4}^{8}(2 j), 0 \leq j \leq 4$. By (1.2), we have that $K_{4}^{8}(0)=K_{4}^{8}(8)=\binom{8}{4}=70$. By Theorem [2.2, for $j=1,3$ we have

$$
K_{4}^{8}(2 j)=\sum_{\ell=0,2,4} 2^{\ell}\binom{4-\ell}{\frac{4-\ell}{2}} K_{\ell}^{4}(j)=\binom{4}{2} K_{0}^{4}(j)+2^{2}\binom{2}{1} K_{2}^{4}(j)+2^{4}\binom{0}{0} K_{4}^{4}(j)=-10
$$

where we have used the second and fourth columns of $K_{4}$ in Table 1 .
Let us now compute $K_{4}^{8}(4)$ with Theorem 3.1. By taking $m=2$ and $j=1$ in (3.1) we have

$$
K_{4}^{8}(4)=\sum_{\substack{0 \leq k \leq \ell \leq 4 \\ k, \ell \text { even }}} 2^{k+\ell}\left(\frac{4-\ell}{\frac{4-\ell}{2}}\right)\left(\frac{2-k}{\frac{\ell-k}{2}}\right) K_{k}^{2}(1) .
$$

We only have to sum over the pairs $(k, \ell)$ of the form $(0,0),(0,2),(2,2)$ and $(0,4)$ since $(2,4)$ and $(4,4)$ do not contribute to the sum because of the appearance of $\binom{0}{1}=0$ and $K_{4}^{2}(1)=0$ in the corresponding terms. Thus, we get

$$
\begin{aligned}
K_{4}^{8}(4) & =\binom{4}{2}\binom{2}{0} K_{0}^{2}(1)+2^{2}\binom{2}{1}\binom{2}{1} K_{0}^{2}(1)+2^{4}\binom{2}{1}\binom{0}{0} K_{2}^{2}(1)+2^{4}\binom{0}{0}\binom{2}{2} K_{0}^{2}(1) \\
& =(6+16+16) K_{0}^{2}(1)+32 K_{2}^{2}(1)=38-32=6
\end{aligned}
$$

where we have used (the second column of) $K_{2}$ in this case.
The computations we have obtained for $K_{4}^{8}(2 j)$ are in coincidence with the values in (the fifth row of) $K_{8}$. Similarly, one can compute $K_{p}^{8}(2 j)$ for any $0 \leq p \leq 8, p \neq 4$.

We now present a more involved example to show how Theorem3.1really works.
Example 3.5. Let us compute $K_{6}^{48}(40)=K_{6}^{2^{4} 3}\left(2^{3} 5\right)$. So $m=3, r=4$ and $s=3$. Thus, $\nu=3$, and by (3.3) we have

$$
\begin{equation*}
K_{p}^{48}(40)=\sum_{\substack{0 \leq p_{3} \leq p_{2} \leq p_{1} \leq 6 \\ p_{3}, p_{2}, p_{1} \text { even }}} 2^{p_{1}+p_{2}+p_{3}} \cdot \pi\left(p_{3}, p_{2}, p_{1}\right) \cdot K_{p_{3}}^{6}(5) \tag{3.7}
\end{equation*}
$$

since $f(3,4)=1$ and $f(4,3)=0$, where we have used the notation

$$
\begin{equation*}
\pi\left(p_{3}, p_{2}, p_{1}\right):=\prod_{k=1}^{\nu}\binom{2^{4-k} 3-p_{k}}{\frac{p_{k-1}-p_{k}}{2}}=\binom{24-p_{1}}{\frac{6-p_{1}}{2}}\binom{12-p_{2}}{\frac{p_{1}-p_{2}}{2}}\binom{6-p_{3}}{\frac{p_{2}-p_{3}}{2}} . \tag{3.8}
\end{equation*}
$$

In this way, $K_{6}^{48}(40)$ is only expressed in terms of $K_{p}^{6}(5)$ with $p$ even, which we know have the values $K_{0}^{6}(5)=K_{6}^{6}(5)=1, K_{2}^{6}(5)=2$ and $K_{4}^{6}(5)=-5$, by Table 1 .

The 20 allowed triplets in (3.7) are $(0,0,0),(0,0,2),(0,0,4),(0,2,2),(0,0,6)$, $(0,2,4),(2,2,2),(0,2,6),(0,4,4),(2,2,4),(0,4,6),(2,2,6),(2,4,4),(0,6,6),(2,4,6)$,
$(4,4,4),(2,6,6),(4,4,6)$ and $(6,6,6)$. Therefore, we have

$$
\begin{aligned}
K_{6}^{48}(40)= & \pi(0,0,0)+2^{2} \pi(0,0,2)+2^{4}(\pi(0,0,4)+\pi(0,2,2)) \\
& +2^{6}(\pi(0,0,6)+\pi(0,2,4))+2^{8}(\pi(0,2,6)+\pi(0,4,4)) \\
& \quad+2^{10} \pi(0,4,6)+2^{12} \pi(0,6,6)+2^{18} \pi(6,6,6) \\
+ & 5\left\{2^{6} \pi(2,2,2)+2^{8} \pi(2,2,4)+2^{10}(\pi(2,2,6)+\pi(2,4,4))\right. \\
& \left.\quad+2^{12}(\pi(2,4,6)-\pi(4,4,4))-2^{14} \pi(4,4,6)-2^{16} \pi(4,6,6)\right\}
\end{aligned}
$$

where all the $\pi\left(p_{3}, p_{2}, p_{1}\right)$ can be easily computed by using (3.8).

## 4. Applications to binomial coefficients

As a direct consequence of Theorem 3.1 we get basic identities between binomial coefficients, expressing $\binom{2^{r} m}{k}$ in terms of numbers $\binom{2^{t} m}{j}$ with $t<r$ for some $j^{\prime}$ s. In fact, under the assumptions of Theorem 3.1, taking $j=0$ in (3.3) and using that $K_{p}^{n}(0)=\binom{n}{p}$, for any $1 \leq s, r$ we have

$$
\begin{equation*}
\binom{2^{r} m}{p}=\sum_{\substack{0 \leq p_{\nu} \leq \cdots \leq p_{2} \leq p_{1} \leq p \\ p_{\nu} \equiv \cdots \equiv p_{1} \equiv p(\bmod 2)}} 2^{p_{1}+\cdots+p_{\nu}}\left(\prod_{k=1}^{\nu}\binom{2^{r-k} m-p_{k}}{\frac{p_{k-1}-p_{k}}{2}}\right)\binom{2^{f(s, r)} m}{p_{\nu}} \tag{4.1}
\end{equation*}
$$

with $\nu=\min \{r, s\}$, where $\binom{2^{f(s, r)} m}{p_{\nu}}$ equals $\binom{2^{r-s} m}{p_{\nu}}$ or $\binom{m}{p_{\nu}}$ depending on whether $s<r$ or $s \geq r$. A much simpler expression is obtained when $s=1$, namely

$$
\begin{equation*}
\binom{2^{r} m}{p}=\sum_{\substack{0 \leq \ell \leq p \\ \ell \equiv p(2)}} 2^{\ell}\binom{2^{r-1} m-\ell}{\frac{p-\ell}{2}}\binom{2^{r-1} m}{\ell} \tag{4.2}
\end{equation*}
$$

For the simplest case, i.e. $r=s=1$, we can give more explicit expressions.
Lemma 4.1. For any integers $0 \leq q \leq m$ we have

$$
\begin{align*}
\binom{2 m}{2 q} & =\sum_{j=0}^{q} 4^{j}\binom{m-2 j}{q-j}\binom{m}{2 j}=\sum_{j=0}^{q} 4^{j}\binom{m}{q+j}\binom{q+j}{2 j},  \tag{4.3}\\
\binom{2 m}{2 q+1} & =2 \sum_{j=0}^{q} 4^{j}\binom{m-2 j-1}{q-j}\binom{m}{2 j+1}=2 \sum_{j=0}^{q} 4^{j}\binom{m}{q+j+1}\binom{q+j+1}{2 j+1},
\end{align*}
$$

where $q<m$ in the second identity.
Proof. By taking $j=0$ in (2.7), or taking $r=1$ in (4.2), and considering the cases $p=2 q$ and $p=2 q+1$, we get the first equalities in each of the expressions in (4.3). To see the remaining equalities, notice that

$$
\binom{m-2 j}{q-j}\binom{m}{2 j}=\binom{m}{2 j}\binom{m-2 j}{q+j-2 j}=\binom{m}{q+j}\binom{q+j}{2 j}
$$

where in the second equality we have applied the relation $\binom{r}{s}\binom{s}{t}=\binom{r}{t}\binom{r-t}{s-t}$ with $r=m, s=q+j$ and $t=2 j$. Thus, we get the second equality in the first row of (4.3). Proceeding similarly for the odd case, one gets the desired expression in the statement.

