Gauss-Manin Connection in Disguise: Dwork Family [1]

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Abstract

We study the enhanced moduli space T of the Calabi-Yau n-folds arising from Dwork family and describe a unique vector field R in T with certain properties with respect to the underlying Gauss-Manin connection. For $n = 1, 2$ we compute explicit expressions of R and give a solution of R in terms of quasi-modular forms.

1 Introduction

The project Gauss-Manin connection in disguise started in the articles [Mov15, AMSY16] and the book [Mov16] aims to unify modular and automorphic forms with topological string partition functions of string theory. The first group has a vast amount of applications in number theory and so it is highly desirable to seek for such applications for the second group. The main ingredient of this unification is a natural generalization of Ramanujan relations between Eisenstein series interpreted as a vector field in a certain moduli space. This has been extensively used in transcendental number theory, see [NePh, Zud01] for an overview of some results. The starting point is either a Picard-Fuchs equation or a family of algebraic varieties. In direction of the first case, in [Mov16] the author has described the construction of vector fields attached to Calabi-Yau equations of the list in [GAZ10]. In direction of the second case, in this article we are going to consider the family of n-dimensional Calabi-Yau varieties $X = X_\psi,\ \psi \in \mathbb{P}^1 - \{0, 1, \infty\}$ obtained by a quotient and desingularization of the so-called Dwork family:

$$x_0^{n+2} + x_1^{n+2} + \ldots + x_{n+1}^{n+2} - (n+2)\psi x_0 x_1 \ldots x_n = 0, \tag{1.1}$$

and from now on we call any X_ψ a *mirror (Calabi-Yau) variety* (see Section 2 for more details). This family and its periods are also the main object of study in some physics articles like [GMP95]. In the present article we discuss this unification in the case of Dwork family, namely we explain a construction of a modular vector field $\mathsf{R}_n = \mathsf{R}$ attached to X_ψ such that for $n = 1, 2$ it has solutions in terms of (quasi)-modular forms, for $n = 3$ the topological partition functions are rational functions in the coordinates of a solution of R, and for $n \geq 4$ one gets q-expansions beyond the so-far well-known special functions. It is worth pointing out that we can consider the modular vector field R as an extension of the systems of differential equations introduced by G. Darboux [Dar78], G. H. Halphen [Hal81] and S. Ramanujan [Ram16] (for more details see [Mov12], [Nik15, § 1]).

For the purpose of the Introduction, we need only to know that for any mirror variety X, $\dim H^n_{\mathrm{dR}}(X) = n + 1$, where $H^n_{\mathrm{dR}}(X)$ is the n-th algebraic de Rham cohomology of X,

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[1] MSC2010: 14J15, 14J32, 11Y55.
Keywords: Gauss-Manin connection, Dwork family, Picard-Fuchs equation, Hodge filtration, quasi-modular form, q-expansion.
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and its Hodge numbers $h^{ij},\ i+j=n$, are all one. For $n=3$ this is also called the family of mirror quintic. Let $\mathsf{T}=\mathsf{T}_n$ be the moduli of pairs $(X,[\alpha_1,\cdots,\alpha_n,\alpha_{n+1}])$, where

$$\alpha_i \in F^{n+1-i}\setminus F^{n+2-i},\quad i=1,\cdots,n,n+1,$$

$$[\langle \alpha_i,\alpha_j\rangle]=\Phi_n.$$

Here F^i is the i-th piece of the Hodge filtration of $H^n_{\rm dR}(X)$, $\langle\cdot,\cdot\rangle$ is the intersection form in $H^n_{\rm dR}(X)$ and $\Phi=\Phi_n$ is an explicit constant matrix given by

$$\Phi_n := \begin{pmatrix} 0_{\frac{n+1}{2}} & J_{\frac{n+1}{2}} \\ -J_{\frac{n+1}{2}} & 0_{\frac{n+1}{2}} \end{pmatrix}, \tag{1.2}$$

if n is an odd positive integer, and

$$\Phi_n := J_{n+1}, \tag{1.3}$$

if n is an even positive integer, where by $0_k, k\in\mathbb{N}$, we mean a $k\times k$ block of zeros, and J_k is the following $k\times k$ block

$$J_k := \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}. \tag{1.4}$$

We construct the universal family $\mathsf{X}\to\mathsf{T}$ together with global sections $\alpha_i,\ \ i=1,\cdots,n+1$ of the relative algebraic de Rham cohomology $H^n_{\rm dR}(\mathsf{X}/\mathsf{T})$. Let

$$\nabla: H^n_{\rm dR}(\mathsf{X}/\mathsf{T}) \to \Omega^1_{\mathsf{T}}\otimes_{\mathcal{O}_{\mathsf{T}}} H^n_{\rm dR}(\mathsf{X}/\mathsf{T}),$$

be the algebraic Gauss-Manin connection on $H^n_{\rm dR}(\mathsf{X}/\mathsf{T})$. Then we state below our main theorem.

Theorem 1.1. *There is a unique vector field $\mathsf{R}:=\mathsf{R}_n$ in T such that the Gauss-Manin connection of the universal family of n-fold mirror variety X over T composed with the vector field R, namely ∇_{R}, satisfies:*

$$\nabla_{\mathsf{R}}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \vdots\\ \alpha_n\\ \alpha_{n+1}\end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \mathsf{Y}_1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \mathsf{Y}_2 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \mathsf{Y}_{n-2} & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & -1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}}_{\mathsf{Y}} \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \vdots\\ \alpha_n\\ \alpha_{n+1}\end{pmatrix}, \tag{1.5}$$

for some regular functions Y_i in T such that $\mathsf{Y}\Phi+\Phi\mathsf{Y}^{\mathsf{tr}}=0$. In fact,

$$\mathsf{T} := \mathrm{Spec}(\mathbb{Q}[t_1,t_2,\ldots,t_{\mathsf{d}},\frac{1}{t_{n+2}(t_{n+2}-t_1^{n+2})\check{t}}]), \tag{1.6}$$

where

$$\mathsf{d}=\mathsf{d}_n=\begin{cases}\frac{(n+1)(n+3)}{4}+1, & \text{if } n \text{ is odd}\\ \frac{n(n+2)}{4}+1, & \text{if } n \text{ is even}\end{cases}, \tag{1.7}$$

and $\check{t}$ *is a product of* s *variables among* t_i*'s,* $i=1,2,\ldots,\mathsf{d}$, $i\neq 1,n+2$ *and* $s=\frac{n-1}{2}$ *if* n *is an odd integer and* $s=\frac{n-2}{2}$ *if* n *is an even integer.*

In the proof of Theorem 1.1 we will show more than what we declared in the statement of the theorem. Indeed we will give the regular functions Y_i's explicitly, and we will find an algorithm to express the modular vector field R. An explicit expression for R_3 has been given in [Mov15, Mov16] by the first author. In the next theorem we find R_1 and R_2 explicitly and express their solutions in terms of quasi-modular forms.

Theorem 1.2. *For* $n=1,2$ *the vector field* R *as an ordinary differential equation is respectively given by*

$$\mathsf{R}_1:\begin{cases}\dot{t}_1=-t_1t_2-9(t_1^3-t_3)\\ \dot{t}_2=81t_1(t_1^3-t_3)-t_2^2\\ \dot{t}_3=-3t_2t_3\end{cases}, \tag{1.8}$$

where $\dot{*}=3\cdot q\cdot\frac{\partial *}{\partial q}$, *and*

$$\mathsf{R}_2:\begin{cases}\dot{t}_1=t_3-t_1t_2\\ \dot{t}_2=2t_1^2-\frac{1}{2}t_2^2\\ \dot{t}_3=-2t_2t_3+8t_1^3\\ \dot{t}_4=-4t_2t_4\end{cases}, \tag{1.9}$$

in which $\dot{*}=-\frac{1}{5}\cdot q\cdot\frac{\partial *}{\partial q}$, *and the following polynomial equation holds among* t_i*'s*

$$t_3^2=4(t_1^4-t_4). \tag{1.10}$$

Moreover, for any complex number τ *with* $\mathrm{Im}\tau>0$, *if we set* $q=e^{2\pi i\tau}$, *then we find the following solutions of* R_1 *and* R_2 *respectively:*

$$\begin{cases}t_1(q)=\frac{1}{3}(2\theta_3(q^2)\theta_3(q^6)-\theta_3(-q^2)\theta_3(-q^6)),\\ t_2(q)=\frac{1}{8}(E_2(q^2)-9E_2(q^6)),\\ t_3(q)=\frac{\eta^9(q^3)}{\eta^3(q)},\end{cases} \tag{1.11}$$

and

$$\begin{cases}\frac{10}{6}t_1(\frac{q}{10})=\frac{1}{24}(\theta_3^4(q^2)+\theta_2^4(q^2)),\\ \frac{10}{4}t_2(\frac{q}{10})=\frac{1}{24}(E_2(q^2)+2E_2(q^4)),\\ 10^4t_4(\frac{q}{10})=\eta^8(q)\eta^8(q^2),\end{cases} \tag{1.12}$$

where E_2, η and θ_i's are the classical Eisenstein, eta and theta series, respectively, given as follows:

$$E_2(q) = 1 - 24\sum_{k=1}^{\infty}\sigma(k)q^k \text{ with } \sigma(k) = \sum_{d|k} d, \tag{1.13}$$

$$\eta(q) = q^{\frac{1}{24}}\prod_{k=1}^{\infty}(1-q^k), \tag{1.14}$$

$$\theta_2(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(\frac{k+1}{2})^2}\ ,\ \ \theta_3(q) = 1 + 2\sum_{k=1}^{\infty} q^{\frac{1}{2}k^2}. \tag{1.15}$$

