Convergence of the Gutt star product

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Chapter 1

Introduction

Mathematical Physics

Throughout history, the fields of mathematics and physics have been closely linked to each other. The great physicists of the past have always been great mathematicians and vice versa: Carl Friedrich Gauss, one of the most brilliant mathematicians, did not only find plenty of mathematical relations, prove a lot of theorems and develop many new ideas, which should become rich and fruitful new fields in later mathematics, but he also has a large number of credits in physics: the recovery of the dwarf planet Ceres in astronomy, new results in electromagnetism (like the Gauss's law or a representation for the unit of magnetism, which was named after him) and the Gaussian lens formula in geometric optics are just some of his best known merits. Isaac Newton on the other hand may have been rather a physicist, but it was his so-called second law of motion that used for the first time something like differential calculus and therefore opened up the door for new branches of mathematics. Even if one want claims that differential calculus was actually invented by Gottfried Wilhelm Leibniz, this does not change the basic observation, since Leibniz wrote a lot of essays on physics and can be considered as the inventor of the concept of kinetic energy (or, as he called it, the vis viva) and its conservation in certain mechanical systems. Of course one has to name Joseph-Louis Lagrange, who was an ingenious mathematician with rich contributions to number theory and algebra, but also to the fields of analytical mechanics and astronomy. Still today, every physics student has to learn his Lagrangian formalism in the lecture on theoretical mechanics and how one can derive the laws of motion for various mechanical systems from it. A last name we want to mention here is Paul Dirac, who is certainly one of the founding fathers of quantum mechanics. He provided the ideas for a lot of structures and relations in differential geometry, functional analysis and distribution theory. Many of the concept he introduced using his physicist's intuition have been later proven to be correct or used as starting points for new theories by mathematicians.

A lot of developments in mathematics can be seen as triggered by physics: they were necessary to describe the physical behaviour of our world and therefore pushed forward by scientists. We already mentioned differential calculus, without whom modern analysis, the theory of ordinary or partial differential equations or differential geometry would not be possible. Besides the also named field of functional analysis, also Lie theory and many parts of geometry provide examples for mathematics which was inspired by physics. Of course, this correspondence is not a one way street since the understanding of nature made great progress due to a better knowledge of the mathematical laws that describe it. A good example fir this is Lebesgue's theory of integration and its application to quantum mechanics: the space $L^2(\mathbb{R}^{3n})$ is the state space of standard

n-particle system in quantum mechanics.

There are good reasons to say that this extremely tight binding of mathematics and physics persisted until the 20th century. Without any doubt, these two areas are still closely linked, but one could say that at a certain point in history they started walking away from each other. Of course, there have always been mathematicians who did not take their motivation from physics and physicists who did not use elaborate mathematics or even find new theorems to describe aspects of the world around them, but for a long time, the vast majority of both groups showed at least an interest for the other domain. This definitely changed during the 20th century. The main reason for this can surely be found in the extremely fast development which both of the domains experienced in this time. It is already impossible for one person to overview the whole field of mathematics or the one of physics, since there are too many new things coming up every day and one has to specialize for being able to do research. Another reason is surely the fact that modern mathematics is strongly influenced by the desire to formulate things as clean as possible, without using handwaving "physical arguments". This is a principle which surely allowed many new and fruitful evolutions in the last decades and which is mostly due to the Bourbaki movement in the middle of the last century, but it also forgets about the fact that physical intuition was often a powerful tool for new ideas or also for heuristics which led to proofs of important theorems. Another reason, which is more situated in the domain of physics, is certainly the incredibly fast development of the knowledge about semiconductors. This became possible due to quantum mechanics which forms the foundation of this theory, but for very most of modern applications a basic understanding of the quantum theory behind is enough or one can even get new results with so called semi-classical approaches. Here, a lot of new results can be established without going deep into mathematics and hence without giving a new stimulus to it. In this sense, it is enough for many modern physicists to acquire a certain amount of mathematical knowledge and then they never have to care about mathematical theories again.

Certainly, this situation is due to a natural development does not present a problem, although it is a bit of a pity. However, it would be too much to say that those fields are falling apart: there are still a lot of intersections of the two sciences and these contact areas provide rich domains of research. The range of topics, where either mathematics takes its motivation from physics, or where theoretical physics needs very elaborate mathematical methods, is usually grouped under the name *mathematical physics*.

A Mathematical Theory of Quantization

One of the younger branches in this field is the theory of quantization. It belongs to the area of pure mathematics, but takes its inspiration from physics and is therefore a part of mathematical physics. The idea is, roughly spoken, to find a correspondence between the quantum and the classical world in physics. The mathematical description of their laws are different but yet there are a lot of similarities. It is more or less clear, how the classical world emerges out of a huge number of quantum objects and the mathematics of classical mechanics can be understood as a limit case the behaviour of n quantum objects where $n \longrightarrow \infty$. The other way round, it is not clear how one can create the mathematical description of a quantum system out of the one of a classical system. This reversed process is usually called *quantization* and its understanding is a mathematical task, not a physical one.

One has to say that the question of *how to quantize a given classical system* is still far from being well understood and there are various approaches to it. We will give an overview of them in Chapter 2.2.3. However, all these attempts to create a mathematical theory of quantization have led to many new developments in mathematics, also in other fields: they had a strong drawback on differential and algebraic geometry, Lie theory and functional analysis. In particular the theory of Poisson Lie groups is completely due to them.

Deformation Quantization and how this Master Thesis fits in

Deformation quantization is not the only one, but maybe the most developed theory in this area. Roughly speaking, it tries to *deform* the (idealized) commutative algebra of smooth functions on a manifold (of classical physical observables) by making it noncommutative and wants to get an (idealized) algebra (of quantum mechanical observables) this way. This is done by replacing the pointwise product of functions with a noncommutative product, which takes into account certain derivatives of the functions and plugs in a formal parameter which is called $\hbar$ and which should correspond to Planck's constant in physics. This new product is then called a star product. There are basically two different ways of constructing such a star product: either by integral formulas, or by formal power series in $\hbar$. These two approaches are linked by the fact that one gets a formal series out of the integral formulas by an asymptotic expansion. This mostly algebraic theory of formal power series is called formal deformation quantization, and it is quite well understood: although there are still a lot of open questions remaining, many existence and classification results for star products were found, there is also a light physical interpretation: The zeroth order of the formal parameter represents classical mechanics and the first order quantum mechanics. Different mechanical systems allow different star products, and of course it is interesting to ask whether some star products are more "natural" for a given system than others.

Besides the algebraic aspects of this theory, one can also speak about convergence of these formal series or study the integral formulas: so one can ask the question, if the deformation is continuous or smooth in some sense. This leads to the field of *strict deformation quantization*. Here, no closed theory exists yet and there are just very few things known about *continuous* star products. The approach via integral formulas is better developed at the moment, but it is only possible for finite-dimensional systems. For a formal power series, the first question is of course in what topology one wants it to converge. Once one has an answer to it, one has to do some more or less complicated analysis of the formal power series, in order to control it somehow.

For this second approach via formal power series, there are just very few examples of continuous star products known. They are all related to classical systems, which can be described as symplectic vector spaces. This master thesis presents a new example of a formal star product, which can be turned into a convergent series and which does not come from a symplectic vector spaces, but from a class of vector spaces with a more general (Poisson) structures. So it enlarges this theory of strict deformation quantization by a very substantial and big group of examples and provides new scientific results. The core part of it is hence made public as a preprint [42] by now. This preprint will not be cited again in this thesis, but it is clear that the latter contains the whole content of the former and even a bit more.

Besides the fact, that this work presents a contribution to the theory of quantization, its main parts are also of independent interest. The topologized, deformed algebra, which is treated here is nothing else but the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. So it also has an impact on Lie theory, since there is no canonical topology on $\mathscr{U}(\mathfrak{g})$: we give a candidate for it by an explicit and functorial construction, which also applies to some infinite-dimensional Lie algebras. To the best of our knowledge, such a topology for infinite-dimensional cases has not been known before.