Remark 4.2. By Pascal's identity, one can get similar formulas for $\binom{2 m+1}{2 q+1}$ and $\binom{2 m+1}{2 q}$ as in Lemma 4.1, by combining the expressions in (4.3).
4.1. Recursions. Next, we will give alternative expressions for $\binom{2 m}{2 q},\binom{2 m}{2 q+1},\binom{2 m+1}{2 q}$ and $\binom{2 m+1}{2 q+1}$ in terms of $\binom{m}{q}$. We will need to make use of the double factorial of $n$

$$
n!!=n(n-2)(n-4) \cdots
$$

i.e. $n!!=\prod_{k=0}^{m}(n-2 k)$ with $m=\left\lceil\frac{n}{2}\right\rceil-1$, and the Pochhammer symbol

$$
(n)_{j}=\prod_{k=0}^{j-1}(n-k)=n(n-1)(n-2) \cdots(n-j+1)
$$

also known as falling factorial.
Theorem 4.3. If $q, m \in \mathbb{N}_{0}$ then we have

$$
\begin{align*}
\binom{2 m}{2 q} & =\binom{m}{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!}(q)_{j}(m-q)_{j},  \tag{4.4}\\
\binom{2 m}{2 q+1} & =2(m-q)\binom{m}{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j+1)!!}(q)_{j}(m-q-1)_{j},  \tag{4.5}\\
\binom{2 m+1}{2 q} & =(2 q+1)\binom{m}{q} \sum_{j=0}^{q} \frac{2^{j}}{2^{j}(2 j+1)!!}(q)_{j}(m-q)_{j},  \tag{4.6}\\
\binom{2 m+1}{2 q+1} & =(2(m-q)+1)\binom{m}{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j+1)!!!}(q)_{j}(m-q)_{j}, \tag{4.7}
\end{align*}
$$

where the sums are in $\mathbb{Q}$.
Proof. Our starting point will be the expressions for $\binom{2 m}{2 q}$ and $\binom{2 m}{2 q+1}$ in (4.3). First, notice that

$$
\begin{equation*}
(2 j)!=2^{j} j!(2 j-1)!! \tag{4.8}
\end{equation*}
$$

Thus, the general term in the first summation for $\binom{2 m}{2 q}$ can be written as follows

$$
4^{j}\binom{m-2 j}{q-j}\binom{m}{2 j}=4^{j} \frac{(m-2 j)!}{(q-j)!(m-q-j)!} \frac{m!}{(2 j)!(m-2 j)!}=\frac{2^{j}}{j!(2 j-1)!!} \frac{m!}{(q-j)!(m-q-j)!}
$$

and hence, we get

$$
\binom{2 m}{2 q}=m!\sum_{j=0}^{q} \frac{2^{j}}{!!(2 j-1)!!}((q-j)!(m-q-j)!)^{-1} .
$$

After performing the sum of the fractions involved, we get

$$
\begin{aligned}
\binom{2 m}{2 q} & =\frac{m!}{q!(m-q)!} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!}\left(\prod_{k=0}^{j-1}(q-k)(m-q-k)\right) \\
& =\binom{m}{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!}\left(\prod_{k=0}^{j-1}(q-k)\right)\left(\prod_{k=0}^{j-1}(m-q-k)\right),
\end{aligned}
$$

and by using the Pochhammer symbols we finally obtain (4.4).

For $\binom{2 m}{2 q+1}$ we proceed similarly as before. By (4.8),

$$
\begin{equation*}
(2 j+1)!=(2 j+1)(2 j)!=2^{j} j!(2 j+1)!! \tag{4.9}
\end{equation*}
$$

and hence, from the expression in the second row of (4.3) we get

$$
\begin{aligned}
\binom{2 m}{2 q+1} & =2 m!\sum_{j=0}^{q} \frac{2^{j}}{j!(2 j+1)!!}((q-j)!(m-q-1-j)!)^{-1} \\
& =\frac{2 m!}{q!(m-1-q)!} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j+1)!!}\left(\prod_{k=0}^{j-1}(q-k)\right)\left(\prod_{k=0}^{j-1}(m-q-1-k)\right)
\end{aligned}
$$

from which (4.5) readily follows.
Finally, since $\binom{2 m+1}{2 q}=\binom{2 m}{2 q-1}+\binom{2 m+1}{2 q}$ and $\binom{2 m+1}{2 q+1}=\binom{2 m}{2 q}+\binom{2 m}{2 q+1}$, expressions (4.6) and (4.7) follow directly from (4.4) and (4.5), after some tedious but straightforward computations.

Remark 4.4. (i) Theorem 4.3 gives expressions for $\binom{2 m+\varepsilon}{2 q+\varepsilon^{\prime}} /\binom{m}{q}$, with $\varepsilon, \varepsilon^{\prime} \in\{0,1\}$.
(ii) It is known that $(q)_{j}=\sum_{i=0}^{j}(-1)^{j-i} s(j, i) q^{i}$. Hence, the Theorem4.3relates binomial coefficients with Stirling numbers of the first kind (see A048994 in [23]). For instance, (4.4) takes the form

$$
\begin{equation*}
\binom{2 m}{2 q}=\binom{m}{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!} \sum_{k, l=0}^{j}(-1)^{k+l} s(j, k) s(j, l) q^{k}(m-q)^{l} . \tag{4.10}
\end{equation*}
$$

Theorem 4.3 implies an expression for any product of $m-q$ consecutive odd (resp. even) positive numbers in terms of fractions of factors of small order.

Corollary 4.5. For $q \leq m-1$, with the convention $(-1)!!=1$, the product of $m-q$ consecutive odd numbers is

$$
N:=\prod_{j=q}^{m-1}(2 j+1)=(2(m-q)-1)!!\sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!}(q)_{j}(m-q)_{j}
$$

and hence the product of $m-q$ consecutive even numbers is

$$
M:=\prod_{j=q}^{m-1}(2 j)=\frac{(2 m-1)!}{(2 q-1)!N} .
$$

Proof. First note that, by (4.8), we have

$$
\binom{2 m}{2 q}=\frac{(2 m)!}{(2 q)!(2(m-q))!}=\frac{2^{m} m!(2 m-1)!!}{2^{q} q!(2 q-1)!2^{m-q}(m-q)!(2(m-q)-1)!!}=\binom{m}{q} \frac{(2 m-1)!!}{(2(m-q)-1)!!(2 q-1)!!} .
$$

Comparing this with (4.4) we have

$$
\frac{(2 m-1)(2 m-3) \cdots(2 q+1)}{(2(m-q)-1)!!}=\sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!}(q)_{j}(m-q)_{j}
$$

from which the expression for $N$ follows. Since $M N=\frac{(2 m-1)!}{(2 q-1)!}$ we are done.