Remark 1.1. *We recall that η and θ_i's are modular forms, and E_2 is a quasi-modular form.*

By studying of the coefficients of q-expansions of the solutions given in (1.11) and (1.12), we find some interesting enumerative properties. For example, in (1.11) the coefficients of $t_1(q) = \sum_{k=0}^{\infty} t_{1,k}q^k$ have the following enumerative property: *Let k be a non-negative integer. If $k = 4m,\ \ m \in \mathbb{N}$, then the equation $x^2 + 3y^2 = k$ has $3t_{1,k}$ integer solutions. Otherwise the equation has $t_{1,k}$ integer solutions. For more properties of this type see Section 8.*

The article is organized in the following way. First, in Section 2 we review and summarize some basic facts, without proofs, about the structure of Dwork family from which the mirror variety X_ψ arises. In Section 3 we introduce the notion of moduli space of holomorphic n-form S, and we see that S is two dimensional and present a coordinate chart for it. Section 4 deals with the calculation of intersection form matrix of a given basis of the de Rham cohomology of mirror variety. In Section 5 we present the moduli space T and construct a complete coordinate system for T. Section 6 is devoted to the computing of Gauss-Manin connection of the families X/S and X/T. In Section 7 Theorem 1.1 is proved and the modular vector field is explicitly computed for $n = 1, 2, 4$. Finally, in Section 8 after finding the solutions of R_1 and R_2 in terms of quasi-modular forms, we proceed with the studying of enumerative properties of the q-expansions of the solutions.

Acknowledgment. The second author thanks the "Instituto de Matemática da Universidade Federal do Rio de Janeiro" of Brazil, where he did a part of this work during his Postdoctoral research with the grant of "CAPES", and in particular he would likes to thank Bruno C. A. Scardua for his supports.

2 Dwork Family

Let f_ψ be the polynomial in the left hand side of (1.1). Let W_ψ be an n-dimensional hypersurface in $\mathbb{P}^{n+1}$ given by f_ψ. We know that the first Chern class of W_ψ is zero, from which follows that W_ψ is a Calabi-Yau manifold. Thus we have a family of Calabi-Yau manifolds given by $\pi : \mathcal{W} \to \mathbb{P}^1$, where $\mathcal{W} \subset \mathbb{P}^n \times \mathbb{C}$, $W_\psi = \pi^{-1}(\psi)$ and

$$W_\infty = \{(x_0, x_1, \ldots, x_{n+1}) \mid\ x_0x_1\ldots x_{n+1} = 0\}.$$

This family, which is known as *n-fold Dwork family*, was a favorite example of Dwork, where he was developing his "deformation theory" about zeta function of a nonsingular hypersurface in a projective space (see [Dwo62, Dwo66]). One can easily see that the singular points of this family are $\psi^{n+2} = 1, \infty$. Let G be the following group

$$G := \{(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \mid \ \zeta_i^{n+2} = 1,\ \zeta_0\zeta_1 \ldots \zeta_{n+1} = 1\}, \tag{2.1}$$

which acts on W_ψ as follow

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}).(x_0, x_1, \ldots, x_{n+1}) = (\zeta_0 x_0, \zeta_1 x_1, \ldots, \zeta_{n+1} x_{n+1}). \tag{2.2}$$

Evidently we see that this action is well defined. We denote by $Y_\psi := W_\psi/G$ the quotient space of this action, which is quite singular. Indeed Y_ψ is singular in any $x \in W_\psi$ that its stabilizer in G is nontrivial. For $\psi^{n+2} \neq 1, \infty$ there exist a resolution $X_\psi \to Y_\psi$ of singularities of Y_ψ, such that X_ψ is a Calabi-Yau n-fold with $h^{i,j}(X_\psi) = 1,\ i + j = n$. Therefore we have a new family where the fibers are Calabi-Yau n-folds X_ψ which is the mirror family of W_ψ (see [GMP95]). The standard variable which is used in literatures is defined by $z := \psi^{-(n+2)}$. By this change of variables, f_ψ changes to f_z given by

$$f_z(x_0, x_1, \ldots, x_{n+1}) := z x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2) x_0 x_1 x_2 \cdots x_{n+1}.$$

The new set of singularities is given by $z = 0, 1$ and ∞, and we have the families W_z and its mirror X_z as well. From now on we call X_z (or X_ψ) the *mirror variety*. There is a global holomorphic $(n, 0)$-form $\eta \in H^n_{\mathrm{dR}}(X_z)$ which is given by

$$\eta := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_z}.$$

in the affine chart $\{x_0 = 1\}$. The periods $\int_\delta \eta, \quad \delta \in H_n(X_z, \mathbb{Z})$ satisfy the well-known Picard-Fuchs equation

$$\mathsf{L}\left(\int_\delta \eta\right) = 0, \quad \text{where} \tag{2.3}$$

$$\mathsf{L} \ := \ \vartheta^{n+1} - z(\vartheta + \frac{1}{n+2})(\vartheta + \frac{2}{n+2}) \ldots (\vartheta + \frac{n+1}{n+2}), \tag{2.4}$$

in which $\vartheta = z\frac{\partial}{\partial z}$. Note that if $n = 1, 2$ or 3 respectively, then X_z is a family of elliptic curves, $K3$-surfaces or mirror quintic 3-folds, respectively.

3 Moduli Space of holomorphic n-forms

By *moduli space of holomorphic n-forms* S we mean the moduli of the pair (X, α), where X is an n-dimensional mirror variety and α is a holomorphic n-form on X. We know that the family X_z is a one parameter family and the n-form α is unique, up to multiplication by a constant, therefore $\dim \mathsf{S} = 2$. The multiplicative group $\mathbb{G}_m := (\mathbb{C}^*, \cdot)$ acts on S by:

$$(X, \alpha) \bullet k = (X, k^{-1}\alpha),\ \ k \in \mathbb{G}_m,\ \ (X, \alpha) \in \mathsf{S}.$$

We present a chart (t_1, t_{n+2}) for S. To do this, for any $(t_1, t_{n+2}) \in \mathbb{C}^2$ we define the following polynomial

$$f_{t_1,t_{n+2}}(x_0, x_1, \ldots, x_{n+1}) := t_{n+2} x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2) t_1 x_0 x_1 x_2 \cdots x_{n+1}.$$

The discriminant of $f_{t_1,t_{n+2}}$ is given by $\Delta_{t_1,t_{n+2}} = (t_{n+2} - t_1^{n+2})t_{n+2}$. Let $\mathsf{W}_{t_1,t_{n+2}}$ be the following two parameter family of Calabi-Yau manifolds

$$\mathsf{W}_{t_1,t_{n+2}} := \{(x_0, x_1, \ldots, x_{n+1}) \mid f_{t_1,t_{n+2}}(x_0, x_1, \ldots, x_{n+1}) = 0\} \subset \mathbb{P}^{n+1}.$$

$\mathsf{W}_{t_1,t_{n+2}}$ is singular if and only if $\Delta_{t_1,t_{n+2}} = 0$. For any

$$(t_1, t_{n+2}) \in \mathbb{C}^2 \setminus \{(t_1, t_{n+2}) \mid \Delta_{t_1,t_{n+2}} = 0\}$$

we let $\mathsf{X}_{t_1,t_{n+2}}$ to be the resolution of the singularities of $\mathsf{W}_{t_1,t_{n+2}}/G$ where the group G and the group action are given by (2.1) and (2.2). Next we fix the n-form ω_1 on the family X_{t_1,t_n+2}, where ω_1 in the affine space $\{x_0 = 1\}$ is given by

$$\omega_1 := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_{t_1,t_{n+2}}}.$$

Proposition 3.1. *We have*

$$\mathsf{S} = \mathrm{Spec}(\mathbb{Q}[t_1, t_{n+2}, \frac{1}{(t_1^{n+2} - t_{n+2})t_{n+2}}])$$

and the morphism $\mathsf{X} \to \mathsf{S}$ is the the universal family of (X, α), where X is an n-dimensional mirror variety and α is a holomorphic n-form on X. Moreover, the $\mathbb{G}_m$-action on S is given by

$$(t_1, t_{n+2}) \bullet k = (kt_1, k^{n+2}t_{n+2}), \ (t_1, t_{n+2}) \in \mathsf{S}, \ k \in \mathbb{G}_m. \tag{3.1}$$