An Outlook

Although (or maybe because) we found many new results, we have to face a lot of open questions. The first one is of course, how far we can generalize the construction of the topology on the universal enveloping algebra and how large its completion can be. It is almost answered by this thesis, but some limiting cases are left to consider, as pointed out in the Chapters 6.2 and 6.4. Another question considers the representation theory of $\mathscr{U}(\mathfrak{g})$: now we have a topology and can ask for continuous representations, for example. Some more questions rather touch the field of deformation quantization again. First: we can think of continuous star products not only on vector spaces, but also on some really geometric manifolds, namely on the coadjoint orbits of a Lie group G. The results of this work are a very big step into this direction and may allow new examples of strict deformations. There is of course still work to be done, but this would be the very first class of such examples. Second: we can think of more complicated classes of quantized classical mechanical systems on vector spaces and try to make those more general star products convergent, too. The final aim would be to find results for the so-called Kontsevich construction, which is somehow the biggest result in formal deformation quantization. The question, how this can be transferred into a strict setting is completely open. Third: a big class of star products can be generated using twists. That is a technique, which uses the Hopf algebra structure of $\mathscr{U}(\mathfrak{g})$. Since we now have a topology, we can think of convergent twists and ask, if this leads to a convergent (non-formal) power series, hence to a continuous star product. This question is closely linked to symmetries of the quantized system and could lead to a better understanding of such things as “quantized symmetries” which have yet not been understood at all.

Summary and Organization

Summary

This work focusses on a particular star product, the so-called Gutt star product. It can be established on a certain class of classical mechanical systems, which turn out to be vector spaces with a particular type of biderivation (a Poisson bracket) on their algebra of smooth functions. The aim of this thesis is to find a large subalgebra of the smooth functions and a (locally convex) topology on them, such that the Gutt star product is continuous and that the commutative classical algebra can be deformed smoothly into the noncommutative quantum algebra. Of course, we must give a proper definition how this smoothness is meant. We also establish many helpful properties of this construction, like a certain functoriality. Moreover, this work relates to other fields of mathematics, such as Lie theory. For example, the quantized algebra of smooth functions is closely linked to a universal enveloping algebra of a Lie algebra, which naturally appears for the considered classical systems. Therefore, it carries the structure of a Hopf algebra, which is also continuous with respect to the constructed topology.

Organization

This master thesis is organized as follows: In Chapter 2, we will introduce the most important concepts of classical and quantum mechanics, explain their relations and give an overview over the field of deformation quantization, its history and its current state. We will also classify it this thesis into the theory of deformation quantization. We will also give an outlook on the next steps, which can be done using the results of this work. In the third chapter, we will explain in more detail the kind of Poisson systems this thesis deals with and see that they are in fact Lie algebras. We will construct the Gutt star product, which is characteristic for those systems, in different ways and show that these constructions are equal. We will explain the link to Lie theory, the Poincaré-Birkhoff-Witt and the Baker-Campbell-Hausdorff theorem and will introduce those results on the Baker-Campbell-Hausdorff series, which we will need for our later work. Chapter four is devoted to finding explicit formulas for the Gutt star product and explaining them using two examples, as well as to some easy conclusions one can draw from those formulas. Chapter five is the core of this thesis. First, we will introduce briefly the concept of locally convex topologies and explain why they are the convenient setting for our task. We will show in detail how the locally convex topology for the Gutt star product is constructed, what their properties are and what kind of topology our Poisson system must have had such that the construction of our topology is possible. At this point, we will introduce the concept of *asymptotic estimate algebras*, which can be seen as a concept between locally multiplicatively convex algebras a general locally convex ones. Then we will show that the Gutt star product is indeed continuous with respect to our topology, that the deformation is analytic (even entire, if the underlying field is $\mathbb{C}$), that the construction is functorial and we will analyse the completion of this algebra. We will also show that this topology is optimal in a certain way. The sixth chapter is devoted to particular systems, namely to nilpotent Lie algebras. We will show how the results we found previously can be improved, but we will also see the limits of our construction. We will establish the link to a work of Stefan Waldmann's and we will show that we come to the same conclusion by taking a different way using the Gutt star product. Finally, we will see that those stronger results are not limited to the very case of strictly nilpotent Lie algebras, but that there are (in infinite dimensions) weaker notions of nilpotency which lead to the same result, when the construction of the topology is adapted a bit. In the end, Chapter seven treats the Hopf algebraic part which is very short due to the algebraic properties of the deformation. We will see that the coproduct and the antipode remain undeformed and continuous with respect to our topology, too.

Summary and Organization (German)

Diese Masterarbeit befasst sich mit dem Gutt-Sternprodukt. Es kann für bestimmte Poissonmannigfaltigkeiten definiert werden, nämlich für Vektorräume mit linearem Poissontensor. Die Poissonmannigfaltigkeit stellt dann den Dualraum einer Lie-Algebra dar. Auf der Polynomalgebra über diesem Dualraum, bzw. der symmetrischen Tensoralgebra über der Lie-Algebra, lässt sich dann das Gutt-Sternprodukt definieren, welches als formale Deformation der symmetrischen Tensoralgebra aufgefasst werden kann. Im Rahmen dieser Arbeit wird mit Hilfe einer von Waldmann entwickelten Konstruktion die symmetrische Tensoralgebra lokal-konvex topologisiert und dann der Nachweis erbracht, dass das Gutt-Sternprodukt in dieser Topologie tatsächlich stetig ist. Das besondere bei diesem Vorgehen ist, dass es sich insbesondere auch auf eine große Klasse von unendlich-dimensionalen Lie-Algebren anwenden lässt. Damit trägt diese Arbeit neue wissenschaftliche Ergebnisse zum Themengebiet der strikten Deformationsquantisierung bei.

Im nächsten Abschnitt, Kapitel 2, wird der Begriff "Quantisierung" und insbesondere der Themenbereich der Deformationsquantisierung näher erläutert und die Fragestellung motiviert. Im dritten Kapitel werden die notwendigen Vorkenntnisse über Kirillov-Kostant-Souriau-Strukturen, das Gutt-Sternprodukt und die beiden Theoreme nach Poincaré, Birkhoff und Witt sowie Baker, Campbell und Hausdorff vermittelt. Kapitel 4 beschäftigt sich mit der Herleitung verschiedener Formeln für das Gutt-Sternprodukt. Im fünften Kapitel wird dann die lokal-konvexe Topologie auf der Tensoralgebra eingeführt sowie die notwendige Unterkategorie lokal-konvexer Lie-Algebren erklärt, mit der wir uns befassen. Danach wird die Stetigkeit des Sternprodukts bewiesen, ebenso wie die analytische Abhängigkeit vom Deformationsparameter, die Funktorialität

der Konstruktion und die Optimalität des Ergebnisses. Kapitel 6 beschäftigt sich mit nilpotenten Lie-Algebren und damit, wie weit sich die bisherigen Ergebnisse in diesem Fall ausbauen lassen. Darüber hinaus wird der Zusammenhang mit einer Arbeit von Waldmann bezüglich der Weyl-Algebra aufgezeigt. Im siebten Kapitel wird gezeigt, dass neben der assoziativen Algebra- auch eine Hopf-Struktur vorliegt, die ebenfalls lokal-konvex topologiert werden kann und stetig ist.

Thanks to ...

I want to thank my advisor, Stefan Waldmann, for the time he invested in this work and for his intense supervision. I am also very grateful to Chiara Esposito, who also spent a lot of time in discussion with me and was so kind to offer me a place in her office, so that I had a place at the chair where I could work. It is mainly due to both of them, that this work progressed so quickly and finally became a preprint which will hopefully become a publication in the next time. I am also grateful to Matthias Schötz and Thorsten Reichert for many fruitful discussions and their patience with me bothering them with questions. Moreover, I am grateful for the support of the Studienstiftung and the Max-Weber Programm, who financed my studies and without whom my life would have looked very differently during the past six years. Finally, I want to say thank you to my parents who supported me all the time of school and studies, who encouraged me and who awakened my interest for the understanding of nature by their way of educating me. A last thank you is dedicated to my girlfriend for encouraging and supporting me when I was unmotivated and for not being angry when I spent a lot of time working on this thesis. :-)

Chapter 2

Deformation quantization

The starting point for every theory of quantization is without any doubt the theory of mechanics. Here, mechanics means both, the classical and the quantum theory. Since we want to explain how one can link those two together, we will give a short overview of both theories and show their similarities and their differences in the first section of this chapter. Afterwards we will explain what a quantization should actually be and collect the existing approaches to it. In Section three, we will focus on Deformation quantization and give an overview of this rather young theory.