Example 4.6. We want to compute $N=13 \cdot 15 \cdot 17 \cdot 19 \cdot 21$. Hence $q=6, m=11$ and $m-q=5$. By Corollary 4.5 we have $N=9!!\sum_{j=0}^{5} \frac{2^{j}}{j!(2 j-1)!!}(6)_{j}(5)_{j}$. Thus,

$$
\begin{aligned}
N=9!!(1+2 \cdot 6 \cdot 5 & +\frac{4}{2 \cdot 3!!}(6 \cdot 5)(5 \cdot 4)+\frac{8}{3!5!!}(6 \cdot 5 \cdot 4)(5 \cdot 4 \cdot 3) \\
& \left.\quad+\frac{16}{4!7!!!}(6 \cdot 5 \cdot 4 \cdot 3)(5 \cdot 4 \cdot 3 \cdot 2)+\frac{32}{5!9!!}(6 \cdot 5 \cdot 4 \cdot 3 \cdot 2)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)\right)
\end{aligned}
$$

and after some easy calculations we get

$$
N=9 \cdot 7 \cdot 5 \cdot 3 \cdot(1+60+400+640)+9 \cdot 5 \cdot 3 \cdot 1920+32 \cdot 720=1.322 .685 .
$$

Thus, we can also compute $M=12 \cdot 14 \cdot 16 \cdot 18 \cdot 20$ by doing $M=\frac{21!}{11!N}=967.680$.
4.2. Congruences mod powers of 2 . Using the previous results we will obtain the values of $\binom{2^{r} m}{2^{r} q}$ and $\binom{2^{r} m}{2^{r} q+1}$ modulo $2^{t}$, for any $r$ and small values of $t$.

We have the equalities

$$
\begin{align*}
\binom{2 m}{2 q} & =\binom{m}{q} \frac{(2 m-1)!!}{(2 q-1)!!(2(m-q)-1)!!} \\
\binom{2 m}{2 q+1} & =2(m-q)\binom{m}{q} \frac{(2 m-1)!}{(2 q+1)!!(2(m-q)-1)!!} . \tag{4.11}
\end{align*}
$$

Since all the double factorials above are odd, we deduce that

$$
\begin{equation*}
\binom{2 m}{2 q} \equiv\binom{m}{q} \quad(\bmod 2) \quad \text { and } \quad\binom{2 m}{2 q+1} \equiv 0 \quad(\bmod 2) . \tag{4.12}
\end{equation*}
$$

Actually, it is well known that $\binom{n}{k} \equiv 0 \bmod 2$ for $n$ even and $k$ odd, and $\binom{n}{k} \equiv\binom{\lfloor n / 2\rfloor}{\lfloor k / 2\rfloor}$ $\bmod 2$ otherwise (hence by taking $n=2 m$ and $k=2 q, 2 q+1$ one recovers (4.12)).

Obviously, we have $\binom{2^{r} m}{2^{r} q+1} \equiv 0 \bmod 2$ for every $s \geq 0$. However, by iterating (4.12), we deduce that for any $r \in \mathbb{N}$ we have

$$
\begin{equation*}
\binom{2^{r} m}{2^{r} q} \equiv\binom{m}{q} \quad(\bmod 2) . \tag{4.13}
\end{equation*}
$$

As a corollary to Theorem 4.3 we have the following result improving (4.13).
Proposition 4.7. Let $m, q \in \mathbb{N}_{0}$.
(a) For any $r \geq 1$ we have

$$
\binom{2^{r} m}{2^{r} q} \equiv \begin{cases}\binom{m}{q} & (\bmod 2,4), \\ \binom{m}{q}(1+2 q(m-q)) & (\bmod 8),(\bmod 16) \text { with } r=1, \\ \binom{m}{q}(1+10 q(m-q)) & (\bmod 16), r \geq 2 .\end{cases}
$$

(b) For $1 \leq r \leq 3$ we have

$$
\binom{2^{r} m}{2^{r} q+1} \equiv \begin{cases}0 & \left(\bmod 2^{r}\right) \\ 2^{r}(m-q)\binom{m}{q} & \left(\bmod 2^{r+1}\right)\end{cases}
$$

Proof. (a) By expanding the expressions in (4.3) for $m, q \in \mathbb{N}_{0}$ we have

$$
\begin{aligned}
\binom{2 m}{2 q} & =\binom{m}{q}+4\binom{m-2}{q-1}\binom{m}{2}+16\binom{m-4}{q-2}\binom{m}{4}+\text { terms divisible by } 64, \\
\binom{2 m}{2 q+1} & =2 m\binom{m-1}{q}+8\binom{m-3}{q-1}\binom{m}{3}+\text { terms divisible by } 32 .
\end{aligned}
$$

From this, since $4\binom{m-2}{q-1}\binom{m}{2}=2 q(m-q)\binom{m}{q}$ and $2 m\binom{m-1}{q}=2(m-q)\binom{m}{q}$, it follows that

$$
\begin{align*}
\binom{2 m}{2 q} & \equiv \begin{cases}\binom{m}{q} & (\bmod 2,4), \\
\binom{m}{q}(1+2 q(m-q))\end{cases}  \tag{4.14}\\
\binom{2 m}{2 q+1} & \equiv \begin{cases}0 & (\bmod 8,16), \\
2(m-q)\binom{m}{q} & (\bmod 4,8),\end{cases}
\end{align*}
$$

which improves (4.12).
The congruences for $\binom{2^{r} m}{2^{r} q}$ and $\binom{2^{r} m}{2^{r} q+1}$ in the statement will follow directly from induction on $r$, (4.14) being the inicial step. It is clear that $\binom{2^{r} m}{2^{r} q} \equiv\binom{m}{q} \bmod 4$ and $\binom{2^{r} m}{2^{r} q+1} \equiv 0 \bmod 2$, for any $r$. By using (4.14) twice we have

$$
\binom{4 m}{4 q} \equiv\binom{2 m}{2 q}(1+8 q(m-q)) \equiv\binom{2 m}{2 q} \equiv\binom{m}{q}(1+2 q(m-q)) \quad(\bmod 8),
$$

and hence, by induction, for any $r \geq 2$ we have

$$
\binom{2^{r} m}{2^{r} q} \equiv\binom{2^{r-1} m}{2^{r-1} q}\left(1+2^{2 r-1} q(m-q)\right) \equiv\binom{2^{r-1} m}{2^{r-1} q} \equiv\binom{m}{q}(1+2 q(m-q)) \quad(\bmod 8) .
$$

For modulo 16 we proceed similarly, but now

$$
\binom{4 m}{4 q} \equiv\binom{2 m}{2 q}(1+8 q(m-q)) \equiv\binom{m}{q}(1+2 q(m-q))(1+8 q(m-q)) \quad(\bmod 16)
$$

and hence $\binom{4 m}{4 q} \equiv\binom{m}{q}(1+10 q(m-q)) \bmod 16$. Thus, for any $r \geq 3$ we have

$$
\binom{2^{r m}}{2^{r} q} \equiv\binom{2^{r-1} m}{2^{r-1} q}\left(1+2^{2 r-1} q(m-q)\right) \equiv\binom{2^{r-1} m}{2^{r-1} q} \equiv\binom{m}{q}(1+10 q(m-q)) \quad(\bmod 16) .
$$

(b) By (4.14), we have $\binom{4 m}{4 q+1} \equiv 4(m-q)\binom{2 m}{2 q} \bmod 4,8$, and hence $\binom{4 m}{4 q+1} \equiv 0 \bmod 4$ and $\binom{4 m}{4 q+1} \equiv 4(m-q)(1+2 q(m-q))\binom{m}{q} \equiv 4(m-q)\binom{m}{q} \bmod 8$. Similarly, one proves that $\binom{8 m}{8 q+1} \equiv 0 \bmod 8$ and $\binom{8 m}{8 q+1} \equiv 8(m-q)\binom{m}{q} \bmod 16$, and we are done.
Example 4.8. Consider $\binom{48}{16}=2.254 .848 .913 .647$ and $\binom{56}{17}=$ 97.997.533.741.800. Since $\binom{48}{16}=\binom{2^{4} \cdot 3}{2^{4} .1}$, by Proposition4.7, with $m=3$ and $q=1$, we have $\binom{48}{16} \equiv\binom{3}{1}=3$ $\bmod 4,\binom{48}{16} \equiv 3(1+4) \equiv 7 \bmod 8$ and $\binom{48}{16} \equiv 3(1+20) \equiv 15 \bmod 16$. Similarly, since $\binom{56}{17}=\binom{2^{3} \cdot 7}{2^{3} \cdot 2}$, we have that $\binom{56}{17} \equiv 0 \bmod 8$ and $\binom{56}{17} \equiv 40\binom{7}{2} \equiv 8 \bmod 16$.
Remark 4.9. Following the same procedure that lead us to (4.14), congruences with bigger moduli can be obtained provided we impose some extra conditions on $q$ or $m$. For instance,

$$
\binom{2 m}{2 q} \equiv\binom{m}{q}\left\{1+2 q(m-q)+\frac{2}{3} q(q-1)(m-q)(m-q-1)\right\} \quad(\bmod 32,64)
$$

if $q \equiv 0,1 \bmod 3$ or $m-q \equiv 0,1 \bmod 3$, and

$$
\binom{2 m}{2 q+1} \equiv 2(m-q)\binom{m}{q}\left\{1+\frac{2}{3} q(m-q-1)\right\} \quad(\bmod 16,32)
$$

if $q$ or $m-q-1$ are divisible by 3 .
We will next need the arithmetic function $\varepsilon: \mathbb{N}_{0} \rightarrow \mathbb{N}_{0}$, where $\varepsilon(k)$ is the biggest power of 2 dividing $k!$, that is

$$
\begin{equation*}
k!=2^{\varepsilon(k)} \ell_{k}, \quad \ell_{k} \text { odd. } \tag{4.15}
\end{equation*}
$$