Proof. We have the map f which maps a point $(t_1, t_{n+2}) \in \mathsf{S}$ to the pair $(\mathsf{X}_{t_1,t_{n+1}}, \omega_1)$ in the moduli space S as a set. Its inverse is given by

$$(X_z, a\eta) \mapsto (a^{-1}, za^{-(n+2)}).$$

Note that $(\mathsf{X}_{t_1,t_{n+2}}, \omega_1)$ and $(X_z, t_1^{-1}\eta)$, where $z = \frac{t_{n+2}}{t_1^{n+2}}$, in the moduli space S represent the same element. The affirmation concerning the $\mathbb{G}_m$-action follows from the isomorphism:

$$\begin{aligned} &(\mathsf{X}_{kt_1,k^{n+2}t_{n+2}}, \ k\omega_1) \cong (\mathsf{X}_{t_1,t_{n+2}}, \omega_1), \\ &(x_1, x_2, \cdots, x_{n+1}) \mapsto (k^{-1}x_1, k^{-1}x_2, \cdots, k^{-1}x_{n+1}), \end{aligned} \tag{3.2}$$

given in the affine coordinates $x_0 = 1$. □

4 Intersection form and Gauss-Manin connection

Let X be an n-dimensional mirror variety and $\xi_1, \xi_2 \in H^n_{\mathrm{dR}}(X)$. Then in the context of de Rham cohomology, the *intersection form* of ξ_1 and ξ_2, denoted by $\langle \xi_1, \xi_2 \rangle$, is given by

$$\langle \xi_1, \xi_2 \rangle = \frac{1}{(2\pi i)^n} \int_X \xi_1 \wedge \xi_2.$$

We recall that $\langle ., . \rangle$ is a non-degenerate $(-1)^n$-symmetric form, and

$$\langle F^i, F^j \rangle = 0, \ i + j \geq n + 1, \tag{4.1}$$

where

$$F^{\bullet}:\{0\}=F^{n+1}\subset F^{n}\subset\ldots\subset F^{1}\subset F^{0}=H^{n}_{\mathrm{dR}}(X),\quad \dim F^{i}=n+1-i\ ,$$

is the Hodge filtration of $H^n_{\mathrm{dR}}(X)$.

Let

$$\nabla: H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{S})\to\Omega^1_{\mathsf{S}}\otimes_{\mathcal{O}_{\mathsf{S}}}H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{S}) \tag{4.2}$$

be Gauss-Manin connection of the two parameter family of varieties X/S, and $\frac{\partial}{\partial t_1}$ be a vector field on the moduli space S. By abuse of notation, we use the same notion $\frac{\partial}{\partial t_1}$, to show $\nabla_{\frac{\partial}{\partial t_1}}$ which is the composition of Gauss-Manin connection ∇ with the vector field $\frac{\partial}{\partial t_1}$. Now we define new n-forms ω_i, $i=1,2,\ldots,n+1$, as follows

$$\omega_i:=\frac{\partial^{i-1}}{\partial t_1^{i-1}}(\omega_1). \tag{4.3}$$

Later, in Lemma 4.1 we will see that $\omega_1,\omega_2,\ldots,\omega_{n+1}$ form a basis of $H^n_{\mathrm{dR}}(X)$ compatible with its Hodge filtration, i.e.

$$\omega_i\in F^{n+1-i}\setminus F^{n+2-i}, i=1,2,\ldots,n+1. \tag{4.4}$$

We write the Gauss-Manin connection of X/S in the basis ω as follow

$$\nabla\omega=\tilde{\mathsf{A}}\omega, \tag{4.5}$$

and we denote by

$$\Omega=\Omega_n:=(\langle\omega_i,\omega_j\rangle)_{1\leq i,j\leq n+1}\ , \tag{4.6}$$

the intersection form matrix in the basis ω. We have

$$d\Omega=\tilde{\mathsf{A}}\Omega+\Omega\tilde{\mathsf{A}}^{\mathsf{tr}}. \tag{4.7}$$

The entries of $\tilde{\mathsf{A}}$ and Ω are respectively regular differential 1-forms and functions in S. For arbitrary n, we do not have a general formula for Ω and $\tilde{\mathsf{A}}$. We have only an algorithm which computes the entries of Ω and $\tilde{\mathsf{A}}$ recursively. For $n=1,2,3,4$ the Picard-Fuchs equation associated with the n-form ω_1 is given by

$$\begin{aligned}\frac{\partial^{n+1}}{\partial {t_1}^{n+1}}&=-S_2(n+2,n+1)\frac{t_1^{n+1}}{t_1^{n+2}-t_{n+2}}\frac{\partial^n}{\partial {t_1}^{n}}-S_2(n+2,n)\frac{t_1^{n}}{t_1^{n+2}-t_{n+2}}\frac{\partial^{n-1}}{\partial {t_1}^{n-1}}-\ldots\\&-S_2(n+2,2)\frac{t_1^{2}}{t_1^{n+2}-t_{n+2}}\frac{\partial}{\partial t_1}-S_2(n+2,1)\frac{t_1}{t_1^{n+2}-t_{n+2}}\ ,\end{aligned} \tag{4.8}$$

where $S_2(r,s)$, $r,s\in\mathbb{N}$, refers to Stirling number of the second kind which is given by

$$S_2\left(r,s\right)\ =\frac{1}{s!}\sum_{i=0}^{s}(-1)^i\left(\begin{array}{c}s\\ i\end{array}\right)(s-i)^r. \tag{4.9}$$

This must be true for arbitrary n, however, we are only interested to compute this for explicit n's and so we skip the proof for arbitrary n.

Lemma 4.1. *We have*

(i) $\langle\omega_i,\omega_j\rangle = 0$, *if* $i+j \le n+1$.

(ii) $\langle\omega_1,\omega_{n+1}\rangle = (-(n+2))^n \frac{c_n}{t_1^{n+2}-t_{n+2}}$, *where* c_n *is a constant.*

(iii) $\langle\omega_j,\omega_{n+2-j}\rangle = (-1)^{j-1}\langle\omega_1,\omega_{n+1}\rangle$, *for* $j = 1,2,\ldots,n+1$.

(iv) *We can determine all the rest of* $\langle\omega_i,\omega_j\rangle$*'s in a unique way.*

Proof. Note that the intersection form is well-defined for all points in S, and so, $\langle\omega_i,\omega_j\rangle$'s are regular functions in S. This implies that they have poles only along $t_{n+2} = 0$ and $t_{n+2} - t_1^{n+2} = 0$.

(i) The Griffiths transversality implies that

$$\omega_i \in F^{n+1-i}, i = 1,2,\ldots,n+1.$$

This property and the property given in (4.1) complete the proof of **(i)**.

(ii) If we present the Picard-Fuchs equation associated with holomorphic n-form η as follow:

$$\vartheta^{n+1} = a_0(z) + a_1(z)\vartheta + \ldots + a_n(z)\vartheta^n, \tag{4.10}$$

then in account of (2.4) we find

$$a_n(z) = \frac{n+1}{2}\frac{z}{1-z} \ .$$

One can verify the differential equation given bellow

$$\vartheta\langle\eta,\vartheta^n\eta\rangle + \frac{2}{n+1}a_n(z)\langle\eta,\vartheta^n\eta\rangle = 0,$$

from which we get $\langle\eta,\vartheta^n\eta\rangle = c_n \exp\left(-\frac{2}{n+1}\int_0^z a_n(v)\frac{dv}{v}\right)$, where c_n is a constant. This yields

$$\langle\eta,\vartheta^n\eta\rangle = \frac{c_n}{1-z}. \tag{4.11}$$

On the other hand in Section 3 we saw $z = \frac{t_{n+2}}{t_1^{n+2}}$, which gets $\vartheta = z\frac{\partial}{\partial z} = -\frac{1}{n+2}t_1\frac{\partial}{\partial t_1}$. One can easily see that $\eta = t_1\omega_1$, hence

$$\begin{aligned}\vartheta^n\eta &= (-\frac{1}{n+2}t_1\frac{\partial}{\partial t_1})^n(t_1\omega_1)\\ &= b_1\omega_1 + \ldots + b_n\omega_n + (-\frac{1}{n+2})^n t_1^{n+1}\omega_{n+1},\end{aligned}$$

where b_j's are rational functions in t_1, t_{n+1}. Therefore **(i)** implies

$$\langle\eta,\vartheta^n\eta\rangle = \langle t_1\omega_1, (-\frac{1}{n+2})^n t_1^{n+1}\omega_{n+1}\rangle \ ,$$

which completes the proof of **(ii)**.