2.1 Mechanics: The Classical and the Quantum World

2.1.1 Classical Mechanical Systems

We want to briefly recall the notions of classical mechanics. There are many good books on this subject and also the notation for the basic concepts is more or less uniform everywhere. Nevertheless, we want to refer to the books of Marsden and Ratiu [63] and Arnold [3], which give very good introductions and overviews of the theory. Imagine the simplest model for a mechanical system, which is not trivial: a single particle with mass m moving in $\mathbb{R}^3$ in a scalar potential V. We will denote its position by $q = (q^1, q^2, q^3) \in Q = \mathbb{R}^3$ and call the set Q of all possible positions the *configuration space*. Since we also have a time coordinate $t \in \mathbb{R}$, we can describe the path on which the particle moves by a parametrized curve $q(t)$. The state of the particle is completely described by its position and its velocity $(q(t), \dot{q}(t))$ and the velocity should be understood as a tangent vector $\dot{q}(t) \in T_{q(t)}Q = \mathbb{R}^3$. Therefore, the tangent bundle TQ, which is in this case $\mathbb{R}^6$, is sometimes called the state space. In classical mechanics, we can describe the movement of the particle using the Euler-Lagrange equations, which are derived from the so-called Lagrange function by a variational principle. The Lagrange function of this system reads

$$\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - V(q, \dot{q})$$

and $T(q, \dot{q}) = \frac{m}{2} \sum_{i=1}^{3} \left(\dot{q}^i\right)^2$ is the kinetic energy. The action along a path is defined as the integral

$$\mathcal{S}(q, \dot{q}) = \int_{t_0}^{t_1} \mathcal{L}(q, \dot{q}) dt.$$

It is a physical observation, similar to a mathematical axiom, that this action functional is stationary along the trajectories of the particle. So by fixing a starting point q_0 and an end point

q_1 of a trajectory, we find

$$\delta \mathcal{S}\big|_{q(t_0)=q_0, q(t_1)=q_1} = 0.$$

From this, one finds the Euler-Lagrange equations for $i = 1, 2, 3$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0.$$

In our case this means

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^i} - \frac{\partial V}{\partial q^i} = m\ddot{q} - \frac{\partial V}{\partial q^i} = 0$$

and we can integrate the equations to get the trajectory. So far, we described the system in the state space. This description is called the Lagrange formalism. It is not the only picture, which describes to behaviour of the system: one can go to the so-called Hamilton formalism, which is based on the conjugate momenta $p = (p_1, p_2, p_3) \in T^*_{q(t)}Q$

$$p_i(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

and linked to the Lagrange formalism by a Legendre transformation. The set T^*Q also describes all possible states of the system and is called the phase space. Now we can define the Hamilton function

$$H(q, p) = p_i q^i - \mathcal{L}(q, \dot{q}),$$

which is the crucial quantity in this setting. It represents the energy of the system. One finds the equations

$$\frac{\partial H}{\partial p_i} = \frac{dq^i}{dt} \quad \text{and} \quad -\frac{\partial H}{\partial q^i} = \frac{dp_i}{dt}.$$

They strongly remind of a symplectic structure and indeed this is the case: using the standard sympectic matrix

$$\omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

one finds

$$\frac{d}{dt}\begin{pmatrix} q^i \\ p_i \end{pmatrix} = \omega \begin{pmatrix} \frac{\partial H}{\partial q^i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix}.$$

More generally, one can define the Poisson bracket $\{f, g\}$ for two functions $\mathscr{C}^\infty(T^*Q)$ by

$$\{f, g\} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \right)$$

and finds the time evolution of a function f given by

$$\frac{d}{dt} f = \{f, H\}.$$

All of those objects have, of course, geometrical interpretations which allow generalizations from this very easy example. For a more complex system than one particle in a scalar potential, the configuration space Q is a smooth manifold, the state space TQ is described by the tangent bundle and the phase space T^*Q by the cotangent bundle. Points in the phase space, which can also be understood as Dirac measures in T^*Q, describe the possible states of the system. This interpretation allows us to speak of positive Borel measures on T^*Q as mixed states, which describe a probabilistic distribution of the state of the system. The transition from the Lagrange to the Hamilton formalism is done by a fiber derivative, the kinetic energy has a mathematical interpretation as a Riemannian metric and the Poisson bracket is the one which is due to the canonical symplectic form on the cotangent bundle. So in some sense, classical mechanics can be described by symplectic geometry.

The last conclusion, however, was a bit too fast. There are mechanical systems like the rigid body, which can not be described in the symplectic formalism, or which have certain symmetries and therefore allow a *reduction.* If a given system has a symmetry (which is described by a certain type of Lie group action), there will be mathematical tools, which allow to divide out a part of phase space. One gets a reduced phase space, which is a quotient of T^*Q and which, in general, is not symplectic any more. In these cases, we end up with a more general structure than a symplectic manifold. The Poisson bracket however is still be there. So the objects we actually want to use in order to describe classical mechanics are *Poisson manifolds.*

Definition 2.1.1 (Poisson Manifold) *A Poisson manifold is a pair $(M, \{\cdot,\cdot\})$ of a smooth manifold M and a Poisson bracket $\{\cdot,\cdot\}$. The bracket is a biderivation*

$$\{\cdot,\cdot\}\colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M),$$

which is anti-symmetric and fulfils the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

It is worth mentioning that sometimes, it will be more convenient not to talk about the Poisson bracket $\{\cdot,\cdot\}$, but about the *Poisson tensor* $\pi \in \Gamma^\infty(\Lambda^2(TV))$, which is a equivalent description. Like the bracket, the tensor has to fulfil the Jacobi identity. We have for all $f, g \in \mathscr{C}^\infty(M)$ and $x \in M$

$$\{f,g\}(x) = \pi\big|_x(\mathrm{d}f, \mathrm{d}g)(x).$$

The "physical observables" in this setting are the smooth functions on the Poisson manifold. Together with the Poisson bracket, they form the prototype of a *Poisson algebra*: an algebra with a bracket, which is an antisymmetric biderivation and which fulfils the Jacobi identity. The range, sometimes also called the spectrum, of a given smooth function has the physical interpretation as the measurable values of this observable. So in full generality, we could say that a classical mechanical system is a Poisson manifold together with a Hamilton function H, and the time evolution of every observable $f \in \mathscr{C}^\infty(M)$ is given by the relation

$$\frac{d}{dt} f(t) = \{f(t), H\}.$$

We will soon see that this formalism is as close as we can get to the one of quantum mechanics.

2.1.2 Quantum Mechanics

In quantum theory, the model of a physical system looks completely different on the first sight. There is indeed no obvious way to describe how this formalism was derived out of the one describing classical mechanics. It was a hard and non-straightforward way, taken in small steps of which some were more or less obvious and some were ingenious guesses. Built up on first ideas by Max Planck, who described the radiation of a black body, Albert Einstein, who explained the photo-electrical effect and Niels Bohr, who solved the problem of atomic spectra, a hand full of physicists including Max Born, Werner Heisenberg, Erwin Schrödinger, Paul Dirac, Wolfgang Pauli and Pascual Jordan developed a complex, counter-intuitive but extremely well working theory, which was able to give precise answers to very most of the open questions at that time. However, this theory, mainly risen between 1925 and 1928, did not have a satisfying mathematical fundament. It took around ten more years until mathematicians, mainly John von Neumann, worked out a mathematical theory for quantum mechanics and gave also a physical interpretation to their mathematical formulation [94]. Today, quantum mechanics is usually introduced as a mathematical theory based on axioms and most of the textbooks (at least most of the more mathematical ones) use this axiomatical approach to explain the theory. Two very nice mathematical introductions are given by Bongaarts [19] and Hall [52].

The state of a physical system is described by a vector ψ on a Hilbert space $\mathfrak{H}$ and the observables are self-adjoint (and usually unbounded) operators on it. The spectra of these operators describe their measurable values. The theory is probabilistic, but has a deterministic time evolution: the pair (A, ψ) gives a probabilistic distribution, from which we can calculate the probability to measure A with the value $a \in \operatorname{spec}(A)$ in the state ψ using the spectral resolution of A. We want to look again at the example of a particle in $\mathbb{R}^3$ moving in a potential. Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ be a coordinate vector, then we have the so-called Schrödinger representation given by the Hilbert space $\mathfrak{H} = \mathrm{L}^2(\mathbb{R}^3)$ of square integrable functions with the Lebesgue measure. The state ψ of a particle is a time-dependent element of this Hilbert space which fulfils the Schrödinger equation

$$i\hbar \frac{d}{dt}\psi(x,t) = \hat{H}\psi(x,t) \tag{2.1.1}$$

where $\hat{H}$ denotes the Hamilton operator which is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(x), \tag{2.1.2}$$

where Δ denotes the Laplace operators and V the potential. Similar to the Hamilton function in classical mechanics, the Hamilton operator (or more precisely: its spectrum) describes the energy of this system, m is the mass of the particle and V the potential. The operators which correspond to the coordinate x_i and the momentum p_j of the particle are given by

$$\begin{aligned}(\hat{q}_i\psi)(x,t) &= x_i\psi(x,t)\\ (\hat{p}_j\psi)(x,t) &= -i\hbar\frac{\partial}{\partial x_j}\psi(x,t).\end{aligned}$$