That is, $\varepsilon(k)=\nu_{2}(k!)$ the 2-adic valuation of $k$ !. By de Polignac's formula we have $\varepsilon(k)=\left\lfloor\frac{k}{2}\right\rfloor+\left\lfloor\frac{k}{2^{2}}\right\rfloor+\cdots+\left\lfloor\frac{k}{2^{t}}\right\rfloor$, with $t=\left\lfloor\log _{2}(k)\right\rfloor$. It is thus clear that

$$
\begin{equation*}
\varepsilon\left(2^{t}\right)=2^{t-1}+2^{t-2}+\cdots+2+1=2^{t}-1 . \tag{4.16}
\end{equation*}
$$

In general, we have the following.
Lemma 4.10. For any $k, r, m \in \mathbb{N}_{0}$, $m$ odd, we have
(a) $\varepsilon(k) \leq k-1$, with equality if $k$ is a power of 2 ;
(b) $\varepsilon\left(2^{r} m\right)=\varepsilon(m)+\left(2^{r}-1\right) m$;
(c) $\varepsilon\left(2^{r}+1\right)=\varepsilon\left(2^{r}-1\right)+r$; and
(d) $\varepsilon$ is an increasing function (monotonic when restricted to even or to odd numbers).

Proof. First note that for any $k \geq 0$ we have

$$
\begin{equation*}
\varepsilon(2 k)=\varepsilon(2 k+1) \quad \text { and } \quad \varepsilon(2 k)=\varepsilon(k)+k . \tag{4.17}
\end{equation*}
$$

The first relation is obvious and the second one follows from $(2 k)!=2^{\varepsilon(2 k)} \ell_{2 k}$ and $(2 k)!=2^{k} k!(2 k-1)!!=2^{\varepsilon(k)+k} \ell_{k}(2 k-1)!!$ and the fact that $\ell_{k}, \ell_{2 k}$ and $(2 k-1)!!$ are all odd numbers.

Now, the inequality in (a) follows directly by applying strong induction, since

$$
\varepsilon(2(k+1))=\varepsilon(k+1)+k+1 \leq 2 k+1=2(k+1)-1,
$$

where we have used (4.17). The remaining assertion is clear.
The expression in (b) is obtained by repeated application of the second equality in (4.17). Since $\left(2^{r}+1\right)!=\left(2^{r}+1\right) 2^{r}\left(2^{r}-1\right)$ !, the expression in (c) is straightforward from the definition of $\varepsilon$. Finally, (d) follows from (4.17) and the fact that by definition we have $\varepsilon(k+2) \geq \varepsilon(k)+1$.

The following result, which is probably known, complements the previous proposition. We include a proof for completeness. We will need the following notation

$$
\begin{equation*}
\varepsilon(m, q):=\varepsilon(m)-\varepsilon(q)-\varepsilon(m-q), \quad q \leq m . \tag{4.18}
\end{equation*}
$$

Proposition 4.11. For every $m, q, r \in \mathbb{N}$ we have $\varepsilon(m, q) \geq 0$ and

$$
\begin{align*}
\binom{2^{r} m}{2^{r} q} & \equiv \begin{cases}0 & \left(\bmod 2^{\varepsilon(m, q)}\right), \\
\binom{m}{q} & \left(\bmod 2^{\varepsilon(m, q)+1}\right),\end{cases}  \tag{4.19}\\
\binom{2^{2^{r} m}}{2^{r} q+1} & \equiv \begin{cases}\binom{m}{q} & \left(\bmod 2^{\varepsilon(m, q)}\right), \\
0 & \left(\bmod 2^{\varepsilon(m, q)+r}\right) .\end{cases}
\end{align*}
$$

In particular, $\binom{2^{r} m}{2^{r} q} \equiv\binom{2^{r} m}{2^{r} q+1} \bmod 2^{\varepsilon(m, q)}$.
Proof. First note that

$$
\binom{m}{q}=\frac{2^{\varepsilon(m)}}{2^{\varepsilon(q)} 2^{\varepsilon(m-q)}} \frac{\ell_{m}}{\ell_{q} \ell_{m-q}}=2^{\varepsilon(m)-\varepsilon(q)-\varepsilon(m-q)} \in \mathbb{Z} .
$$

Thus, since $\ell_{m}, \ell_{k}$ and $\ell_{m-q}$ are odd numbers, we have that $\binom{m}{q}=2^{\varepsilon(m, q)} \ell$, with $\ell$ odd, and hence $\varepsilon(m, q) \geq 0$.

Now, by (b) of Lemma 4.10 we have
with $\ell^{\prime}$ odd. In this way, we get $\binom{2^{r} m}{2^{r} q} \equiv 0 \bmod 2^{\varepsilon(m, q)}$ and

$$
\binom{2^{r} m}{2^{r} q}-\binom{m}{q}=2^{\varepsilon(m, q)}\left(\ell^{\prime}-\ell\right),
$$

with $\ell, \ell^{\prime}$ odd, and thus the first congruence is established.
On the other hand, we have

$$
\binom{2^{r} m}{2^{r} q+1}=\frac{\left(2^{r} m\right)!}{\left(2^{r} q\right)!\left(2^{r}(m-q)\right)!} \frac{2^{r}(m-q)}{2^{r} q+1}=2^{\varepsilon(m, q)+r} \frac{(m-q) \ell^{\prime \prime}}{2^{r} q+1}
$$

for some odd integer $\ell^{\prime \prime}$. Also, $\binom{2^{r} m}{2^{r} q+1}-\binom{m}{q} \equiv 0 \bmod 2^{\varepsilon(m, q)+r}$, and the second congruence in the statement follows.

The remaining assertion follows directly from (4.19).
In particular, Proposition 4.11 implies that for any $m, q, r \in \mathbb{N}$ we have

$$
\begin{equation*}
\binom{2^{r} m}{2^{r}+s} \equiv\binom{m}{q}\left(1-\delta_{s, t}\right) \quad\left(\bmod 2^{\varepsilon(m, q)+t}\right) \tag{4.20}
\end{equation*}
$$

where $s, t \in\{0,1\}$ and $\delta_{s, t}$ is the Kronecker $\delta$-function.
In certain cases, Lemma 4.11improves Proposition 4.7. This will be the case, for instance, when $\varepsilon(m, q) \geq 4$.
Proposition 4.12. Let $r$ and $t$ be natural numbers and for fixed t put $m_{t}=2^{t}, q_{t}=2^{t-1}-1$.
(a) For $(m, q)=\left(m_{t}+1, q_{t}\right),\left(m_{t}+1, q_{t}-1\right)$ or $\left(m_{t}, q_{t}-1\right)$ we have

$$
\binom{2^{r} m}{2^{r} q} \equiv 0 \quad\left(\bmod 2^{t-1}\right) \quad \text { and } \quad\binom{2^{r} m}{2^{r} q} \equiv\binom{m}{q} \quad\left(\bmod 2^{t}\right) .
$$