(iii) By **(i)** we have $\langle \omega_j, \omega_{n+1-j} \rangle = 0, \; j = 1, 2, \ldots, n$. Thus we get

$$\begin{aligned}\frac{\partial}{\partial t_1}\langle \omega_j, \omega_{n+1-j}\rangle &= \langle \frac{\partial}{\partial t_1}\omega_j, \omega_{n-j}\rangle + \langle \omega_j, \frac{\partial}{\partial t_1}\omega_{n+1-j}\rangle \\ &= \langle \omega_{j+1}, \omega_{n+1-j}\rangle + \langle \omega_j, \omega_{n+2-j}\rangle = 0,\end{aligned}$$

hence we obtain $\langle \omega_{j+1}, \omega_{n+1-j}\rangle = -\langle \omega_j, \omega_{n+2-j}\rangle$, $j = 1, 2, \ldots, n$, from which follows **(iii)**.

(iv) We present the desired algorithm. So far, we computed the first row of the matrix Ω. Suppose that we have the i-th row of Ω, $1 \le i \le n$, and then determine $(i+1)$-th row. To compute $\langle \omega_{i+1}, \omega_j \rangle$, $n + 2 - i \le j \le n + 1$, we apply $\frac{\partial}{\partial t_1}\langle \omega_i, \omega_j \rangle$, which implies

$$\langle \omega_{i+1}, \omega_j \rangle = \frac{\partial}{\partial t_1}\langle \omega_i, \omega_j \rangle - \langle \omega_i, \omega_{j+1} \rangle \, .$$

Note that if $j = n + 1$, then $\omega_{n+2} = \frac{\partial^{n+1}}{\partial t_1^{n+1}}(\omega_1)$ and we compute it by using of Picard-Fuchs equation given in (4.8).

□

The intersection form matrix for $n = 1, 2, 4$ are respectively given as follows:

$$\Omega_1 = \begin{pmatrix} 0 & -\frac{3c_1}{t_1^3 - t_3} \\ \frac{3c_1}{t_1^3 - t_3} & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & 0 & \frac{16c_2}{t_1^4 - t_4} \\ 0 & -\frac{16c_2}{t_1^4 - t_4} & \frac{32c_2t_1^3}{(t_1^4 - t_4)^2} \\ \frac{16c_2}{t_1^4 - t_4} & \frac{32c_2t_1^3}{(t_1^4 - t_4)^2} & \frac{-16c_2t_1^2(5t_1^4 - t_4)}{(t_1^4 - t_4)^3} \end{pmatrix},$$

$$\Omega_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{6^4c_4}{t_1^6 - t_6} \\ 0 & 0 & 0 & -\frac{6^4c_4}{t_1^6 - t_6} & \frac{9\times 6^4c_4t_1^5}{(t_1^6 - t_6)^2} \\ 0 & 0 & \frac{6^4c_4}{t_1^6 - t_6} & -\frac{3\times 6^4c_4t_1^5}{(t_1^6 - t_6)^2} & \frac{6^4c_4t_1^4(7t_1^6 + 20t_6)}{(t_1^6 - t_6)^3} \\ 0 & -\frac{6^4c_4}{t_1^6 - t_6} & -\frac{3\times 6^4c_4t_1^5}{(t_1^6 - t_6)^2} & \frac{6^4c_4t_1^4(14t_1^6 - 5t_6)}{(t_1^6 - t_6)^3} & \frac{-6^4c_4t_1^3(56t_1^{12} + 35t_1^6t_6 - 10t_6^2)}{(t_1^6 - t_6)^4} \\ \frac{6^4c_4}{t_1^6 - t_6} & \frac{9\times 6^4c_4t_1^5}{(t_1^6 - t_6)^2} & \frac{6^4c_4t_1^4(7t_1^6 + 20t_6)}{(t_1^6 - t_6)^3} & \frac{-6^4c_4t_1^3(56t_1^{12} + 35t_1^6t_6 - 10t_6^2)}{(t_1^6 - t_6)^4} & \frac{6^4c_4t_1^2(273t_1^{18} + 238t_1^{2}t_6 + 217t_1^6t_6^2 + t_6^3)}{(t_1^6 - t_6)^5} \end{pmatrix}.$$

5 Enhanced Moduli Space

By *enhanced moduli space* $\mathsf{T} = \mathsf{T}_n$ we mean the moduli of the pair $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, in which X is an n-fold mirror variety and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ is a basis of $H^n_{\mathrm{dR}}(X)$compatible with its Hodge filtration, and such that the intersection matrix of this basis is constant, that is,

$$\left(\, \langle \alpha_i, \alpha_j \rangle \, \right)_{1 \le i,j \le n+1} = \Phi. \tag{5.1}$$

If we denote by $\mathsf{d}_n := \dim \mathsf{T}_n$, then from [Nik15, Theorem 1] we get (1.7). The objective of this section is to construct a coordinates system for T.

In Section 4 we fixed the basis $\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}$ of $H^n_{\mathrm{dR}}(X)$ that is compatible with its Hodge filtration. Let $S = \left(\, s_{ij} \, \right)_{1 \le i,j \le n+1}$ be a lower triangular matrix, whose entries are indeterminates s_{ij}, $\; i \ge j$ and $s_{11} = 1$. We define

$$\alpha := S\omega,$$

where

$$\omega := \begin{pmatrix} \omega_1 & \omega_2 & \dots & \omega_{n+1} \end{pmatrix}^{\mathsf{tr}}.$$

We assume that $\begin{pmatrix} \langle \alpha_i, \alpha_j \rangle \end{pmatrix}_{1\leq i,j\leq n+1} = \Phi$, and so, we get the following equation

$$S\Omega S^{\mathsf{tr}} = \Phi. \tag{5.2}$$

If we set $\Psi = \begin{pmatrix} \Psi_{ij} \end{pmatrix}_{1\leq i,j\leq n+1} := S\Omega S^{\mathsf{tr}}$, then Ψ is a $(-1)^n$-symmetric matrix and $\Psi_{ij} = 0$ for $i = 1, 2, \dots, n$ and $j \leq n+1-i$. Moreover, in the case that n is an odd integer we get $\Psi_{ii} = 0$, $i = 1, 2, \dots, n+1$. Therefore the equation (5.2) gives us $d_0 := \frac{(n+2)(n+1)}{2} - \mathsf{d} - 2$ equations, where d is given by (1.7). The next argument shows that these equations are independent from each other and so we can express d_0 numbers of parameters s_{ij}'s in terms of other $\mathsf{d} - 2$ parameters that we fix them as *independent parameters*. For simplicity we write the first class of parameters as $\check{t}_1, \check{t}_2, \cdots, \check{t}_{d_0}$ and the second class as $t_2, t_3, \dots, t_{n+1}, t_{n+3}, \dots, t_{\mathsf{d}}$. We put all these parameters inside S according to the following rule which we write it only for $n = 1, 2, 3, 4$:

$$\begin{pmatrix} 1 & 0 \\ t_2 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t_2 & \check{t}_2 & 0 \\ \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 \\ t_4 & t_6 & \check{t}_2 & 0 \\ t_7 & \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 \\ t_4 & t_5 & \check{t}_3 & 0 & 0 \\ t_7 & \check{t}_7 & \check{t}_5 & \check{t}_2 & 0 \\ \check{t}_9 & \check{t}_8 & \check{t}_6 & \check{t}_4 & \check{t}_1 \end{pmatrix}.$$

Note that we have already used t_1, t_{n+1} as coordinates system of S in Section 3.

Proposition 5.1. *The equation $S\Omega S^{\mathsf{tr}} = \Phi$ yields*

$$s_{(n+2-i)(n+2-i)} = \frac{(-1)^{n+i+1}}{c_n(n+2)^n} \frac{t_1^{n+2} - t_{n+2}}{s_{ii}}, \tag{5.3}$$

where $i = 1, 2, \dots, \frac{n+1}{2}$ if n is an odd integer, and $i = 1, 2, \dots, \frac{n+2}{2}$ if n is an even integer. Moreover, one can compute $\check{t}_i$'s in terms of t_i's.

