ON THE EXISTENCE OF RATIO LIMITS OF WEIGHTED N-GENERALIZED FIBONACCI SEQUENCES WITH ARBITRARY INITIAL CONDITIONS

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Abstract. We study ratio limits of the consecutive terms of weighted n-generalized Fibonacci sequences generated from arbitrary complex initial conditions by linear recurrences with arbitrary complex weights. We prove that if the characteristic polynomial of such a linear recurrence is asymptotically simple, then the ratio limit exists for *any* sequence generated from arbitrary nontrivial initial conditions and it is equal to the unique zero of the characteristic polynomial.

1. Introduction

Sequences generated by linear recurrences of an arbitrary order $n \geq 2$ and the ratio limits of their consecutive terms have been studied for over a half a century. The unweighted n-generalized Fibonacci sequences* with weights $(1, \ldots, 1)$ and initial conditions $(0, \ldots, 0, 1)$ were introduced and investigated in 1960 by Miles [7]. In 1967, Fielder introduced and studied the unweighted n-generalized Lucas sequences that are generated from the initial conditions $(-1, \ldots, -1, n)$ [3].

Also in 1967, Byrd [1] and, independently, Flores [4] showed that when n goes to infinity, the ratio limits of the unweighted n-generalized Fibonacci sequences converge to 2. We recently extended this result [10] to the weighted n-generalized Fibonacci sequences with weights $(p, \ldots, p)$, $p > 0$, and initial conditions $(0, \ldots, 0, 1)$ by proving that, when n goes to infinity, the ratio limits of these sequences converge to $p + 1$. In a subsequent paper [11], we further showed that for any given p and arbitrary n, these ratio limits can be represented geometrically using dilations of a collection of convex compact sets with rising dimensions n, such as n-balls, n-pyramids, n-cones, or n-simplexes.

An extensive study of ratio limits of weighted n-generalized Fibonacci sequences with complex weights and arbitrary complex initial conditions was conducted in 1997 by Dubeau et al. [2]. They proved among the other things that if the characteristic polynomial of a given linear recurrence is asymptotically simple, i.e., among the polynomial's zeros of maximal modulus there is a unique zero λ_0 of maximal multiplicity ν, then the ratio limit of the sequence $(F_k^0)_{k=-n+1}^{\infty}$, generated by this linear recurrence from the initial conditions $(0, \ldots, 0, 1)$, exists and is equal λ_0. The authors also showed that if a sequence $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty}$ generated from arbitrary complex initial conditions $\mathbf{a} = (a_{-n+1}, \ldots, a_0)$ by a linear recurrence with an asymptotically simple characteristic polynomial satisfies

$$\lim_{k\to\infty} \left(F_k^{\mathbf{a}}/k^{\nu-1}\lambda_0^k\right) \neq 0, \tag{1.1}$$

then the sequence's ratio limit exists and equals λ_0 as well.

In this paper, we show that condition (1.1) is redundant, i.e., we prove that if the characteristic polynomial of a given linear recurrence with complex weights is asymptotically simple, then the ratio limit exist and is equal to λ_0 for *any* sequence generated by this linear recurrence from arbitrary complex initial conditions.

2. Main result

Given a linear recurrence of an order $n \geq 2$ with the characteristic polynomial

$$\lambda^n - b_1\lambda^{n-1} - \cdots - b_n \tag{2.1}$$

with complex weights b_i, $i = 1, \ldots, n$, such that $b_n \neq 0$. We study sequences $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty}$ generated by this liner recurrence from arbitrary nontrivial complex initial conditions $\mathbf{a} = (a_{-n+1}, \ldots, a_0)$, i.e.,

$$F_k^{\mathbf{a}} = b_1F_{k-1}^{\mathbf{a}} + \cdots + b_nF_{k-n}^{\mathbf{a}}, \ \ k > 0, \ \text{ and } \ F_k^{\mathbf{a}} \equiv a_k, \ \ k = -n+1, \ldots, 0. \tag{2.2}$$

*They are also referred to as n-step Fibonacci sequences [6].