This turns Equation (2.1.1) into

$$i\hbar\frac{d}{dt}\psi(x,t) = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\psi(x,t)$$

and looks therefore similar to the classical time evolution. Since the state is a function of the coordinate x, one calls this the representation in *position space*. Sometimes it is more convenient to describe a state in terms of its momentum and one changes to *momentum space* via Fourier transformation. Both representations are equivalent, very similar to the variables q and p in the Hamilton form of classical mechanics, from which this picture is very much inspired. Unlike the classical position and momentum observables being functions, the quantum mechanical observables are *operators* and do not commute any longer. One can check these by calculating commutators and finds the so-called Heisenberg or *classical commutations relations*

$$[p_i, p_j] = [q_i, q_j] = 0, \tag{2.1.3}$$

$$[q_i, p_j] = i\hbar\delta_{ij}. \tag{2.1.4}$$

These relations are the starting point for all quantization theories. The main goal is always to find an algebra which fulfils these (or maybe equivalent) relations and this is what makes a quantum algebra so different from a classical one. Physically spoken, this noncommutativity means that a measurement of the position of a particle influences its momentum and vice versa, such that it becomes impossible to measure both, position and momentum, at the same time with arbitrarily high precision. This still very vague statement can be made precise using the Hilbert norm and the Cauchy-Schwarz inequality and yields the famous Heisenberg uncertainty relation

$$\Delta A_\psi \Delta B_\psi \geq \frac{1}{2}|\langle\psi, [A, B]\psi\rangle|.$$

Here A, B are observables and $\Delta A_\psi, \Delta B_\psi$ denote their standard deviations in the state ψ. For the position and momentum of a particle, this gives

$$\Delta\hat{p}\Delta\hat{q} \geq \frac{\hbar}{2}.$$

However, the Schrödinger picture is not the only possibility to describe the behaviour of a quantum system. Heisenberg proposed a description in which not the particles, but the observables depend on the time. Since quantum mechanics, time evolution is described by a one-parameter group of unitary operators, we have

$$\psi(x, t) = U(t - t_0)\psi(x, t_0),$$

at least in systems where the interactions are not explicitly time dependent. This makes it possible to apply unitary transformations to the states as well as to the operators

$$\psi(x) = \psi(x, t_0) = U(t_0 - t)\psi(x, t) \quad \text{and} \quad A(t) = U(t_0 - t)AU(t - t_0).$$

One gets the so called *Heisenberg picture*, in which the time-dependence is not expressed in the states but in the operators. Both pictures are equivalent and it is just a question of convenience which one is used. In our example, the unitary operators are given by

$$U(t - t_0) = \mathrm{e}^{-\frac{i}{\hbar}\hat{H}t},$$

since they describe the time evolution of solutions of the Schrödinger equation. If one differentiates an operator with respect to t in the Heisenberg picture, one finds the *Heisenberg equation*

$$\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A(t), H].$$

This reminds us of course of the time evolution in terms of Poisson brackets in classical mechanics. The idea that both objects, the quantum mechanical commutator and the Poisson bracket, should somehow be seen as counterparts to each other, is called the *correspondence principle*. It is the second staring point for nearly all theories of quantization.

2.2 Making Things Noncommutative: Quantization

2.2.1 The Task

We have had a brief overview over the mathematical formulation of classical and quantum mechanics. On one hand, if we compare for example the way in which states of a physical system are described (a point in the cotangent bundle of a manifold and a vector in a Hilbert space) we will see that they are very different. On the other hand, we have seen that both theories have certain things in common: the time evolution can be described in a similar way. We want to give a short list of the main concepts of both theories and compare them (a much more detailed list and discussion can be found in Chapter 5 of [95]):

	Classical	**Quantum**
Observables	Poisson(-*)-algebra $\mathscr{A}_{Cl}$ of smooth functions $\mathscr{C}^\infty(M)$ on a Poisson manifold	*-algebra $\mathscr{A}_{QM}$ of self-adjoint operators on a Hilbert space $\mathfrak{H}$
Measurable Values	$\mathrm{spec}(f) \subseteq \mathbb{R}$	$\mathrm{spec}(A) \subseteq \mathbb{R}$
States	Points in the phase space	Vectors in a Hilbert space
Time evolution	Hamilton function H	Hamilton operator $\hat{H}$
Infinitesimal time evolution	$\frac{d}{dt}f(t) = \{f(t), H\}$	$\frac{d}{dt}A(t) = \frac{1}{i\hbar}[A(t), H]$

When we look at this table, we see that it is probably a good guideline to understand the observable algebras as the main concepts of mechanics, rather than the states. We know that the classical theory emerges as a limit of many quantum systems, such that the physical constant $\hbar$ becomes small in enough (in comparison to the characteristic action scale of the system) to be neglected. How to construct a correspondence the other way round is the question which the theory of quantization addresses. Physicists used to solve this by "making a hat on the variables p and q and saying that they the do not commute any more". Surprisingly enough, this approach, which is called *canonical quantization* in physics, works for most of the simple examples. In particular, this is the case when we do not have high powers of $\hat{p}$ and $\hat{q}$ and no mixed terms. That this idea is however not canonical in a *mathematical sense* is almost needless to say and of course it causes severe problems when higher and mixed terms in position and momentum appear. Briefly stated, the problem is that in the classical theory, the expressions q^2p, qpq, pq^2 describe the same polynomial, but in the quantum theory, they do not. So we must think about the question, to which operators we want to map mixed polynomials, and canonical quantization does not provide an answer. To create a mathematical theory of quantization, one needs to impose an ordering on the $\hat{p}$ and $\hat{q}$. Already Hermann Weyl knew about this problem and looked for ways how to solve it by proposing a totally symmetrized expression [99] for such terms. This ordering is now known as the Weyl ordering and his formula became the starting point for the theory of deformation quantization many years later.

2.2.2 A Definition

First, we want to take a step back and try to formalize what we actually mean by a quantization. See also Section 3.2 in [41] or Section 5.1.2 in [95] for a good introduction to this topic. For us, a quantization should be a correspondence

$$\mathcal{Q}\colon \mathscr{A}_{\text{Classical}} \longrightarrow \mathscr{A}_{\text{Quantum}}$$

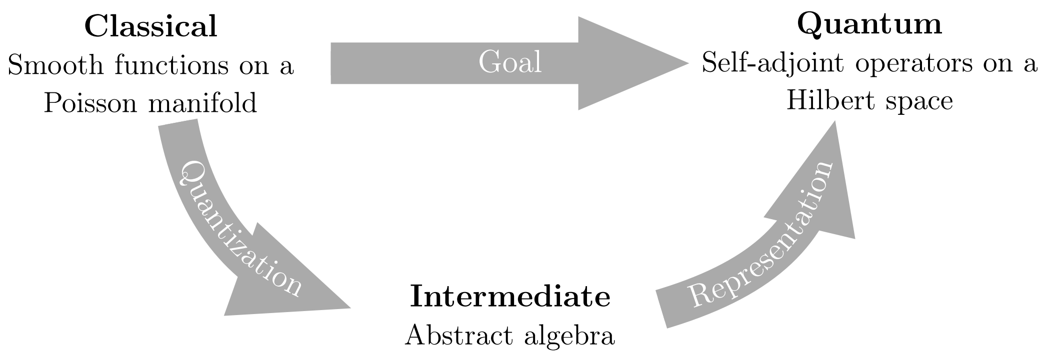
between a commutative algebra $\mathscr{A}_{\text{Classical}}$ and a noncommutative algebra $\mathscr{A}_{\text{Quantum}}$, which is a "bijection" in some sense: all classical observables appear as classical limits from quantum observables, hence we should expect $\mathcal{Q}$ to be "injective". On the other hand, if the quantum algebra was bigger than the classical algebra, the whole concept of quantization would be pointless, since we could never recover all observables, thus we want $\mathcal{Q}$ to be "surjective", too. The correspondence should also keep somehow track of the physical meaning of our observables, or at least tell us, what the observable $\mathcal{Q}(f)$ should be for some $f \in \mathscr{A}_{\text{Classical}}$. Moreover, $\mathcal{Q}^{-1}$ should behave like the classical limit, since this is the rather well-understood part of both. Finally, we want the correspondence to keep associative structures: the classical algebra is associative and we want the product in our noncommutative algebra to correspond to the concatenation of operators on a Hilbert space which is associative, too. One possibility of formulating this would be the following: in the very prototype of a classical observable algebra, a symplectic vector space of dimension $2n$, we want that the following three axioms are fulfilled:

(Q1) We want a "map", which allows a representation in the standard picture of quantum mechanics: $\mathcal{Q}(1) = \mathbb{1}$, $\mathcal{Q}(q^i) = \hat{q}^i$, $\mathcal{Q}(p_j) = -i\hbar\frac{\partial}{\partial q^j}$.