Proof. Let us first count the number of equalities that we get from $S\Omega S^{\mathsf{tr}} = \Phi$. This is $\frac{(n+1)(n+2)}{2} + 1 - \mathsf{d}$. Note that the left upper triangle of this equality consisits of trivial equalities $0 = 0$. The equality (5.3) follows from the $(i, n+2-i)$-th entry of $S\Omega S^{\mathsf{tr}} = \Phi$. We have plugged the parameters $\check{t}_k = s_{ij}$ inside S such that the equality corresponding to the $(n+2-j, i)$-th entry of $S\Omega S^{\mathsf{tr}} = \Phi$ gives us an equation which computes $\check{t}_k$ in terms of $\check{t}_r,\ r < k$ and t_s's. Note that only divisions by s_{ii}'s, $t_{n+2} - t_1^{n+2}$ and t_{n+2} occurs. Another way to see this is to redefine $S := S^{-1}$ and so we will have the equality $S\Phi S^{\mathsf{tr}} = \Omega$. $\square$

For $n = 1, 2, 4$, we express $\check{t}_j$'s in terms of t_i's as follows:

- $n = 1$:

$$\check{t}_1 = -\frac{1}{3c_1}(t_1^3 - t_3).$$

- $n = 2$:

$$\begin{aligned} \check{t}_1 &= \frac{1}{16c_2}(t_1^4 - t_4)\,, \\ \check{t}_2^2 &= -\frac{1}{16c_2}(t_1^4 - t_4)\,, \\ \check{t}_3 &= \frac{1}{16c_2}(-16c_2t_2\check{t}_2 + 2t_1^3)\,, \\ \check{t}_4 &= \frac{1}{32c_2}(-16c_2t_2^2 + t_1^2)\,. \end{aligned} \tag{5.4}$$

- $n=4$:

$$
\begin{aligned}
\check{t}_1 &= \frac{t_1^6-t_6}{1296c_4}\,,\\
\check{t}_2 &= -\frac{t_1^6-t_6}{1296c_4t_3}\,,\\
\check{t}_3^2 &= \frac{t_1^6-t_6}{1296c_4}\,,\\
\check{t}_4 &= \frac{t_1^6t_2+9t_1^5t_3-t_2t_6}{1296c_4t_3}\,,\\
\check{t}_5 &= \frac{-432c_4t_5\check{t}_3-t_1^5}{432c_4t_3}\,,\\
\check{t}_6 &= \frac{1296c_4t_2t_5\check{t}_3-1296c_4t_3t_4\check{t}_3+3t_1^5t_2+20t_1^4t_3}{1296c_4t_3}\,,\\
\check{t}_7 &= \frac{-1296c_4t_5^2-5t_1^4}{2592c_4t_3}\,,\\
\check{t}_8 &= \frac{1296c_4t_2t_5^2-2592c_4t_3^2t_7-2592c_4t_3t_4t_5+5t_1^4t_2+20t_1^3t_3}{2592c_4t_3}\,,\\
\check{t}_9 &= \frac{-2592c_4t_2t_7-1296c_4t_4^2+t_1^2}{2592c_4}\,.
\end{aligned}
\tag{5.5}
$$

6 Gauss-Manin connection

We return to the Gauss-Manin connection ∇, that was introduced in (4.2), and we proceed with the computation of the Gauss-Manin connection matrix $\tilde{\mathsf{A}}$, which is given in (4.5).

If we denote by $A(z)$ the Gauss-Manin connection matrix of the family $\mathsf{X}_{1,z}$ in the basis $\{\eta,\frac{\partial\eta}{\partial z},\ldots,\frac{\partial^n\eta}{\partial z^n}\}$, i.e.,

$$\nabla\begin{pmatrix}\eta & \frac{\partial\eta}{\partial z} & \ldots & \frac{\partial^n\eta}{\partial z^n}\end{pmatrix}^{\mathsf{tr}} = A(z)dz\otimes\begin{pmatrix}\eta & \frac{\partial\eta}{\partial z} & \ldots & \frac{\partial^n\eta}{\partial z^n}\end{pmatrix}^{\mathsf{tr}}\,,$$

then we get

$$A(z)=\begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0\\
0 & 0 & 1 & 0 & \ldots & 0\\
0 & 0 & 0 & 1 & \ldots & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \ldots & 1\\
b_1(z) & b_2(z) & b_3(z) & b_4(z) & \ldots & b_{n+1}(z)
\end{pmatrix}$$

in which the functions $b_i(z)$'s are the coefficients of the Picard-Fuchs equation associated with the n-form η that follows from (2.4) given below:

$$\frac{\partial^{n+1}}{\partial z^{n+1}} = b_1(z)+b_2(z)\frac{\partial^n}{\partial z^n}+\ldots+b_n(z)\frac{\partial^n}{\partial z^n}.$$

We calculate ∇ with respect to the basis (4.3) of $H^n_{\rm dR}(\mathsf{X}/S)$. For this purpose we return back to the one parameter case. For $z:=\frac{t_{n+2}}{t_1^{n+2}}$, consider the map

$$g:\mathsf{X}_{t_1,t_{n+2}}\to\mathsf{X}_{1,z},$$

given by (3.2) with $k=t_1^{-1}$. We have $g^*\eta=t_1\omega_1$, where by abuse of notation we just writ $\eta=t_1\omega_1$, and

$$\frac{\partial}{\partial z}=\frac{-1}{n+2}\frac{t_1^{n+3}}{t_{n+2}}\frac{\partial}{\partial t_1}.$$

From these two equalities we obtain the base change matrix $\tilde{S}=\tilde{S}(t_1,t_{n+2})$ such that

$$\begin{pmatrix} \eta & \frac{\partial\eta}{\partial z} & \ldots & \frac{\partial^n\eta}{\partial z^n}\end{pmatrix}^{\mathsf{tr}}=\tilde{S}^{-1}\begin{pmatrix}\omega_1 & \omega_2 & \ldots & \omega_{n+1}\end{pmatrix}^{\mathsf{tr}}.$$

Thus we find Gauss-Manin connection in the basis $\omega_i,\ i=1,2,\ldots,n+1$ as follow:

$$\tilde{\mathsf{A}}=\left(d\tilde{S}+\tilde{S}\cdot A(\frac{t_{n+2}}{t_1^{n+2}})\cdot d(\frac{t_{n+2}}{t_1^{n+2}})\right)\cdot\tilde{S}^{-1}.$$

Let $\tilde{\mathsf{A}}[i,j]$ be the (i,j)-th entry of the Gauss-Manin connection matrix $\tilde{\mathsf{A}}$. We have

$$\begin{aligned}
&(6.1)\qquad & \tilde{\mathsf{A}}[i,i] &= -\frac{i}{(n+2)t_{n+2}}dt_{n+2}\,,\ \ 1\le i\le n\\
&(6.2)\qquad & \tilde{\mathsf{A}}[i,i+1] &= dt_1-\frac{t_1}{(n+2)t_{n+2}}dt_{n+2}\,,\ \ 1\le i\le n\,,\\
& & \tilde{\mathsf{A}}[n+1,j] &= \frac{-S_2(n+2,j)t_1^j}{t_1^{n+2}-t_{n+2}}dt_1+\frac{S_2(n+2,j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\,,\ \ 1\le j\le n\,,\\
& & \tilde{\mathsf{A}}[n+1,n+1] &= \frac{-S_2(n+2,n+1)t_1^{n+1}}{t_1^{n+2}-t_{n+2}}dt_1+\frac{\frac{n(n+1)}{2}t_1^{n+2}+(n+1)t_{n+2}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\,,
\end{aligned}$$

where $S_2(r,s)$ is the Stirling number of the second kind defined in (4.9), and the rest of the entries of $\tilde{\mathsf{A}}$ are zero. The equalities (6.1) and (6.2) are easy to check and to those with Stirling numbers are checked for $n=1,2,3,4$. It would be interesting to prove this statement for arbitrary n. We will not need such explicit expressions for the proof of our main theorem. The Gauss-Manin connection matrix $\tilde{\mathsf{A}}$ for $n=1,2$ are respectively given as follows:

$$\tilde{\mathsf{A}}_1=\begin{pmatrix} -\frac{1}{3t_3}dt_3 & dt_1-\frac{t_1}{3t_3}dt_3\\ -\frac{t_1}{t_1^3-t_3}dt_1+\frac{t_1^2}{3t_3(t_1^3-t_3)}dt_3 & -\frac{3t_1^2}{t_1^3-t_3}dt_1+\frac{t_1^3+2t_3}{3t_3(t_1^3-t_3)}dt_3\end{pmatrix},$$

$$\tilde{\mathsf{A}}_2=\begin{pmatrix} -\frac{1}{4t_4}dt_4 & dt_1-\frac{t_1}{4t_4}dt_4 & 0\\ 0 & -\frac{2}{4t_4}dt_4 & dt_1-\frac{t_1}{4t_4}dt_4\\ -\frac{t_1}{t_1^4-t_4}dt_1+\frac{t_1^2}{4t_4(t_1^4-t_4)}dt_4 & -\frac{7t_1^2}{t_1^4-t_4}dt_1+\frac{7t_1^3}{4t_4(t_1^4-t_4)}dt_4 & -\frac{6t_1^3}{t_1^4-t_4}dt_1+\frac{3t_1^4+3t_4}{4t_4(t_1^4-t_4)}dt_4\end{pmatrix}.$$

Let A to be the Gauss-Manin connection matrix of the family X/T written in the basis $\alpha_i,\ i=1,2,\ldots,\alpha_{n+1}$, i.e., $\nabla\alpha=\mathsf{A}\alpha$. Then we calculate A as follow:

$$\mathsf{A}=\left(dS+S\cdot\tilde{\mathsf{A}}\right)\cdot S^{-1},\tag{6.3}$$

where S is the base change matrix $\alpha=S\omega$.