If the ratio limit of the consecutive terms of a sequence $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty}$ exists, we denote it as

$$\Phi^{\mathbf{a}} = \lim_{k_0<k\to\infty} \left(F_{k+1}^{\mathbf{a}}/F_k^{\mathbf{a}}\right), \text{ where } F_k^{\mathbf{a}} \neq 0 \text{ for } k > k_0. \tag{2.3}$$

The assumptions that the initial conditions are nontrivial and that the weight $b_n \neq 0$ imply that no more than $n-1$ consecutive elements of the sequence $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty}$ are equal to zero. Let $F_{k'}^{\mathbf{a}}$ be a nonzero element. Obviously, finding the ratio limit $\Phi^{\mathbf{a}}$ of the sequence $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty}$ is equivalent to finding the ratio limit $\Phi^{\mathbf{a}'} = \Phi^{\mathbf{a}}$ of the sequence $(F_k^{\mathbf{a}'})_{k=-n+1}^{\infty}$ with initial conditions $\mathbf{a}' = (a'_{-n+1}, \dots, a'_0) = (F_{k'}^{\mathbf{a}}, \dots, F_{k'+n-1}^{\mathbf{a}})$, i.e, where $a'_{-n+1} \neq 0$.

Theorem 2.1. *Given a linear recurrence with complex weights $(b_1, \dots, b_n)$, $b_n \neq 0$, let $(F_k^{\mathbf{a}})_{k=-n+1}^{\infty}$ be a sequence generated by this linear recurrence from arbitrary complex initial conditions $\mathbf{a} = (a_{-n+1}, \dots, a_{-n+1})$, $a_{-n+1} \neq 0$, and let $(F_k^0)_{k=-n+1}^{\infty}$ be a sequence generated from the initial conditions $(0, \dots, 0, 1)$.*

(i) If there exists k_0 such that $F_k^0 \neq 0$ for $k > k_0$, then $F_k^{\mathbf{a}} \neq 0$ for $k > k_0 + n - 1$.

(ii) If the characteristic polynomial of the linear recurrence is asymptotically simple, then the ratio limit $\Phi^{\mathbf{a}}$ exists for any nontrivial initial conditions $\mathbf{a}$ and it coincides with the characteristic polynomial's unique zero λ_0.

Proof. (i) Dubeau et al. [2] showed that elements of the sequence $(F_k^{\mathbf{a}})_{k=0}^{\infty}$ can be expressed by elements of the sequence $(F_k^0)_{k=-n+1}^{\infty}$ in the following way:

$$F_k^{\mathbf{a}} = a_0 F_k^0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F_{k-j}^0. \tag{2.4}$$

Let us assume that $F_{k_1}^{\mathbf{a}} = 0$ for some $k_1 > k_0 + n - 1$. Due to our assumptions, the sum in the RHS of equation (2.4) with $k = k_1$ includes the nonzero term $a_{-n+1} b_n F_{k_1-n+1}^0$. Thus, the sum can equal zero only if it contains *at least one more* nonzero term so that all the terms add up to zero.

Consequently, the following linear relation between the sequence elements F_l^0, $l = k_1 - n + 1, \dots, k_1$, is implied by our assumption that $F_{k_1}^{\mathbf{a}} = 0$ for $k_1 > k_0 + n - 1$:

$$F_{k_1-n+1}^0 = c_1 F_{k_1}^0 + \dots + c_{n-2} F_{k_1-n+2}^0, \tag{2.5}$$

where coefficients c_i are determined by weights $(b_2, \dots, b_n)$ and initial conditions $(a_{-n+1}, \dots, a_0)$, and *at least one* of the c_is is not equal to zero.

Inserting the relation (2.5) in formula (2.2) with $k = k_1 + 1$ and initial conditions $(0, \dots, 0, 1)$ allows us to express the sequence element $F_{k_1+1}^0$ by its predecessors without using the sequence element $F_{k_1-n+1}^0$, i.e, relying on a new set of weights $(b'_1, \dots, b'_{n-1}, 0)$. We conclude by induction that the latter is also true for all elements of the sequence $(F_k^0)_{k=k_1+1}^{\infty}$. But this contradicts our assumption that characteristic polynomial (2.1) includes $b_n \neq 0$.