(Q2) The correspondence principle should be fulfilled: $[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f,g\})$, for all $f, g \in \mathscr{A}_{\text{Classical}}$.

(Q3) For mathematical simplicity, $\mathcal{Q}$ should be linear.

We finally split up the whole process: in a first step, we *quantize* the system and in a second step, we look for *representations* on a Hilbert space.



2.2.3 Different approaches

There exist various ideas about how to build up such a scheme. For example, in axiomatic quantum field theory one usually wants the quantum algebra to be a C^*-algebra (see for example [50] or [5]). The reason is that the bounded operators $\mathfrak{B}(\mathfrak{H})$ naturally form such an algebra (and actually even more than that). Unfortunately, most of the operators in quantum mechanics are unbounded. This problem is cured by looking at the exponentiated operators

$$A \longmapsto \mathrm{e}^{itA}, \quad t \in \mathbb{R}.$$

This way, one gets a one parameter group for $t \in \mathbb{R}$ and unitary operators are clearly bounded. Moreover, there is a very nice correspondence between C^*-algebras and locally compact Hausdorff spaces, which is known as the Gelfand-Naimark theorem. Roughly stated, this means that there is an equivalence of categories between compact Hausdorff spaces and unital commutative C^*-algebras: every such space gives rise to a commutative algebra of complex-valued, continuous functions, which form a C^*-algebra. On the other hand, from every unital commutative C^*-algebra $\mathscr{A}$ one can construct a compact Hausdorff space X such that $\mathscr{A} \cong \mathscr{C}(X)$. This correspondence can be extended to open and closed subsets of the space, to homeomorphism, locally compact spaces, their compactification, vector bundles and even more. There is a strong link between commutative algebra and topology or geometry. One possibility to think of a quantized system is to think of continuous or smooth functions on a noncommutative space, which then should correspond to a noncommutative C^*-algebra. This idea leads to noncommutative geometry, which is mostly due to Alain Connes. A very detailed, but not necessarily easy to read book is [30] by Connes, another and rather brief introduction is for example [93].

A different approach, called geometric quantization, tries to fulfil all of the three axioms (Q1) - (Q3) from the previous part. Unfortunately, this causes problems: already for a symplectic vector space, it is impossible to have a one-to-one correspondence of the Poisson bracket and the quantum mechanical commutator. This is known as the Groenewold-van Hove theorem, which was found around 1950 [47, 91]. More precisely spoken: no representation of the Lie algebra, which is generated by the q^i and p_j and which is defined by the classical commutation relations, can be extended irreducibly and faithfully to the commutator Lie algebra which comes from the associative unital algebra which is generated by the q^i and p_j. Thus geometric quantization restricts to smaller observable algebras, which are not problematic. At its present state, this approach is however still far from being a general theory, but it provide a procedure for number of exemplary cases. Its ideas are mostly due to Souriau [87], Kostant and Segal.

There are also different approaches. Berezin proposed a quantization scheme for particular Kähler manifolds, [11–13]. Still, new ideas keep coming up, but in the following, we want to concentrate on a particular type of quantization, which is called *deformation quantization.*

2.3 Deformation Quantization

2.3.1 The Concept

Deformation quantization tries to realize the three points (Q1) and (Q2) from the previous section, but weakens the third. If it is not possible to have such a correspondence exactly, we will at least want to have it *asymptotically.* The motivating example is the Weyl quantization which we already talked about. There are actually two such formulas, that can be given: the first maps a function f in the variables $q, p \in \mathbb{R}^n$ to a differential operator on $\mathbb{R}^n$, which is actually a formal power series in the parameter $\hbar$. For polynomial functions, this series is a sum (more precisely: a polynomial again) and well defined, for general functions this will really be a formal power series, hence a truly infinite sequence, which a priori has no analytical, but just an algebraic meaning. The second is given by an integral formula and holds for another class of functions (Schwartz functions), but one gets the first formula out of the second as an asymptotic expansion for $\hbar \longrightarrow 0$. With these quantizations, one can also define a product of two functions f, g, which will necessarily take those two functions to a formal power series in $\hbar$. Moyal showed that the commutator of this product can be understood as a series which approximates the quantum mechanical commutator [66]. The reason why seemingly all of a sudden power series appear is the following: if one wants the correspondence principle to be asymptotically fulfilled, i.e.

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = i\hbar\mathcal{Q}(\{f, g\}) + \mathcal{O}(\hbar^2)$$

and the multiplication of these quantized functions to be associative (as needed for representations on the Hilbert space), one will necessarily get higher and higher orders in $\hbar$. This iteration can never be stopped without loosing associativity. Motivated by this observation, a group of mathematicians, the so called Dijon-school, started working out this idea of products as formal power series. They understood quantization as a *deformation* of the commutative product by a formal parameter (mostly called $\hbar$, λ or ν, in this work we will call it z from now on), which controls the noncommutativity of the theory. These deformed products should moreover fulfil some compatibility conditions with the classical theory. This was the hour of birth of deformation quantization. The main characters of this group were Flato, Lichnerowicz, Bayen, Frønsdal and Sternheimer, who published their ideas in the late 70's [6, 7] and gave a first definition of a star product. These two articles became the starting point for what has now become a rich and fruitful theory. The deformed products are tits corner stone and one defines them in the following way.

Definition 2.3.1 (Star Product) *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A star product on M is a bilinear map*

$$\star_z \colon \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M)[\![z]\!],\ (f, g) \longmapsto f \star_z g = \sum_{n=0}^{\infty} z^n C_n(f, g)$$

such that its $\mathbb{K}[\![z]\!]$-linear extension to $\mathscr{C}^\infty(M)[\![z]\!]$ fulfils the following properties:

i.) $\star_z$ *is associative.*

ii.) $C_0(f, g) = f \cdot g,\ \forall_{f,g\in\mathscr{C}^\infty(M)}$ *(Classical limit).*

iii.) $C_1(f, g) - C_1(g, f) = z\{f, g\},\ \forall_{f,g\in\mathscr{C}^\infty(M)}$ *(Semi-classical limit).*

iv.) $1 \star_z f = f \star_z 1 = f,\ \forall_{f\in\mathscr{C}^\infty(M)}$.

If the C_n are bidifferential operators, the star product is said to be differential and if the order of differentiation of the C_n does not exceed n in both arguments, a differential star product is said to be natural. Moreover, we will say that a star product $\star_z$ is of Weyl-type, if $\overline{f \star_z g} = \overline{g} \star_z \overline{f}$ for all $f, g \in \mathscr{C}^\infty(M)$ where $^-$ denotes the complex conjugation.

They also defined a notion of equivalence of star products. The idea behind is that two equivalent star products should give rise to the same physics.

Definition 2.3.2 (Equivalence of Star Products) *Two star products $\star_z$ and $\widehat{\star}_z$ for a Poisson manifold $(M, \{\cdot, \cdot\})$ are said to be equivalent, if there is a formal power series*

$$T = \mathsf{id} + \sum_{n=0}^{\infty} z^n T_n$$

of linear maps $T_n \colon \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M)$, which extends $\mathbb{K}[\![z]\!]$-linearly to $\mathscr{C}^\infty(M)[\![z]\!]$, such that the following statements hold:

$$f \star_z g = T^{-1}(T(f)\widehat{\star}_z T(g)),\ \forall_{f,g\in\mathscr{C}^\infty(M)[\![z]\!]} \quad \text{and} \quad T(1) = 1.$$

For differential or natural star products, we accordingly speak of differential or natural equivalences.

Note that these definitions are purely algebraic, since we do not ask for the convergence of those power series. Hence the theory which was developed from this in the following years is also a mostly algebraic theory. Like for the Weyl product, there were also integral formulas around for other types of star products for which one can also make sense of convergence. But as already pointed out, one has to strongly restrict the algebra of functions, for example to the Schwartz space.