7 Proof of Theorem 1.1

As we saw in (6.3), the Gauss-Manin connection matrix of the family X/T in the basis α is given by

$$\mathsf{A} = dS \cdot S^{-1} + S \cdot \tilde{\mathsf{A}} \cdot S^{-1}. \tag{7.1}$$

For a moment, let us consider the entries $s_{ij},\ j \leq i, (i,j) \neq (1,1)$ of S as independent parameters with only the following relation:

$$s_{(n+1)(n+1)} + s_{nn}s_{22} = 0. \tag{7.2}$$

We denote by $\tilde{\mathsf{T}}$ and $\tilde{\alpha}$ the corresponding family of varieties and a basis of differential forms.

The existence of a vector field in $\tilde{\mathsf{T}}$ with the desired property in relation with the Gauss-Manin connection is equivalent to solve the equation

$$\dot{S} = \mathsf{Y}S - S \cdot \tilde{\mathsf{A}}(\mathsf{R}). \tag{7.3}$$

Note that here $\dot{x} := dx(\mathsf{R})$ is the derivation of the function x along the vector field R in $\tilde{\mathsf{T}}$. The equalities corresponding to the entries $(i,j), j \leq i, \quad (i,j) \neq (1,1)$ serves as the definition of $\dot{s_{ij}}$. The equality corresponding to $(1,1)$-th and $(1,2)$-th entries give us respectively

$$\dot{t}_1 = t_3 - t_1t_2, \quad \dot{t}_{n+2} = -(n+2)t_2t_{n+2}.$$

Recall that $t_2 = s_{21}$ and $t_3 = s_{22}$. The equalities corresponding to $(i, i+1)$-th, $i = 2, \cdots, n-1$, entries compute the quantities Y_i's:

$$\mathsf{Y}_{i-1} = \frac{t_3 s_{ii}}{s_{(i+1)(i+1)}}, \quad i = 2, 3, \ldots, n-1. \tag{7.4}$$

Finally the equality corresponding to the $(n, n+1)$-th entry is given by (7.2) which is already implemented in the definition of $\tilde{\mathsf{T}}$. All the rest are trivial equalities $0 = 0$. We conclude the statement of Theorem 1.1 for the moduli space $\tilde{\mathsf{T}}$.

Now, let us prove the main theorem for the moduli space T. First, note that we have a map

$$\tilde{\mathsf{T}} \to \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{C}), \quad (t_1, t_{n+2}, S) \mapsto S\Omega S^{\mathsf{tr}} \tag{7.5}$$

and T is the fiber of this map over the point Φ. We prove that the vector field R is tangent to the fiber of the above map over Φ. This follows from

$$\begin{aligned}
\overbrace{(S\Omega S^{\mathsf{tr}})}^{\cdot} &= \dot{S}\Omega S^{\mathsf{tr}} + S\dot{\Omega}S^{\mathsf{tr}} + S\Omega\dot{S}^{\mathsf{tr}} \\
&= (\mathsf{Y}S - S\tilde{\mathsf{A}})\Omega S^{\mathsf{tr}} + S(\tilde{\mathsf{A}}\Omega + \Omega\tilde{\mathsf{A}}^{\mathsf{tr}})S^{\mathsf{tr}} + S\Omega(S^{\mathsf{tr}}\mathsf{Y}^{\mathsf{tr}} - \tilde{\mathsf{A}}^{\mathsf{tr}}S^{\mathsf{tr}}) \\
&= \mathsf{Y}\Phi + \Phi\mathsf{Y}^{\mathsf{tr}} \\
&= 0.
\end{aligned}$$

where $\dot{x} := dx(\mathsf{R})$ is the derivation of the function x along the vector field R in T. The last equality follows from (7.4) and Proposition 5.1. It follows that if n is an even integer then

$\mathsf{Y}_{i-1} = -\mathsf{Y}_{n-i},\ i = 2, \ldots, \frac{n}{2}$ and if n is an odd integer then $\mathsf{Y}_{i-1} = -\mathsf{Y}_{n-i},\ i = 2, \ldots, \frac{n-1}{2}$ and

$$\mathsf{Y}_{\frac{n-1}{2}} = (-1)^{\frac{3n+3}{2}} c_n (n+2)^n \frac{t_3 s^2_{\frac{n+1}{2}\frac{n+1}{2}}}{t_1^{n+2} - t_{n+2}}.$$

In other words

$$\mathsf{Y}\Phi + \Phi\mathsf{Y}^{\mathsf{tr}} = 0. \tag{7.6}$$

To prove the uniqueness, first notice that (7.4) guaranties the uniqueness of Y_i's. Suppose that there are two vector fields R and $\hat{\mathsf{R}}$ such that $\nabla_{\mathsf{R}}\alpha = \mathsf{Y}\alpha$ and $\nabla_{\hat{\mathsf{R}}}\alpha = \mathsf{Y}\alpha$. If we set $\mathsf{H} := \mathsf{R} - \hat{\mathsf{R}}$, then

$$\nabla_{\mathsf{H}}\alpha = 0. \tag{7.7}$$

We need to prove that $\mathsf{H} = 0$, and to do this it is enough to verify that any integral curve of H is a constant point. Assume that γ is an integral curve of H given as follow

$$\gamma : (\mathbb{C}, 0) \to \mathsf{T}; \qquad x \mapsto \gamma(x).$$

Let's denote by $\mathcal{C} := \gamma(\mathbb{C}, 0) \subset \mathsf{T}$ the trajectory of γ in T . We know that the members of T are in the form of the pairs $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}])$, in which X is an n-fold mirror variety and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ is a basis of $H^n_{\rm dR}(X)$ compatible with its Hodge filtration and has constant intersection form matrix Φ. Thus, we can parameterize γ in such a way that for any $x \in (\mathbb{C}, 0)$ the vector field H on $\mathcal{C}$ reduces to $\frac{\partial}{\partial x}$, and so, we have $\gamma(x) = (X(x), [\alpha_1(x), \alpha_2(x), \ldots, \alpha_{n+1}(x)])$. We know that $X(x)$ is a member of mirror family that depends only on the parameter z, hence x holomorphically depends to z. From this we obtain a holomorphic function f such that $x = f(z)$. We now proceed to prove that f is constant. Otherwise, by contradiction suppose that $f' \neq 0$. Then we get

$$\nabla_{\frac{\partial}{\partial x}}\alpha_1 = \frac{\partial z}{\partial x}\nabla_{\frac{\partial}{\partial z}}\alpha_1. \tag{7.8}$$

Equation (7.7) gives that $\nabla_{\frac{\partial}{\partial x}}\alpha_1 = 0$, but since $\alpha_1 = \omega_1$, it follows that the right hand side of (7.8) is not zero, which is a contradiction. Thus f is constant and $X(x)$ does not depend on the parameter x. Since $X(x) = X$ does not depends on x, we can write the Taylor series of $\alpha_i(x)$, $\ i = 1, 2, 3, \ldots, n+1$, in x at some point x_0 as $\alpha_i(x) = \sum_j (x - x_0)^j \alpha_{i,j}$, where $\alpha_{i,j}$'s are elements in $H^n_{\rm dR}(X)$ independent of x. In this way the action of $\nabla_{\frac{\partial}{\partial x}}$ on α_i is just the usual derivation $\frac{\partial}{\partial x}$. Again according to (7.7) we get $\nabla_{\frac{\partial}{\partial x}}\alpha_i = 0$, and we conclude that α_i's also do not depend on x. Therefore, the image of γ is a point. $\square$

The modular vector field R for $n = 1, 2, 4$, are given as follows:

- $n = 1$:

$$\mathsf{R}_1 : \begin{cases} \dot{t}_1 = \frac{1}{3c_1}(-3c_1t_1t_2 - (t_1^3 - t_3)) \\ \dot{t}_2 = \frac{1}{9c_1^2}(t_1(t_1^3 - t_3) - 9c_1^2t_2^2) \\ \dot{t}_3 = -3t_2t_3 \end{cases}. \tag{7.9}$$