(b) Moreover,

$$
\binom{2^{r} m_{t}}{2^{r} q_{t}} \equiv 0 \quad\left(\bmod 2^{t}\right) \quad \text { and } \quad\binom{2^{r} m_{t}}{2^{r} q_{t}} \equiv\binom{m_{t}}{q_{t}} \quad\left(\bmod 2^{t+1}\right) .
$$

Proof. (a) If $t=1$ the result is trivial for $\bmod 2^{t-1}$ and holds by Proposition 4.7 for $\bmod 2^{t}$. For $t \geq 2$ fixed, consider the numbers $m=m_{t}+1=2^{t}+1$ and $q=q_{t}=2^{t-1}-1$; hence $m-q=2^{t-1}+2$. By using (4.16) and (4.17) we have $\varepsilon(m)=\varepsilon\left(2^{t}+1\right)=\varepsilon\left(2^{t}\right)=2^{t}-1$,

$$
\begin{aligned}
\varepsilon(q) & =\varepsilon\left(2^{t-1}-2\right)=\varepsilon\left(2\left(2^{t-2}-1\right)\right)=\varepsilon\left(2^{t-2}-1\right)+2^{t-2}-1, \\
\varepsilon(m-q) & =\varepsilon\left(2^{t-1}+2\right)=\varepsilon\left(2\left(2^{t-2}+1\right)\right)=\varepsilon\left(2^{t-2}+1\right)+2^{t-2}+1 .
\end{aligned}
$$

Now, by Lemma 4.10 (c) we have $\varepsilon(q)=\varepsilon\left(2^{t-2}+1\right)-(t-2)+2^{t-2}+1$. In this way, by (4.18), we have

$$
\begin{aligned}
\varepsilon(m, q) & =2^{t}-1-\left\{2 \varepsilon\left(2^{t-2}+1\right)+2^{t-1}-(t-2)\right\} \\
& =2^{t}-1-\left\{2\left(2^{t-2}-1\right)+2^{t-1}-(t-2)\right\} \\
& =2^{t}-1-\left(2^{t-1}+2^{t-1}-t\right)=t-1 .
\end{aligned}
$$

The result now follows directly by (4.19) in this case. Finally, by using Lemma 4.10, it is easy to check that

$$
\varepsilon\left(m_{t}+1, q_{t}\right)=\varepsilon\left(m_{t}+1, q_{t}-1\right)=\varepsilon\left(m_{t}, q_{t}\right)
$$

and hence the statement in (a) follows.
(b) Proceeding similarly as above we have $\varepsilon\left(m_{t}\right)=2^{t}-1, \varepsilon\left(q_{t}\right)=2^{t-1}-1-(t-1)$ and $\varepsilon\left(m_{t}-q_{t}\right)=2^{t-1}-1$. Thus, $\varepsilon\left(m_{t}, q_{t}\right)=t$, as we wanted to see.

## 5. Consequences for central binomial coefficients

We will apply the formulas for BKP's of the previous sections to obtain some new explicit and recursive expressions for the numbers

$$
c_{m}=\binom{2 m}{m}, \quad m \geq 0,
$$

known as central binomial coefficients. For $0 \leq m \leq 12$ we have $1,2,6,20,70,252,924$, $3432,12870,48620,184756,705432$ and 2704156 (see A000984 in [23]).

By taking $r=1$ and $p=m$ in (4.2) we get the expression

$$
\begin{equation*}
c_{m}=\sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell}\binom{m-\ell}{\frac{m-\ell}{2}}\binom{m}{\ell}=m!\sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} \frac{2^{\ell}}{\ell\left\{\left(\frac{m^{-\ell}}{2}\right)!\right\}^{2}}, \tag{5.1}
\end{equation*}
$$

or, distinguishing the cases $m$ even or odd,

$$
\begin{align*}
c_{2 q} & =(2 q)!\sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!(q-j)!\}^{2}}, \\
c_{2 q+1} & =2(2 q+1)!\sum_{j=0}^{q} \frac{2^{j}}{j!(2 j+1)!!\{(q-j)!\}^{2}} . \tag{5.2}
\end{align*}
$$

Also, by taking $m=2 q, 2 q+1$ in Lemma 4.1 we get

$$
\begin{align*}
c_{2 q} & =\sum_{j=0}^{q} 4^{j}\binom{2 q}{2 j} c_{q-j},  \tag{5.3}\\
c_{2 q+1} & =2 \sum_{j=0}^{q} 4^{j}\left(\begin{array}{c}
\binom{2 q+1}{2 j+1}
\end{array} c_{q-j} .\right.
\end{align*}
$$

The above identities recursively express $c_{2 q}$ and $c_{2 q+1}$ in terms of the first $q+1$ central binomial coefficients $c_{0}, c_{1}, \ldots, c_{q}$. In other words, we have

$$
\begin{equation*}
c_{2 q+1}=2\left\{\binom{2 q+1}{1} c_{q}+4\binom{2 q+1}{3} c_{q-1}+4^{2}\binom{2 q+1}{5} c_{q-2}+\cdots+4^{q-1}\binom{2 q+1}{2 q-1} c_{1}+4^{q}\right\}, \tag{5.4}
\end{equation*}
$$

since $c_{0}=1$. It is well known that $\binom{2 m}{m}$ are even numbers for any $m \geq 1$. This is trivial from (5.4), since $c_{1}=2$.

Furthermore, by taking $m=2 q$ in (4.4) and (4.10) we get the reduction formulas

$$
\begin{equation*}
c_{2 q}=c_{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!}(q)_{j}^{2}=c_{q} \sum_{j=0}^{q} \frac{2^{j}}{j!(2 j-1)!!} \sum_{k, l=0}^{q}(-1)^{k+l} s(j, k) s(j, l) q^{k+l} \tag{5.5}
\end{equation*}
$$

expressing $c_{2 q}$ in terms of $c_{q}$. This allows to give nice expressions for $\binom{4 q}{2 q} /\binom{2 q}{q}$ and $\binom{4 q}{2 q}-\binom{2 q}{q}$, and by iteration for $\binom{2^{r+1} q}{2^{r} q} /\binom{2^{r} q}{2^{r-1} q}$ and $\binom{2^{r+1} q}{2^{r} q}-\binom{2^{r} q}{2^{r-1} q}$, for any $r$.

We now give alternative expressions to (5.3) for the central binomial coefficients, recursively expressing $c_{2 q}$ and $c_{2 q+1}$ in terms of fractions involving $c_{0}, c_{1}, \ldots, c_{q}$.

Proposition 5.1. For any $q \in \mathbb{N}$ we have

$$
\begin{align*}
c_{2 q} & =\frac{4 q-1}{2 q^{2}} \sum_{j=1}^{q} 4^{j} j\binom{2 q}{2 j} c_{q-j}, \\
c_{2 q+1} & =\frac{2(4 q+1)}{(2 q+1)^{2}} \sum_{j=0}^{q} 4^{j}(2 j+1)\binom{2 q+1}{2 j+1} c_{q-j} . \tag{5.6}
\end{align*}
$$

Note. Notice that $c_{2 q}$ depends only on $c_{0}, c_{1}, \ldots, c_{q-1}$; compare with (5.3).
Proof. The result will follow directly by considering $p=m$ in Theorem [2.2 and evaluating the resulting expressions (2.4) at $j=0$ and $j=1$. In fact, by (1.2) we have $K_{\ell}^{m}(0)=\binom{m}{\ell}$ and $K_{\ell}^{m}(1)=\left(1-\frac{2 \ell}{m}\right)\binom{m}{\ell}$. Taking $j=0$ and $j=1$ in (2.4) respectively we have

$$
\begin{equation*}
\binom{2 m}{m}=\sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell}\binom{m-\ell}{\frac{m-\ell}{2}}\binom{m}{\ell} \quad \text { and } \quad K_{m}^{2 m}(2)=\sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell}\binom{m-\ell}{\frac{m-\ell}{2}} K_{\ell}^{m}(1) . \tag{5.7}
\end{equation*}
$$