2.3.2 A Mathematical Theory

Deformation quantization is a good example for a mathematical theory, which is motivated by a physical idea. It is not really talking about a physical problem, since the world does not need to be quantized – it already is. It is talking about a mathematical problem: how to recover the quantized (and very counter-intuitive) mathematics, which describe the world on very small scales out of the classical (and much more intuitive) mathematics, which describe the world on our scale? Deformation quantization tries to give an answer to that, but is unfortunately (at least at its present state) still far from doing so completely. Like many such theories, it started of from a more or less precise physical background and developed into something very different: a mostly algebraic, purely mathematical theory. That is also due to the fact that the questions, which had to be answered in the beginning, were very hard and of mathematical nature. It took a lot of time to find answers and meanwhile, the mathematicians working on them were interested other aspects of the theory. We want to give a short overview of those questions and their answers very briefly and summarize a bit the history of deformation quantization next. A more detailed summary can be found in section 6.1 of [95].

The prototype is, as already mentioned, a symplectic vector space, for which Weyl proposed a star product with a certain (symmetric) ordering (although the definition of a star product did not exist at his time). However, other orderings are possible: one can have a standard or an anti-standard ordering, where the $\hat{p}$'s are all ordered to the right or to the left, respectively, or something in between. One of the first questions was, if one could also construct star products on symplectic manifolds and if these products will be standard or Weyl ordered, if they will be differential or natural and so on. Locally, the answer was yes, but it took some time and many small steps, until DeWilde and Lecomte could show that every cotangent bundle of a smooth manifold has star products [34] and then extended this result to arbitrary smooth manifolds [33]. Another proof was given independently from that by Omori, Maeda and Yoshioka [72] and then by Fedosov, who presented a simple and very geometric construction [43], which always give rise to natural star products, as shown in [21] or more generally in [49]. Moreover, every star product on a symplectic manifold is equivalent to a Fedosov star product [14]. A lot of results were found for Kähler manifolds and also the already mentioned procedure, which is due to Berezin, gives rise to star products. There are moreover standard, anti-standard ordered and many other types of star products on every cotangent bundle. The next question was the one concerning the equivalence classes of star products in the symplectic case. One can show that locally, two star products on a symplectic manifold are always equivalent. Hence a classification result should depend on global phenomena. Indeed, this is the case and it can be shown that star products on a symplectic manifold are classified by its second deRham cohomology $\mathrm{H}^2_{\mathrm{dR}}(M)$. This result is due to Deligne [32], a different proof was given by Nest and Tsygan [69], another one by Bertelson, Cahen and Gutt [14]. More precisely: the choice of an equivalence class of closed, nondenerate 2-forms $\omega \in \Omega^2(M)$ determines a Fedosov star product and from every Fedosov star product one can calculate such an equivalence class. This already determines all star products on symplectic manifolds, since every star product on such a manifold is equivalent to a Fedosov star product. The case of Poisson manifolds took longer and was much harder to solve, since associativity turned out to be a complicated condition to fulfil. There were some examples of star products known for particular Poisson structure, like the Gutt star product [48], which was also found by Drinfel'd [37] independently, but the general existence (and also the classification) result was proven by Kontsevich [59, 60] many years later. His classification result is known as the formality theorem and needs the notion of L_∞-algebras, which are fairly involved objects. He gave an explicit construction, how star products can be built out of Poisson brackets on $\mathbb{R}^d$. This construction was extended by Cattaneo, Felder and Tomassini to Poisson manifolds [28] and indepedently from that by Dolgushev [36]. Another and easier formulation of the Kontsevich construction on $\mathbb{R}^d$ in terms of operads was later given by Tamarkin [88].

2.3.3 From Formal to Strict

So far, one could say that the big cornerstones of the theory are already there and that it is somehow "finished". For two reasons, this is not the case. First, a mathematical theory is never actually "finished", since there are always a lot of new things which can be found. There are still many different types of star products to classify, like invariant or equivariant star products in the case that one has Lie group or Lie algebra actions. A very recent result concerning the classification of equivariant star products on symplectic manifolds is, for example, due to Reichert and Waldmann [76]. Second, the theory of deformation quantization still has a different aspect: all we talked about so far was purely algebraic and there is no notions of convergence of these formal power series. If some day, this theory shall have a real drawback on physics, it will be necessary to talk about the convergence properties of these star products, since in physics $\hbar$ is *not* a formal parameter but a constant with a dimension and a fixed value and therefore the question of convergence matters. When we dace those problems and speak about continuous star products, we leave the field of *formal* deformation quantization and come to *strict* deformation quantization.

Although it is closer to physics, strict deformation quantization is still a mathematical theory. There are two different approaches to it: we already mentioned integral formulas, which allow to speak of continuous star products. The second approach uses the formal power series instead and wants to construct a topology on the polynomial algebra, such that the star product becomes continuous. Then one completes the tensor algebra over the vector space to a subalgebra of the smooth functions, on which the star product will still be continuous.

The first approach is mostly due to Rieffel, who developed these ideas in some of his papers [77–79]. He wants to realize a strict deformation quantization by actions of an abelian Lie group on a C^*-algebra of classical observables. Later Rieffel formulated a list of open questions, which strict deformation quantization should try to answer [80] in the next years. His approach was carried on by Bieliavsky and Gayral [15, 16], who extended these concepts to much more general Lie groups and different manifolds. To get reasonably big observable algebras, they used oscillatory integrals and pushed this theory forward. A similar idea was realized by Natsume, Nest and Peter [68], who could show that under certain topological conditions, symplectic manifolds always admit strict deformations.

The second approach is due to Beiser and Waldmann [8, 9, 96]. They restrict to the local situation, that means to Poisson structures on vector spaces. Then, they look at the polynomial functions on this vector spaces and try to find continuity estimates for them by constructing an explicit locally convex topology on the symmetric tensor algebra (which is isomorphic to the polynomial algebra). The aim is to make the topology as coarse as possible, to get then a large completion and hence a big quantized algebra of observables. There are two big advantages of this approach: the first one is that it can be applied to infinite dimensional vector spaces, what is necessary for quantum field theory, which deals with infinitely many degrees of freedom. The second is that we can really speak about all observables, also those which will correspond to unbounded operators, without having to exponentiate. In this sense, this idea is somehow more fundamental. The disadvantage is, however, that it is just a local theory at the moment. The idea is worked out just for one type of star products by now, which are star products of exponential type like the Weyl product. This means until now one can only control star products on symplectic vector spaces which come from a constant Poisson tensor [96]. This theory was carried on in the master thesis of Matthias Schötz [82], who rephrased it using semi-inner product, which are a somehow more physical notion, since one can interpret spaces with such a topology as projective limits of pre-Hilbert spaces. This also allows a slightly coarser topology and hence a larger completion of the symmetric tensor algebra.

In this work, we will follow the second approach and apply it to another type of star product, the Gutt star product, which comes from a linear Poisson tensor on a vector space. Of course, this is the next logical step after constant Poisson tensors. However, these are also the first *non-symplectic* Poisson structures, which will be strictly quantized this way. Thus this master thesis really contributes something new to the theory of strict deformation quantization: a second example, in which Waldmann's locally convex topology on the tensor algebra leads to a continuous star product, when considered as a power series and not as an integral. Note that this also has a certain effect on Lie theory: the result can be seen as a functorial construction for a locally convex topology on the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ of a (possibly infitely-dimensional) Lie algebra $\mathfrak{g}$. Therefore they may have applications to, for example, the representation theory of $\mathscr{U}(\mathfrak{g})$.

Chapter 3

Algebraic Preliminaries

3.1 Linear Poisson Structures in Infinite Dimensions

As we have seen before, there has already been done some work on how to strictly quantize Poisson structures on vector spaces. Star products of exponential type on locally convex vector spaces were topologized by Stefan Waldmann in [96] and then investigated more closely by Matthias Schötz in [82]. The Poisson tensor which cooresponds to the Poisson bracket in these cases, is constant. Hence, as a the next step, we want to investigate linear Poisson structures on locally convex vector spaces. This will give a new big class of Poisson structures, which will be deformable in a strict way. Before we do so in the rest of this master thesis, we recall briefly some basics on linear Poisson structures.