- $n=2$: We know that $\dim \mathsf{T}_2=3$, hence the modular vector field R_2 should have three components, but to avoid the second root of $\check{t}_2$ that comes from (5.4) we add one more variable $t_3 := \check{t}_2$. Thus we find R_2 as follow:

$$\mathsf{R}_2 : \begin{cases} \dot{t}_1 = -t_1 t_2 + t_3 \\ \dot{t}_2 = -\frac{1}{32c_2}(t_1^2 + 16c_2 t_2^2) \\ \dot{t}_3 = -\frac{1}{8c_2}(16c_2 t_2 t_3 + t_1^3) \\ \dot{t}_4 = -4t_2 t_4 \end{cases}, \tag{7.10}$$

such that following equation holds among t_i's

$$t_3^2 = -\frac{1}{16c_2}(t_1^4 - t_4). \tag{7.11}$$

- $n=4$: Here, analogously of the case $n=2$, to avoid the second root of $\check{t}_3$ given in (5.5), we add the variable $t_8 := \check{t}_3$ and we find:

$$\mathsf{R}_4 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = \frac{1296c_4 t_3^2 t_4 t_8 - t_1^6 t_2^2 + t_2^2 t_6}{t_1^6 - t_6} \\ \dot{t}_3 = \frac{1296c_4 t_3^2 t_5 t_8 - 3t_1^6 t_2 t_3 + 3t_2 t_3 t_6}{t_1^6 - t_6} \\ \dot{t}_4 = \frac{-1296c_4 t_3^2 t_7 t_8 - t_1^6 t_2 t_4 + t_2 t_4 t_6}{t_1^6 - t_6} \\ \dot{t}_5 = \frac{1296c_4 t_3 t_5^2 t_8 - 4t_1^6 t_2 t_5 - 2t_1^6 t_3 t_4 + 5t_1^4 t_3 t_8 + 4t_2 t_5 t_6 + 2t_3 t_4 t_6}{2(t_1^6 - t_6)} \\ \dot{t}_6 = -6t_2 t_6 \\ \dot{t}_7 = \frac{1296c_4 t_4^2 - t_1^2}{2592c_4} \\ \dot{t}_8 = \frac{-3t_1^6 t_2 t_8 + 3t_1^5 t_3 t_8 + 3t_2 t_6 t_8}{t_1^6 - t_6} \end{cases}, \tag{7.12}$$

where

$$t_8^2 = \frac{1}{1296c_4}(t_1^6 - t_6). \tag{7.13}$$

In this case the functions Y_1 and Y_2 are given by

$$\mathsf{Y}_1^2 = (-\mathsf{Y}_2)^2 = \frac{1296c_4 t_3^4}{t_1^6 - t_6}. \tag{7.14}$$

8 Enumerative properties of q-expansions

Here to find the q-expansion of the solution of R we follow the process given in [Mov16, § 5.2] for the case $n=3$. Consider the vector field R as follow

$$\mathsf{R} : \begin{cases} \dot{t}_1 = f_1(t_1, t_2, \ldots, t_{\mathsf{d}}) \\ \dot{t}_2 = f_2(t_1, t_2, \ldots, t_{\mathsf{d}}) \\ \vdots \\ \dot{t}_{\mathsf{d}} = f_{\mathsf{d}}(t_1, t_2, \ldots, t_{\mathsf{d}}) \end{cases}, \tag{8.1}$$

where for $1 \le j \le \mathsf{d}$,

$$f_j \in \mathbb{Q}[t_1, t_2, \ldots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}],$$

and $\check{t}$ is the same as Theorem 1.1. Let us assume that

$$t_j = \sum_{k=0}^{\infty} t_{j,k} q^k, \quad j = 1, 2, \ldots, \mathsf{d},$$

form a solution of R, where $t_{j,k}$'s are subject to be constants, and $\dot{*} = a \cdot q \cdot \frac{\partial *}{\partial q}$ in which a is an unknown constant. By comparing the coefficients of $q^k, k \geq 2$ in both sides of (8.1) we find recursions for $t_{j,k}$'s. By notation, set

$$p_k := (t_{1,k}, t_{2,k}, \ldots, t_{\mathsf{d},k}), \quad k = 1, 2, 3, \ldots.$$

By comparing the coefficients of q^0 we get that p_0 is a singularity of R. The same for q^1, gives us some constrains on $t_{j,1}$. Therefore, some of the coefficients $t_{j,k},\ k = 0, 1$ are free initial parameters of the recursion and we have to fix them by other means.

After finding the solutions, we proceed with the studying of the enumerative properties of the q-expansions. Following we state the results in the cases $n = 1, 2, 4$.

8.1 The case $n = 1$

Considering the modular vector field R_1 given in (7.9), we find $Sing(\mathsf{R}_1) = Sing_1 \cup Sing_2$, where

$$\begin{aligned} Sing_1 &: t_2 = t_1^3 - t_3 = 0, \\ Sing_2 &: t_3 = t_1^2 + 3c_1 t_2 = 0. \end{aligned}$$

Thus we get

$$p_0 = (t_{1,0}, -\frac{1}{3c_1} t_{1,0}^2, 0) \in Sing_2.$$

The comparison of the coefficients of q^1 yields $a = \frac{1}{c_1} t_{1,0}^2$ and we find p_1 as follow:

$$p_1 = (\frac{2}{9} \frac{t_{3,1}}{t_{1,0}^2}, -\frac{1}{27c_1} \frac{t_{3,1}}{t_{1,0}}, t_{3,1}).$$

If we choose $c_1 = 3^{-3}$, $t_{1,0} = \frac{1}{3}$ and $t_{3,1} = 1$, then we find the solution given in (1.11) for R_1 (to find this solution we use a $\mathsf{Singular}$ code).

In order to study the enumerative properties we first state the following lemma.

Lemma 8.1. *The coefficient of q^k, $k = 0, 1, 2, 3, \ldots$, in $\theta_3(q^{2r})\theta_3(q^{2s})$, $r, s \in \mathbb{N}$, gives the number of integer solutions of equation $rx^2 + sy^2 = k$, in which x and y are unknown variables.*

Proof. We know that $\theta_3(q^2) = 1 + 2\sum_{j=1}^{\infty} q^{j^2}$, hence

$$\theta_3(q^{2r})\theta_3(q^{2s}) = 1 + 2\sum_{i=1}^{\infty} q^{ri^2} + 2\sum_{j=1}^{\infty} q^{sj^2} + 4\sum_{i,j=1}^{\infty} q^{ri^2+sj^2}. \qquad (8.2)$$

If $(i, 0)$ or $(0, j)$ is a solution, then $(-i, 0)$ or $(0, -j)$, respectively, is another solution, and if (i, j), with $i \neq 0,\ j \neq 0$, is a solution, then $(-i, j)$, $(i, -j)$ and $(-i, -j)$ are other solutions. Therefore, on account of the above fact, the proof follows from equation (8.2). $\square$

Corollary 8.1. *The coefficient of q^k, $k=0,1,2,3,\ldots$, in $\theta_3(q^2)\theta_3(q^6)$ gives the number of integer solutions of equation $x^2+3y^2=k$.*

For more information about the number of integer solutions of equation $x^2+3y^2=k$, one can see [Oei, A033716] and the references given there.

As we saw in (1.11), $t_1(q)=\frac{1}{3}(2\theta_3(q^2)\theta_3(q^6)-\theta_3(-q^2)\theta_3(-q^6))$. If we denote by $t_1(q):=\sum_{k=0}^{\infty}t_{1,k}q^k$, then in the following proposition we state enumerative property of $t_{1,k}$.

Proposition 8.1. *Let k be a non-negative integer. If $k=4m$ for some $m\in\mathbb{Z}$, then the equation $x^2+3y^2=k$ has $3t_{1,k}$ integer solutions, otherwise the equation has $t_{1,k}$ integer solutions.*

Proof. Suppose that $\theta_3(q^2)\theta_3(q^6)=\sum_{k=0}^{\infty}a_kq^k$ and $\theta_3(-q^2)\theta_3(-q^6)=\sum_{k=0}^{\infty}b_kq^k$. Fix a non-negative integer k. If $k=4m$ for some $m\in\mathbb{Z}$, then $a_k=b_k$, otherwise $a_k=-b_k$. This fact together with Corollary 8.1 complete the proof. $\square$

Y. Martin in [Mar96] studied a more general class of η-quotients. By definition an *η-quotient* is a function $f(q)$ of the form

$$f(q)=\prod_{j=1}^{s}\eta^{r_j}(q^{t_j})$$

where t_j's are positive integers and r_j's are arbitrary integers. He gives an explicit finite classification of modular forms of this type which is listed in [Mar96, Table I]. In (1.11) we found

$$t_3(q)=\frac{\eta^9(q^3)}{\eta^3(q)}, \tag{8.3}$$

which is the multiplicative η-quotient $\sharp 3$ presented by Y. Martin in Table I of [Mar96]. For more details and references about this η-quotient the reader is referred to the Web page [Oei, A106402].

Remark 8.1. *If we define $\sum_{k=0}^{\infty}a_kq^k:=t_2(q)=\frac{1}{8}(E_2(q^2)-9E_2(q^6))$, then one can see that $3\mid a_k$ for integers $k\geq 1$.*

8.2 The case $n=2$

From (7.10) we get

$$Sing(\mathsf{R}_2)=\{(t_1,t_2,t_3,t_4)\,|\,t_4=t_3-t_1t_2=t_1^2+16c_2t_2^2=0\},$$

hence we find

$$p_0=(t_{1,0},\frac{1}{4}\,k_0t_{1,0},\frac{1}{4}\,k_0t_{1,0}^2,0)\in Sing(\mathsf{R}_2),$$

in which $k_0=\frac{1}{\sqrt{-c_2}}$. By comparing of the coefficients of q^1 we get $a=-t_{1,0}k_0$ and

$$p_1=(-\frac{6}{5}\frac{t_{3,1}}{k_0t_{1,0}},\frac{1}{10}\frac{t_{3,1}}{t_{1,0}},t_{3,1},-\frac{64}{5}\frac{t_{1,0}^2t_{3,1}}{k_0}),$$

where the equality $t_{4,1}=-\frac{64}{5}\frac{t_{1,0}^2t_{3,1}}{k_0}$ follows from (7.11). By considering $c_2=-\frac{1}{64}$, $t_{1,0}=\frac{1}{40}$ and $t_{3,1}=-1$, we find the solution given in (1.12) for R_2 (here also we use a $\mathsf{Singular}$ code).