(ii) If the characteristic polynomial of the linear recurrence is asymptotically simple, the ratio limit $\Phi^0 = \lim_{k_0<k\to\infty} \left(F_{k+1}^0/F_k^0\right)$, where $F_k^0 \neq 0$ for $k > k_0$, of the sequence $(F_k^0)_{k=k_0+1}^{\infty}$ exists and equals λ_0. Moreover, part (i) of the theorem implies that for $k > k_0 + n - 1$, we have $F_k^{\mathbf{a}} \neq 0$. Thus, it follows from formula (2.4) that

$$\begin{aligned}
\lim_{k_0<k\to\infty} \left(F_{k+n}^{\mathbf{a}}/F_{k+n-1}^{\mathbf{a}}\right) &= \lim_{k_0<k\to\infty} \frac{a_0 F_{k+n}^0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F_{k+n-j}^0}{a_0 F_{k+n-1}^0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F_{k+n-1-j}^0} = \\
&= \lim_{k_0<k\to\infty} \frac{a_0 F_{k+n}^0/F_{k+n-1}^0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F_{k+n-j}^0/F_{k+n-1}^0}{a_0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} F_{k+n-1-j}^0/F_{k+n-1}^0} = \\
&= \lim_{k_0<k\to\infty} \frac{a_0 \Phi^0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} (\Phi^0)^{-j+1}}{a_0 + \sum_{i=1}^{n-1} a_{-i} \sum_{j=1}^{n-i} b_{i+j} (\Phi^0)^{-j}} = \Phi^0,
\end{aligned} \tag{2.6}$$

i.e., the limit $\Phi^{\mathbf{a}}$ exists and equals λ_0.

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3. Conclusions

A criterion proven in 1966 by Ostrowski [8, Theorem 12.2] states that a linear recurrence with weights $b_i \geq 0$, $i = 1, \ldots, n$, has an asymptotically simple polynomial with unique dominant zero λ_0 of multiplicity $\nu = 1$ if the gcd of indices j corresponding to positive weights b_j equals 1. Thus, in the case of a linear recurrence with nonnegative weights, our theorem allows establishing whether *all* sequences generated by such a linear recurrence from arbitrary nontrivial initial conditions have the same ratio limit λ_0 by simply finding the gcd of the positive weights.

In particular, our theorem implies that for a given n, all sequences with weights $(1, \ldots, 1)$ and arbitrary initial conditions, e.g., the n-generalized Lucas sequence, have the same ratio limit as the n-generalized Fibonacci sequence. Also, it follows from our theorem that over 340 integer sequences with signatures $(m, \ldots, m)$, $m \in \mathbb{N}$, and a variety of initial conditions, cataloged in the Sloane's *Online Encyclopedia of Integer Sequences* [9], have their ratio limits equal to the dominant zero of the corresponding characteristic polynomial. Moreover, according to our theorem, the results proven by us for the weighted n-generalized Fibonacci sequences with weights $(p, \ldots, p)$, $p > 0$, and initial conditions $(0, 0 \ldots, 1)$ [10, 11], are, in fact, applicable for all sequences with such weights and arbitrary nontrivial initial conditions.

The determination whether all sequences generated by a linear recurrences with complex weights have the same ratio limit equal to the unique zero λ_0 of its characteristic polynomial can be achieved by using a criterion introduced by Dubeau et al. [2] that, as the authors showed, is valid in numerous cases, cf. [2, Theorem 15]. According to this criterion, if for any characteristic polynomial's zero λ it holds

$$\sum_{j=1}^{n-1} \Big| \sum_{i=j}^{n-1} \frac{b_{1+i}}{\lambda^{1+i}} \Big| < 1, \tag{3.1}$$

then the characteristic polynomial is asymptotically simple with the dominant zero $\lambda = \lambda_0$ of multiplicity $\nu = 1$.

Of course, our theorem implies that *all integer* sequences that are generated by a linear recurrence that satisfies either of the criteria described above have the same ratio limit equal to the dominant zero λ_0 of the linear recurrence's characteristic polynomial. The OEIS [9] and Khovanova's website *Recursive Sequences* [5] catalogs and describes applications of thousands of such integer sequences.

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MSC2010: 11B37, 11B39

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