In the following, we always consider a vector space V and study Poisson structures on the coordinates which are elements of the dual space V^*. In order to cover most of the physically interesting examples by our reflections, we want to allow vector spaces of very general type and therefore assume that V is a locally convex vector space. Every finite-dimensional vector space is normable and hence locally convex, so it fits in this framework. In this case, it is clear what V^* should be and there is just one interesting topology on it. For infinite-dimensional spaces, the situation is more delicate: we have to think about what coordinates should be and how a Poisson structure on them could look like. A priori, it is not clear which dual we should consider: the algebraic dual V^* of all linear forms on V, or the topological dual V' which contains just the continuous linear forms? Here, one could argue that only V' is of real interest, since otherwise we would encounter the very strange effect of having discontinuous polynomials, and the aim of constructing a continuous star product on them seems somehow pointless. But even if we stick to V', the question of the topology still remains: do we want to consider the weak or the strong topology there and why one of them should be more interesting. In any case, we have to choose a topology on this space. Once this is done, we have to think about a good notion of Poisson structures in this context. However, we encounter quite a number of question, which have no trivial answer. For this reason, it is worth looking at some equivalent formulations of $\mathrm{Pol}^\bullet(V^*)$ in the finite-dimensional case, since they may allow better generalizations.

Let V be a finite-dimensional vector space. Now, there is no question about the dual or its topology, since $V^* = V'$ is finite-dimensional, too, and we deal with polynomials on it. It is easy to see that a linear Poisson structure on V^* is something very familiar: it is equivalent to a Lie algebra structure on V.

Proposition 3.1.1 *Let V be a vector-space of dimension $n \in \mathbb{N}$ and $\pi \in \Gamma^\infty(\Lambda^2(TV^*))$. Then the two following things are equivalent:*

i.) π *is a linear Poisson tensor.*

ii.) V *has a uniquely determined Lie algebra structure.*

PROOF: We choose a basis $\{e_1, \ldots, e_n\} subset V$ and denote its dual basis $\{e^1, \ldots, e^n\} \subset V^*$. Then we call the linear coordinates in these bases $x_1, \ldots, x_n \in \mathscr{C}^\infty(V^*)$ and $\xi^1, \ldots, \xi^n \in \mathscr{C}^\infty(V)$, such that for all $\xi \in V, x \in V^*$

$$\xi = \xi^i(\xi)e_i \quad \text{and} \quad x = x_i(x)e^i.$$

In these coordinates, the Poisson tensor reads

$$\pi = \frac{1}{2}\pi_{ij}(x)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

where π is linear in the coordinates. Since $\pi_{ij}(x)$ is linear in the coordinates, we can write it as

$$\pi_{ij}(x) = c^k_{ij}x_k.$$

This gives for $f, g \in \mathscr{C}^\infty(V^*)$

$$\{f, g\}(x) = \pi(df, dg)(x) = x_k c^k_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}, \tag{3.1.1}$$

using the identification $T^*V^* \cong V^{**} \cong V$. But now, the statement is obvious, since antisymmetry of π means antisymmetry of the c^k_{ij} in the indices i and j and the Jacobi identity for the Poisson tensor gives

$$c^\ell_{ij}c^m_{\ell k} + c^\ell_{jk}c^m_{\ell i} + c^\ell_{ki}c^m_{\ell j} = 0 \tag{3.1.2}$$

for all i, j, k, m, since it must be fulfilled for all smooth functions. Vicely versa, (3.1.2) ensures the Jacobi identity of π in (3.1.1). Hence the map

$$[\cdot, \cdot]\colon V \times V \longrightarrow V \quad (e_i, e_j) \longmapsto c^k_{ij}e_k \tag{3.1.3}$$

defines a Lie bracket, since the c^k_{ij} are antisymmetric and fulfil the Jacobi identity and are therefore structures constants. Conversely, the structure constants of a Lie algebra on V define a Poisson tensor on V^* via (3.1.1). □

A more detailed explanation of Poisson manifolds in general, their correspondence to Lie algebroids this special correspondence of linear Poisson structures and a vector space and Lie algebras can be found in the textbook of Waldmann [95] or in his lecture notes [98]. The fact that V is a Lie algebra in the situation we want to consider motivates a change of notation: from now on, we will call the original vector space $\mathfrak{g}$, which is more intuitive for a Lie algebra. Since this kind of Poisson systems has a particular structure, there is a proper name for them.

Definition 3.1.2 (Kirillov-Kostant-Souriau bracket) *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra. We call the Poisson bracket, which is given on* g^* *by Equation* (3.1.1)*, the Kirillov-Kostant-Souriau bracket and denote it by* $\{\cdot, \cdot\}_{KKS}$.

Proposition 3.1.1 gives us a hint how we could think of infinite-dimensional vector spaces with linear Poisson tensor: We take $\mathfrak{g}$ to be an infinite-dimensional Lie algebra, which gives something like a linear Poisson structure on $\mathfrak{g}'$. If we chose directly $\mathfrak{g}'$ to have a linear Poisson tensor, we would get a Lie algebra structure on $\mathfrak{g}''$. Of course, we could think of using this structure on

$\mathfrak{g}$, since it canonically injects into $\mathfrak{g}''$. The problem is that in general, this will *not* be closed: taking the Lie bracket of $\xi, \eta \in \mathfrak{g}$, we might drop out of $\mathfrak{g}$ and have $[\xi, \eta] \subseteq \mathfrak{g}'' \backslash \mathfrak{g}$. Usually, such a behaviour will not be of interest, since the algebras of physical systems are closed objects and the double-dual is not what we are aiming for. This is why we will translate the term "linear Poisson structure on $\mathfrak{g}^*$" by "$\mathfrak{g}$ is a Lie algebra" in infinite dimensions. Remark however that, from a mathematical point of view, this way of thinking about infinite-dimensional Poisson structures is a choice, not a logical necessity and other choices would have been possible.

The next task are the polynomials on $\mathfrak{g}'$. As already mentioned, it is not easy to find a good generalization for them, since for a locally convex Lie algebra $\mathfrak{g}$, even $\mathfrak{g}'$ will be a huge vector space. Again, it is helpful to go back to the finite-dimensional case, where we have the following result:

Proposition 3.1.3 *Let $\mathfrak{g}$ be a vector space of dimension $n \in \mathbb{N}$. Then the algebras $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ are canonically isomorphic.*

PROOF: Since this is a very well-known result, we just want to sketch the proof briefly: take a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ and its linear coordinates $x_1, \ldots, x_n \in \mathscr{C}^\infty(\mathfrak{g}^*)$ with $x_i(x) = e_i(x)$ for $x \in \mathfrak{g}^*$. On homogeneous symmetric tensors this yields the map

$$\mathcal{J} \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{Pol}^\bullet(\mathfrak{g}^*), \quad e_1^{\mu_1} \cdots e_n^{\mu_n} \longmapsto \xi_1^{\mu_1} \cdots \xi_n^{\mu_n}.$$

We see immediately that this is an isomorphism, but note that we have used the identification $\mathfrak{g}^{**} \cong \mathfrak{g}$ via

$$e_i(x) = \langle x, e_i \rangle. \qquad \square$$

In infinite dimensions, the last identification we used in the last step will not work in both directions any more, but just in one: we have a canonical injection $\mathrm{S}^\bullet(\mathfrak{g}) \subseteq \mathrm{Pol}^\bullet(\mathfrak{g}')$, so every symmetric tensor still gives a polynomial. Anyway, this gives an idea how to avoid speaking about $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ and its topology: we restrict from the beginning to $\mathrm{S}^\bullet(\mathfrak{g})$. For finite-dimensional systems, both points of view are equivalent, but in infinite dimensions, this becomes a choice. However, we have good reasons to think that this is enough: we get a closed and reasonably big subalgebra of the polynomials. Moreover, the symmetric tensor algebra is defined on infinite-dimensional spaces exactly in the same way as on finite-dimensional ones, and the construction is identical.

So finally, we found a suitable way of speaking about our object of interest: we replace linear Poisson structures on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ by $\mathrm{S}^\bullet(\mathfrak{g})$.

3.2 The Gutt Star Product

The aim of this chapter is to endow the symmetric algebra, and as a consequence the polynomial algebra, with a new, noncommutative product. This is possible in a very natural way, due to the Poincaré-Birkhoff-Witt theorem. It links the symmetric tensor algebra $\mathrm{S}^\bullet(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$.