We now compute $K_{m}^{2 m}(2)$. By (1.1), after some computations, we have

$$
K_{p}^{n}(2)=\binom{n-2}{p}-2\binom{n-2}{p-1}+\binom{n-2}{p-2}=\binom{n}{p} \frac{(n-p)(n-p-1)-2 p(n-p)+p(p-1)}{n(n-1)}
$$

and hence, taking $p=m$ and $n=2 m$ we get

$$
K_{m}^{2 m}(2)=\binom{2 m}{m} \frac{m(m-1)-2 m^{2}+m(m-1)}{2 m(2 m-1)}=\frac{1}{1-2 m}\binom{2 m}{m} .
$$

Putting all these things together into the second equality in (5.7) we have

$$
\frac{1}{1-2 m}\binom{2 m}{m}=\sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell}\binom{m-\ell}{\frac{m-\ell}{2}}\left(1-\frac{2 \ell}{m}\right)\binom{m}{\ell}=\binom{2 m}{m}-\frac{2}{m} \sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell} \ell\binom{m-\ell}{\frac{m-\ell}{2}}\binom{m}{\ell},
$$

where we have used the first equation in (5.7). From this we get

$$
\binom{2 m}{m}=\frac{2 m-1}{m^{2}} \sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell} \ell\binom{m-\ell}{\frac{m-\ell}{2}}\binom{m}{\ell}
$$

from which, by taking $m=2 q$ and $m=2 q+1$, the expressions in (5.6) follows directly after some trivial computations.

By combining the previous expressions obtained for $c_{2 q}$ and $c_{2 q+1}$ we get two other recursions for $c_{q}$ in terms of all the previous $c_{0}, \ldots, c_{q-1}$, one in terms of 'even' binomials $\binom{2 q}{2 j}$ and the other in terms of 'odd' binomials $\binom{2 q+1}{2 j+1}$, for $0 \leq j \leq q$.

Corollary 5.2. For every $q \in \mathbb{N}$ we have

$$
\begin{equation*}
c_{q}=\sum_{j=1}^{q} 4^{j}\binom{2 q}{2 j}\left\{\frac{4 q-1}{2 q^{2}+1} j-1\right\} c_{q-j}=\sum_{j=1}^{q} 4^{j}\binom{2 q+1}{2 j+1}\left\{\frac{4 q+1}{2 q^{2}} j-1\right\} c_{q-j} . \tag{5.8}
\end{equation*}
$$

Proof. By equating the expressions for $c_{2 q}$ in (5.3) and (5.6) and isolating the contribution for $j=0$ in the sums we get the first equality in (5.8). Proceeding similarly with $c_{2 q+1}$, after some more calculations, we get the second equality in (5.8).

Example 5.3. We now compute $c_{8}=\binom{16}{8}$ by using (5.3) and (5.6) and the values $c_{0}=1, c_{1}=2, c_{2}=6, c_{3}=20, c_{4}=70$. Taking $q=4$, by (5.3) and (5.6) we respectively have

$$
\begin{aligned}
& c_{8}=\sum_{j=0}^{4} 4^{j}\binom{8}{2 j} c_{4-j}=c_{4}+4\binom{8}{2} c_{3}+4^{2}\binom{8}{4} c_{2}+4^{3}\binom{8}{6} c_{1}+4^{4}, \\
& c_{8}=\frac{15}{32} \sum_{j=1}^{4} 4^{j}\binom{8}{2 j} j c_{4-j}=\frac{15}{32}\left\{4\binom{8}{2} c_{3}+4^{2}\binom{8}{4} 2 c_{2}+4^{3}\binom{8}{6} 3 c_{1}+4^{5}\right\},
\end{aligned}
$$

hence $c_{8}=12.870=\frac{15 \cdot 27456}{32}$, as one can easily check.
Central binomial coefficients and Krawtchouk polynomials. We will now give a mixed relation between central binomial coefficients and BKP's of the form $K_{2 t}^{2 q}(q)$. For $q$ even, we will get a cancellation rule; while, if $q$ is odd, we will get a recursive formula for $c_{2 q}$ in terms of $c_{0}, c_{1}, \ldots, c_{q-1}$. We will first need the following result.
Lemma 5.4. For any $q \in \mathbb{N}_{0}$ we have

$$
K_{2 q}^{4 q}(2 q)=(-1)^{q}\binom{2 q}{q} \quad \text { and } \quad K_{2 q+1}^{4 q+2}(2 q+1)=0 .
$$

Proof. By (1.1) we have

$$
K_{p}^{2 p}(p)=\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}\binom{p}{p-j}=\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}^{2},
$$

and it is well known that

$$
\sum_{j=0}^{p}(-1)^{j}\binom{p}{j}^{2}= \begin{cases}0 & p \text { odd } \\ (-1)^{p / 2}\binom{p}{p / 2} & p \text { even } .\end{cases}
$$

Hence, the result follows directly by considering the cases $p=2 q$ and $p=2 q+1$.
It is known that $K_{k}^{n}\left(\frac{n}{2}\right)=0$ for $k$ odd and $K_{k}^{n}\left(\frac{n}{2}\right)=(-1)^{k / 2}\binom{n / 2}{k / 2}$ for $k$ even (see for instance (7) in [13]). We have included a direct proof of the case that we need for completeness.

Proposition 5.5. Let $q$ be natural number.
(a) If $q$ is even then

$$
\begin{equation*}
\sum_{t=1}^{q} 4^{t} c_{q-t} K_{2 t}^{2 q}(q)=0 \tag{5.9}
\end{equation*}
$$

(b) If $q$ is odd then

$$
\begin{equation*}
c_{2 q}=-\sum_{t=1}^{q} 2^{2 t-1} c_{q-t} K_{2 t}^{2 q}(q) . \tag{5.10}
\end{equation*}
$$

Proof. By applying Theorem 2.2] with $m=2 q=p, j=q$, we get

$$
K_{2 q}^{4 q}(2 q)=\sum_{\substack{0 \leq \ell \leq 2 q \\
\ell \text { even }}} 2^{\ell}\left(\begin{array}{c}
\frac{2 q-\ell}{2 q-\ell}
\end{array}\right) K_{\ell}^{2 q}(q) .
$$

Also, by the previous Lemma we have $K_{2 q}^{4 q}(2 q)=(-1)^{q}\binom{2 q}{q}$. Thus, by equating these values we obtain

$$
(-1)^{q}\binom{2 q}{q}=\binom{2 q}{q}+\sum_{\substack{2 \leq \ell \leq 2 q \\
\ell \text { even }}} 2^{\ell}\left(\begin{array}{c}
\frac{2 q-\ell}{2}
\end{array}\right) K_{\ell}^{2 q}(q),
$$

that is to say

$$
\binom{2 q}{q}\left((-1)^{q}-1\right)=\sum_{t=1}^{q} 4^{t}\binom{2(q-t)}{q-t} K_{2 t}^{2 q}(q) .
$$

It is clear from this identity that we get the expressions in the statement, taking $q$ even or odd respectively.

Example 5.6. If $q=3$, the sum in (5.10) equals

$$
\binom{6}{3}=-\left(2\binom{4}{2} K_{2}^{6}(3)+2^{3}\binom{2}{1} K_{4}^{6}(3)+2^{5}\binom{0}{0} K_{6}^{6}(3)\right)=-(12(-3)+16 \cdot 3-32)=20 .
$$

For $q=4$, the sum in (5.9) is $4\binom{6}{3} K_{2}^{8}(4)+4^{2}\binom{4}{2} K_{4}^{8}(4)+4^{3}\binom{2}{1} K_{6}^{8}(4)+4^{4}\binom{0}{0} K_{8}^{8}(4)$ which equals $80(-4)+96 \cdot 6+128(-4)+256=0$, as it should be.