R_1	t_1	t_2	t_3	R_2	$\frac{10}{6}t_1(\frac{q}{10})$	$\frac{10}{4}t_2(\frac{q}{10})$	$10^4t_4(\frac{q}{10})$
q^0	1/3	-1	0	q^0	1/24	1/8	0
q^1	2	-3	1	q^1	1	-1	1
q^2	0	-9	3	q^2	1	-5	-8
q^3	2	15	9	q^3	4	-4	12
q^4	2	-21	13	q^4	1	-13	64
q^5	0	-18	24	q^5	6	-6	-210
q^6	0	45	27	q^6	4	-20	-96
q^7	4	-24	50	q^7	8	-8	1016
q^8	0	-45	51	q^8	1	-29	-512
q^9	2	69	81	q^9	13	-13	-2043
q^{10}	0	-54	72	q^{10}	6	-30	1680
q^{11}	0	-36	120	q^{11}	12	-12	1092
q^{12}	2	105	117	q^{12}	4	-52	768
q^{13}	4	-42	170	q^{13}	14	-14	1382
q^{14}	0	-72	150	q^{14}	8	-40	-8128
q^{15}	0	90	216	q^{15}	24	-24	-2520

Table 1: Coefficients of $q^k,\ 0 \leq k \leq 15$, in the q-expansion of the solutions of R_1 and R_2.

Remark 8.2. *The demonstration of that* (1.11) *and* (1.12) *are solutions of* (1.8) *and* (1.9)*, respectively, can be done in a similar way as of Ramanujan's or Darboux's case, and in order to keep the article short, we skip the proofs and just mention that we checked the equality of first 100 coefficients of q-expansions, that we find by using of Singular, with the coefficients of (quasi-)modular forms given in* (1.11) *and* (1.12)*. in Table 1 we list the first 16 coefficients.*

The sum of positive odd divisors of a positive integer k, which is also known as *odd divisor function*, was introduced by Glaisher [Gla07] in 1907. Let k be a positive integer. We denote the sum of divisors, the sum of odd divisors and the sum of even divisors of k, by $\sigma(k)$, $\sigma^o(k)$ and $\sigma^e(k)$, respectively, i.e.,

$$\sigma(k) = \sum_{d|k} d \quad \& \quad \sigma^o(k) = \sum_{\substack{d|k \\ \text{d is odd}}} d \quad \& \quad \sigma^e(k) = \sum_{\substack{d|k \\ \text{d is even}}} d\,.$$

It is evident that $\sigma(k) = \sigma^o(k) + \sigma^e(k)$. Also one can find that

$$\sigma^o(k) = \sigma(k) - 2\sigma(k/2),$$

in which $\sigma(k/2) := 0$ if k is an odd integer. The generating function of odd divisor function is given as follow

$$\sum_{k=0}^{\infty} \sigma^o(k)q^k = \frac{1}{24}(\theta_3^4(q^2) + \theta_2^4(q^2)),$$

where by definition $\sigma^o(0) = 1/24$. On account of (1.12), one finds that

$$\sum_{k=0}^{\infty} \sigma^o(k)q^k = \frac{10}{6}t_1(\frac{q}{10}).$$

Therefore we have the following result.

Proposition 8.2. t_1 *generates the odd divisor function.*

For more details about odd divisor function one can see [Oei, A000593].

By comparing the coefficients of t_2 presented in Table 1 with the integers sequence given in the [Oei, A215947] we find that

$$\sum_{k=0}^{\infty}(\sigma^o(2k)-\sigma^e(2k))q^k=\frac{10}{4}t_2(\frac{q}{10})=\frac{1}{24}(E_2(q^2)+2E_2(q^4))$$

where we define $\sigma^o(0)-\sigma^e(0):=1/8$ (we verified this for the first 100 coeficients). Thus we get following proposition.

Proposition 8.3. *t_2 generates the function of difference between the sum of the odd divisors and the sum of the even divisors of $2k$, i.e., $\sigma^o(2k)-\sigma^e(2k)$.*

Another nice observation is about $10^4 t_4(\frac{q}{10})=\eta^8(q)\eta^8(q^2)$. The same as t_3 in the case of elliptic curve (see (8.3)), we see that t_4 is the η-quotient $\sharp 2$ classified by Y. Martin in Table I of [Mar96]. The interested reader can see [Oei, A002288] and the references given there. It is worth to point out that this η-quotient appears in the work of Heekyoung Hahn [Hah07]. She proved that $3 \mid \mu_{3k}$, $k=0,1,2,\ldots$, where μ_k is defined as follow

$$\sum_{k=0}^{\infty}\mu_k q^k := \eta^8(q)\eta^8(q^2).$$

Also she found some partition congruences by using the notion of colored partitions (for more details see [Hah07, §6]).

8.3 The case $n=4$

The set of the singularities of R_4 contains the set of $(t_1,t_2,\ldots,t_8)$'s that satisfy

$$t_6=t_3-t_1t_2=6^4c_4t_4^2-t_1^2=t_8-6^4c_4t_4^3=t_5-3t_1t_4=-t_4^2-t_2t_7=0. \tag{8.4}$$

Hence if we fix $t_{1,0}$ and $t_{2,0}$, then from (8.4) we get

$$p_0=(t_{1,0},t_{2,0},t_{1,0}t_{2,0},-\frac{1}{36k_0}t_{1,0},-\frac{1}{12k_0}t_{1,0}^2,0,-\frac{1}{1296c_4}\frac{t_{1,0}^2}{t_{2,0}},-\frac{1}{36k_0}t_{1,0}^3),$$

in which $c_4=k_0^2$. By comparing coefficients of q^1 we find

$$a=-6t_{2,0},$$

R_4	q^0	q^1	q^2	q^3	q^4	q^5	q^6
$\frac{1}{20}t_1$	$\frac{1}{720}$	1	4131	51734044	918902851011	19562918469120126	465569724397794578388
$\frac{1}{216}t_2$	$-\frac{1}{216}$	9	110703	2248267748	55181044614231	1498877559908208054	43378802521495632926652
$\frac{1}{14}t_3$	$-\frac{1}{504}$	11	115137	2265573692	54820079452449	1477052190387154386	42523861222488896739828
$\frac{1}{24}t_4$	$-\frac{1}{144}$	16	193131	3904146832	95619949713765	2594164605185043648	75018247757143686903060
$\frac{1}{2}t_5$	$-\frac{1}{144}$	45	469872	9215455916	222628516313454	5992746995783064438	172421735348939185816992
-6^6t_6	0	-1	1944	10066356	139857401664	2615615263199250	57453864811412558112
$-\frac{1}{2}t_7$	$-\frac{1}{72}$	7	32859	414746092	7395891627375	157811370338782458	3761184845284146266940
$\frac{18}{7}t_8$	$-\frac{1}{3024}$	7	54855	1034706148	24546181658391	653902684588247058	18687787944102314534628

Table 2: Coefficients of q^k, $0\leq k\leq 6$, in the q-expansion of a solution of R_4.

and

$$p_1 = \Big(\frac{60k_0t_{8,1}}{49t_{1,0}^2}, \frac{-162k_0t_{2,0}t_{8,1}}{49t_{1,0}^3}, \frac{-66k_0t_{2,0}t_{8,1}}{7t_{1,0}^2}, \frac{16t_{8,1}}{147t_{1,0}^2}, \frac{45t_{8,1}}{49t_{1,0}}, \frac{3888k_0t_{1,0}^3t_{8,1}}{49}, \frac{t_{8,1}}{1512k_0t_{1,0}t_{2,0}}, t_{8,1}\Big).$$

After fixing $k_0 = -6^{-3}$, $t_{1,0} = \frac{1}{36}$, $t_{2,0} = -1$ and $t_{8,1} = \frac{49}{18}$ we find the q-expansion of a solution of R_4. We list the first seven coefficients of q^k's in Table 2. As it was expected, after multiplying t_j's by a constant, all the coefficients are integers.

If we compute the q-expansion of Y_1^2 given in (7.14), then we find

$$\begin{aligned}\frac{1}{6}\mathsf{Y}_1^2 = 6 &+ 120960\,q + 4136832000\,q^2 + 148146924602880\,q^3 + 5420219848911544320\,q^4 \\ &+ 200623934537137119778560\,q^5 + 7478994517395643259712737280\,q^6 \\ &+ 280135301818357004749298146851840\,q^7 + 10528167289356385699173014219946393600\,q^8 \\ &+ 396658819202496234945300681212382224722560\,q^9 \\ &+ 14972930462574202465673643937107499992165427200\,q^{10} + \dots\end{aligned}$$

which is 4-point function discussed in [GMP95, Table 1, $d = 4$].

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