## 6. Catalan numbers

For $n \geq 0$, the Catalan numbers

$$
C_{n}=\frac{(2 n)!}{n!(n+1)!}=\frac{(2 n)!}{(n+1)(n!)^{2}},
$$

which appear in several different counting problems, are closely related with central binomial coefficients because of the relation

$$
\begin{equation*}
C_{n}=\frac{1}{n+1}\binom{2 n}{n}=\binom{2 n}{n}-\binom{2 n}{n+1} . \tag{6.1}
\end{equation*}
$$

The first seventeen Catalan numbers are (see A000108 in [23])

$$
\begin{gathered}
C_{0}=1, C_{1}=1, C_{2}=2, C_{3}=5, C_{4}=14, C_{5}=42, C_{6}=132, \\
C_{7}=429, C_{8}=1.430, C_{9}=4.862, C_{10}=16.796, C_{11}=58.786, C_{12}=208.012, \\
C_{13}=742.900, C_{14}=2.674 .440, C_{15}=9.694 .845, C_{16}=35.357 .670
\end{gathered}
$$

Note that $C_{3}, C_{7}$ and $C_{15}$ are odd. It is a classic result that $C_{n}$ is odd if and only if $n$ is a Mersenne number, i.e. $n=M_{a}=2^{a}-1$ for some $a \geq 0$ (see for instance [1]).

Note that by (6.1) we have $c_{n}=(n+1) C_{n}$, hence all the expressions obtained for central binomial coefficients in the previous section give rise to similar expressions for Catalan numbers. For instance, by (5.1) we have

$$
\begin{equation*}
C_{m}=\frac{1}{m+1} \sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} 2^{\ell}\binom{m-\ell}{\frac{m-\ell}{2}}\binom{m}{\ell}=\frac{m!}{m+1} \sum_{\substack{0 \leq \ell \leq m \\ \ell \equiv m(2)}} \frac{2^{\ell}}{\ell!\left\{\left(\frac{m-\ell}{2}\right)!\right\}^{2}} . \tag{6.2}
\end{equation*}
$$

By using (5.2) one gets similar expressions for $C_{2 q}$ or $C_{2 q+1}$.

Recursions. It is known that Catalan numbers satisfy the recursions

$$
C_{n+1}=\frac{2(2 n+1)}{n+2} C_{n} \quad \text { and } \quad C_{n+1}=\sum_{k=0}^{n} C_{k} C_{n-k},
$$

for any $n \geq 0$, expressing $C_{n+1}$ in terms of all the previous numbers $C_{0}, C_{1}, \ldots, C_{n}$. By using recursions between binomial coefficients already obtained, it is possible to give other recursion formulas for Catalan numbers, in which $C_{2 n}$ and $C_{2 n+1}$ are linear combinations of $C_{0}, C_{1}, \ldots, C_{n}$ only.
Proposition 6.1. For any non negative integer $n$ we have

$$
\begin{align*}
C_{2 n} & =\frac{1}{2 n+1} \sum_{k=0}^{n} 4^{k}(n-k+1)\binom{2 n}{2 k} C_{n-k}, \\
C_{2 n+1} & =\frac{1}{n+1} \sum_{k=0}^{n} 4^{k}(n-k+1)\binom{2 n+1}{2 k+1} C_{n-k} . \tag{6.3}
\end{align*}
$$

Proof. The result follows directly from the relation $C_{m}=\frac{1}{m+1}\binom{2 m}{m}$ by applying (5.3) with $m=2 n$ and $m=2 n+1$, and then using the first equality in (6.1) again.

Also from (5.6), by using (6.1), we get the alternative recursive expressions

$$
\begin{align*}
C_{2 n} & =\frac{4 n-1}{(2 n+1) 2 n^{2}} \sum_{k=1}^{n} 4^{k} k(n-k+1)\binom{2 n}{2 k} C_{n-k}, \\
C_{2 n+1} & =\frac{4 n+1}{(n+1)(2 n+1)^{2}} \sum_{k=0}^{n} 4^{k}(2 k+1)(n-k+1)\binom{2 n+1}{2 k+1} C_{n-k} . \tag{6.4}
\end{align*}
$$

Similarly, two more recursive expressions for Catalan numbers can be obtained from (5.8) by using (6.1).

Note that (6.3) and (6.4) are very similar to Touchard's identity

$$
\begin{equation*}
C_{n+1}=\sum_{k=0}^{[n / 2]} 2^{n-2 k}\binom{n}{2 k} C_{k} \quad(n \geq 0) \tag{6.5}
\end{equation*}
$$

(see for instance [24] and the references therein) which also enables one to recursively obtain $C_{2 n}$ from $C_{0}, \ldots, C_{n-1}$ and $C_{2 n+1}$ from $C_{0}, \ldots, C_{n}$. Another Touchardtype identities are

$$
\begin{equation*}
C_{n}=\frac{n+2}{n(n-1)} \sum_{k=1}^{\lfloor n / 2\rfloor} 2^{n-2 k} k\binom{n}{2 k} C_{k} \tag{6.6}
\end{equation*}
$$

for $n \geq 2$, proved by Callan ([2]) and the very similar ones

$$
\begin{equation*}
C_{n+1}=(n+3) \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{1}{k+2} 2^{n-2 k}\binom{n-1}{2 k} C_{k}, \tag{6.7}
\end{equation*}
$$

for $n \geq 0$, due to Hurtado-Noy ([12]), and

$$
\begin{equation*}
C_{n}=\frac{n+3}{2 n} \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \frac{2 k+1}{k+2} 2^{n-2 k}\binom{n}{2 k+1} C_{k}, \tag{6.8}
\end{equation*}
$$

for $n \geq 0$, obtained by Amdeberhan (according to [2]).
Example 6.2. We will compute $C_{8}$ in four ways. If $n=4$, by (6.3) we have

$$
\begin{aligned}
C_{8} & =\frac{1}{9} \sum_{k=0}^{4} 4^{k}(5-k)\binom{8}{2 k} C_{n-k} \\
& =\frac{1}{9}\left\{5\binom{8}{0} C_{4}+16\binom{8}{2} C_{3}+4^{2} 3\binom{8}{4} C_{2}+4^{3} 2\binom{8}{6} C_{1}+4^{4}\binom{8}{8} C_{0}\right\}
\end{aligned}
$$

and by (6.4) we have

$$
C_{8}=\frac{5}{96} \sum_{k=1}^{4} 4^{k} k(5-k)\binom{8}{2 k} C_{4-k}=\frac{5}{96}\left\{16\binom{8}{2} C_{3}+4^{2} 6\binom{8}{4} C_{2}+4^{3} 6\binom{8}{6} C_{2}+4^{5}\binom{8}{8} C_{1}\right\} .
$$

Alternatively, using (6.5) we have $n=7$ and

$$
C_{8}=\sum_{k=0}^{3} 2^{7-2 k}\binom{7}{2 k} C_{k}=2^{7}\binom{7}{0} C_{0}+2^{5}\binom{7}{2} C_{1}+2^{3}\binom{7}{4} C_{2}+2^{1}\binom{7}{6} C_{3},
$$

and also, using (6.6), we get

$$
C_{8}=\frac{5}{28} \sum_{k=1}^{4} 2^{8-2 k} k\binom{8}{2 k} C_{k}=\frac{5}{28}\left\{2^{6}\binom{8}{2} C_{1}+2^{4}\binom{8}{4} 2 C_{2}+2^{2}\binom{8}{6} 3 C_{3}+2^{0}\binom{8}{8} 4 C_{4}\right\} .
$$

It is reassuring that, since $C_{0}=C_{1}=1, C_{2}=2, C_{3}=5$ and $C_{4}=14$, in all the cases we get the value $C_{8}=1430$. One can also use expressions (6.7) and (6.8).

Remark 6.3. By (6.1), Proposition [5.5 relates Catalan numbers with BKP's. In fact, for $q$ even we have the 'orthogonality' relation

$$
\begin{equation*}
\sum_{t=1}^{q} 4^{t}(q-t+1) C_{q-t} K_{2 t}^{2 q}(q)=0 \quad(q \text { even }) \tag{6.9}
\end{equation*}
$$

and for $q$ odd we have the recursion

$$
\begin{equation*}
C_{q}=-\frac{1}{q+1} \sum_{t=1}^{q} 2^{2 t-1}(q-t+1) C_{q-t} K_{2 t}^{2 q}(q) \quad(q \text { odd }) \tag{6.10}
\end{equation*}
$$

both involving integral values of BKP's of the form $K_{2}^{2 q}(q), K_{4}^{2 q}(q), \ldots, K_{2 q}^{2 q}(q)$.
Congruences modulo 2, 4, 8 and 16. Notice that the the recursions (6.3) - (6.6) seem well suited to study congruences of Catalan numbers modulo powers of 2 (this is not the case for (6.7) and (6.8) since they involve fractions). In fact, by expanding these expressions and reducing modulo $2^{r}$, for some $1 \leq r \leq n$, one can obtain congruence relations for $C_{2 n}$ and $C_{2 n+1}$ in terms of $C_{n}$ and $C_{n-1} \bmod 2^{r}$.

The simplest expressions are the ones obtained from Touchard's identity. By considering the cases $n$ even or odd separately in (6.5), we get

$$
C_{2 n}=\frac{1}{2} \sum_{k=0}^{n-1} 4^{n-k}\binom{2 n-1}{2 k} C_{k}, \quad \text { and } \quad C_{2 n+1}=\sum_{k=0}^{n} 4^{n-k}\binom{2 n}{2 k} C_{k},
$$

and by expanding these expressions one can easily deduce that

$$
\begin{array}{llll}
C_{2 n} \equiv 0 & (\bmod 2), & C_{2 n+1} \equiv C_{n} & (\bmod 2), \\
C_{2 n} \equiv 2 C_{n-1} & (\bmod 4), & C_{2 n+1} \equiv C_{n} & (\bmod 4),  \tag{6.11}\\
C_{2 n} \equiv 2(2 n-1) C_{n-1} & (\bmod 8), & C_{2 n+1} \equiv C_{n}-4 n C_{n-1} & (\bmod 8),
\end{array}
$$

and

$$
\begin{align*}
C_{2 n} & \equiv 2(2 n-1) C_{n-1}+8\binom{2 n-1}{3} C_{n-2} & & (\bmod 16),  \tag{6.12}\\
C_{2 n+1} & \equiv C_{n}+4 n(2 n-1) C_{n-1} & & (\bmod 16) .
\end{align*}
$$

From the congruences mod 2 above, considering $n=2 m$ and $n=2 m+1$, we get

$$
C_{4 m+1} \equiv C_{2 m} \equiv 0 \quad(\bmod 2) \quad \text { and } \quad C_{4 m+3} \equiv C_{2 m+1} \equiv C_{m} \quad(\bmod 2) .
$$

Taking $m=2 k$ and $m=2 k+1$ above we get

$$
\begin{gathered}
C_{8 k+5} \equiv C_{8 k+3} \equiv C_{8 k+1} \equiv C_{4 k+2} \equiv C_{4 k+1} \equiv C_{4 k} \equiv C_{2 k} \equiv 0 \quad(\bmod 2), \\
C_{8 k+7} \equiv C_{4 k+3} \equiv C_{2 k+1} \equiv C_{k} \quad(\bmod 2) .
\end{gathered}
$$

Iterating this process, for every $k, \ell \geq 1$ one has that

$$
C_{2^{k} \ell+j} \equiv{ }_{2} \begin{cases}0 & \text { if } 1 \leq j<2^{k}-1,  \tag{6.13}\\ C_{\ell} & \text { if } j=2^{k}-1,\end{cases}
$$

where $\equiv_{2}$ denotes congruence modulo 2 . In particular, if $\ell=1$ in (6.13), for every $k \geq 1$, taking $j=2^{k}-1$ we get

$$
C_{2^{k+1}-1} \equiv C_{1} \equiv 1 \quad(\bmod 2) .
$$

From this and (6.13) we recover the fact that $C_{n}$ is odd if and only if $n$ is a Mersenne number.

Now, considering the cases even and odd separately in Callan's identity (6.6) we obtain

$$
C_{2 n}=\frac{n+1}{n(2 n-1)} \sum_{k=1}^{n} 4^{n-k} k\binom{2 n}{2 k} C_{k} \quad \text { and } \quad C_{2 n+1}=\frac{2 n+3}{n(2 n+1)} \sum_{k=1}^{n} 4^{n-k} k\binom{2 n+1}{2 k} C_{k} ;
$$

and by expanding these expressions one gets

$$
\begin{array}{rlr}
n C_{2 n} & \equiv 0 & (\bmod 2), \\
n(2 n-1) C_{2 n} & \equiv n(n+1) C_{n} & (\bmod 4,8), \\
n(2 n-1) C_{2 n} & \equiv(n+1)\left\{4 n(n-1)(2 n-1) C_{n-1}+n C_{n}\right\} & (\bmod 16), \\
n C_{2 n+1} & \equiv n C_{n} & (\bmod 2), \\
n(2 n+1) C_{2 n+1} & \equiv 3 n C_{n} & (\bmod 4),  \tag{6.15}\\
n(2 n+1) C_{2 n+1} & \equiv 4(n-1)\binom{2 n+1}{3} C_{n-1}-\left(4 n^{2}+3\right) n C_{n} & (\bmod 8,16),
\end{array}
$$

or equivalently $n C_{2 n+1} \equiv n(2 n+3) C_{n} \bmod 4$.

On the other hand, by expanding (6.3) in Proposition 6.1] it follows directly that

$$
\begin{array}{rlr}
C_{2 n} & \equiv(n+1) C_{n} & (\bmod 2), \\
(2 n+1) C_{2 n} & \equiv(n+1) C_{n} & (\bmod 4), \\
(2 n+1) C_{2 n} & \equiv(n+1) C_{n}-4 n^{2} C_{n-1} & (\bmod 8), \\
(2 n+1) C_{2 n} & \equiv(n+1) C_{n}+4 n^{2}(2 n-1) C_{n-1} & (\bmod 16), \\
(n+1) C_{2 n+1} & \equiv(n+1) C_{n} & (\bmod 2), \\
(n+1) C_{2 n+1} & \equiv(n+1)(2 n+1) C_{n} & (\bmod 4),  \tag{6.17}\\
(n+1) C_{2 n+1} & \equiv(n+1)(2 n+1) C_{n}+4 n\binom{2 n+1}{3} C_{n-1} & (\bmod 8,16) .
\end{array}
$$

Congruences modulo 32 and 64 , or even higher powers of 2 , can also be obtained in the same way, although with a fast increasing complexity.

Remark 6.4. The determination of the Catalan numbers mod 4 (resp. 8) is given in Theorems 2.3 (resp. 4.2) in [6] by using ad hoc methods. In the mod 4 case, if we put $\mathcal{C}_{4}(i)=\left\{C_{n}: C_{n} \equiv i(\bmod 4)\right\}$ for $0 \leq i \leq 3$ and $N_{a, b}=2^{a}+2^{b}-1=2^{a}+M_{b}$ then $\mathcal{C}_{4}(0)=\left\{C_{n}: n \neq N_{a, b}, a>b \geq 0\right\}, \mathcal{C}_{4}(1)=\left\{C_{n}: n=M_{a}, a \geq 0\right\}$ $\mathcal{C}_{4}(2)=\left\{C_{n}: n=N_{a, b}, a>b \geq 0\right\}$ and $\mathcal{C}_{4}(3)=\varnothing$. Similar results hold for the mod 8 case. By using (6.5), shorter and easier proofs of these facts can be found in [28], where also a systematic approach to Catalan numbers modulo $2^{r}$ is carried out.

FINAL REMARKS
(a) The method used to obtain Theorem [2.2]does not seem to apply for elements $x \in T_{2 m}$ of order > 2 because the $p$-traces $\chi_{p}(x)$ are not expressible, a priori, in terms of (binary) Krawtchouk polynomial, as in (2.5).
(b) Is there any combinatorial proof for (or explanation to) each of the expressions (6.2) - (6.4) obtained for Catalan numbers?
(c) The expressions for BKP's obtained so far seem well suited to study the values $K_{p}^{2^{r} m}\left(2^{s} j\right)$ modulo high powers of 2.
(d) The techniques and results in this paper could be of some utility in studying recursions and congruences for the Motzkin numbers $M_{n}$ because of the relations $M_{n}=\sum_{k=0}^{\ell}\binom{n}{2 k} C_{k}$ with $\ell=\lfloor n / 2\rfloor$ and $C_{n+1}=\sum_{k=0}^{n}\binom{n}{k} M_{k}$.

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