3.2.1 The Universal Enveloping Algebra

If $\mathscr{A}$ is an associative algebra, one can construct a Lie algebra out of it by using the commutator

$$[a, b] = a \cdot b - b \cdot a, \quad a, b \in \mathscr{A}.$$

This construction is functorial, since it does not only map associative algebras to Lie algebras, but also morphisms of the former to those of the latter. While constructing a Lie algebra out of an associative algebra is easy, the reversed process is more complicated, but also possible. Every Lie algebra $\mathfrak{g}$ can be embedded into a particular associative algebra, known as the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, which is uniquely determined (up to isomorphism) by the universal property: for every unital associative algebra $\mathscr{A}$ and every Lie algebra homomorphism $\phi\colon \mathfrak{g} \longrightarrow \mathscr{A}$ using the commutator on $\mathscr{A}$, one gets a unital homomorphism of associative algebras $\Phi\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{A}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{g}) \arrow[dr, "\Phi"] & \\
& \mathscr{A} \\
\mathfrak{g} \arrow[uu, "\iota"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

where ι denotes the embedding of $\mathfrak{g}$ into $\mathscr{U}(\mathfrak{g})$. The proof of existence and uniqueness of the universal enveloping algebra can be found in standard textbooks on Lie theory like [54] or [92], and we will not explain it here in detail. Just recall that existence is proven by an explicit construction: one takes the tensor algebra $\mathrm{T}^\bullet(\mathfrak{g})$ and considers the two-sided ideal

$$\mathfrak{J} = \langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] \rangle, \qquad \text{for } \xi, \eta \in \mathfrak{g}$$

inside of it. Then one gets the universal enveloping algebra by the quotient

$$\mathscr{U}(\mathfrak{g}) = \frac{\mathrm{T}^\bullet(\mathfrak{g})}{\mathfrak{J}}. \tag{3.2.1}$$

To avoid confusion, we will always denote the multiplication in $\mathscr{U}(\mathfrak{g})$ by $\odot$, whereas the commutative product in $\mathrm{S}^\bullet(\mathfrak{g})$ will be denoted without a sign. It follows from this construction, that $\mathscr{U}(\mathfrak{g})$ is a filtered algebra

$$\mathscr{U}(\mathfrak{g}) = \bigcup_{k \in \mathbb{N}} \mathscr{U}^k(\mathfrak{g}), \quad \mathscr{U}^k(\mathfrak{g}) = \Big\{ x = \sum_i \xi_1^i \odot \cdots \odot \xi_n^i \;\Big|\; \xi_j^i \in \mathfrak{g},\ 1 \leq j \leq n, i \in I \Big\}.$$

Generally, we just get a filtration, not a graded structure, since the ideal $\mathfrak{J}$ is not homogeneous in the symmetric degree. We will get a graded structure on $\mathscr{U}(\mathfrak{g})$, if and only if $\mathfrak{g}$ was commutative. Then $\mathscr{U}(\mathfrak{g})$ is isomorphic to the symmetric tensor algebra and thus also commutative. But $\mathscr{U}(\mathfrak{g})$ is much more than an associative algebra: it is also a Hopf algebra, since one can define a coassociative, cocommutative coproduct on it by

$$\Delta\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}), \quad \xi \longmapsto \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi, \qquad \text{for } \xi \in \mathfrak{g}$$

which extends to $\mathscr{U}(\mathfrak{g})$ via algebra homomorphism, as well as an antipode

$$S\colon \mathscr{U}(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}), \quad \xi \longmapsto -\xi, \qquad \text{for } \xi \in \mathfrak{g}$$

which extends to $\mathscr{U}(\mathfrak{g})$ via algebra antihomomorphism. We will come back to those two maps and to the Hopf structure in Chapter 7, when we talk about their continuity. More details on the algebraic aspect of deformation theory using Hopf algebras can be found in [29] and [61], for example.

3.2.2 The Poincaré-Birkhoff-Witt Theorem

The algebra $\mathscr{U}(\mathfrak{g})$ always admits a basis, which must be infinite. This result is due to the already mentioned theorem of Poincaré, Birkhoff and Witt:

Theorem 3.2.1 (Poincaré-Birkhoff-Witt theorem) *Let $\mathfrak{g}$ be a Lie algebra with a basis $\mathcal{B}_{\mathfrak{g}} = \{\beta_i\}_{i\in I}$. Then the set*

$$\mathcal{B}_{\mathscr{U}(\mathfrak{g})} = \left\{\beta_{i_1}^{\mu_{i_1}} \odot \cdots \odot \beta_{i_n}^{\mu_{i_n}} \;\middle|\; n \in \mathbb{N},\ i_k \in I \textit{ with } i_1 \preccurlyeq \cdots \preccurlyeq i_n \textit{ and } \beta_{i_k} \in \mathcal{B}_{\mathfrak{g}},\ \mu_{i_1}, \dots, \mu_{i_n} \in \mathbb{N}\right\}$$

defines a basis of $\mathscr{U}(\mathfrak{g})$.

There are different proofs for this statement. While a geometrical proof (like e.g. in [98]) is very convenient in the finite-dimensional case, a combinatorial argument must be used for infinite-dimensional Lie algebras. Most textbooks restrict to finite-dimensional Lie algebras and give a version of the latter one, except [23], which does it in full generality. The idea of most of the combinatoric proof works with minor changes also for any Lie algebra, since it relies on ordered index sets which can be defined in any dimension. The PBW theorem allows us to set up an isomorphism between $\mathrm{S}^\bullet(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g})$ immediately, since a basis of the former can be given by almost the same expression

$$\mathcal{B}_{\mathrm{S}^\bullet(\mathfrak{g})} = \left\{\beta_{i_1}^{\mu_{i_1}} \cdots \beta_{i_n}^{\mu_{i_n}} \;\middle|\; n \in \mathbb{N},\ i_k \in I, 1 \le k \le n, i_1 \preccurlyeq \cdots \preccurlyeq i_n \text{ and } \beta_{i_k} \in \mathcal{B}_{\mathfrak{g}},\ \mu_{i_1}, \dots, \mu_{i_n} \in \mathbb{N}\right\}$$

where we just have replaced the noncommutative product in $\mathscr{U}(\mathfrak{g})$ by the symmetric tensor product in $\mathrm{S}^\bullet(\mathfrak{g})$. This allows us to write down an isomorphism between the symmetric tensor algebra and the universal enveloping algebra, just by mapping the basis vectors to each other in a naive way. Of course, this can never be an isomorphism in the sense of algebras, but only of (filtered) vector spaces, because one of the algebras is commutative and the other isn't. Moreover, the symmetric algebra has a graded structure, which comes from the one on the tensor algebra, that the universal enveloping algebra does not have:

$$\mathrm{S}^\bullet(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathrm{S}^n(\mathfrak{g}), \quad \mathrm{S}^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \vee \ldots \vee \mathfrak{g}}_{n \text{ times}}.$$

We will denote by $\pi_n \colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^n(\mathfrak{g})$ the canonical projections of this grading. This induces a filtration by $\mathrm{S}^{(k)}(\mathfrak{g}) = \sum_{j=0}^{k} \mathrm{S}^j(\mathfrak{g})$. Our simple isomorphism will respect the filtration, but not the grading. However, it is not the only isomorphism which one can write down. In [10], Berezin proposed another isomorphism which is more helpful to use:

$$\mathfrak{q}_n \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}^n(\mathfrak{g}), \quad \beta_{i_1} \cdots \beta_{i_n} \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \beta_{i_{\sigma(1)}} \odot \cdots \odot \beta_{i_{\sigma(n)}}, \quad \mathfrak{q} = \sum_{n=0}^{\infty} \mathfrak{q}_n. \tag{3.2.2}$$

We will refer to it as the quantization map, for reasons that will soon become clear. It also respects the filtration and transfers the symmetric product to another symmetric expression. In this sense, we can now switch between both algebras and use the setting, which is more convenient in the current situation: the graded structure of $\mathrm{S}^\bullet(\mathfrak{g})$, or the Hopf algebra structure of $\mathscr{U}(\mathfrak{g})$.

3.2.3 The Gutt Star Product

Since we know, that the universal enveloping and the symmetric tensor algebra are isomorphic as vector spaces, we have a good tool at hand to endow the symmetric tensor algebra, and hence the polynomials, with a noncommutative product by pulling back the product from $\mathscr{U}(\mathfrak{g})$ to $\mathrm{S}^\bullet(\mathfrak{g})$ via $\mathfrak{q}$. This is exactly what Gutt did in [48]. She constructed a star product on $\mathrm{Pol}^\bullet(\mathfrak{g}^*)$ from $\mathscr{U}(\mathfrak{g})$ while encoding the noncommutativity in a formal parameter $z \in \mathbb{C}$ in a convenient way.

Definition 3.2.2 (Gutt star product) *Let $\mathfrak{g}$ be a Lie algebra, $z \in \mathbb{C}$, and $f, g \in \mathrm{S}^\bullet(\mathfrak{g})$ of degree k and ℓ respectively. Then we define the Gutt star product by:*

$$\star_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad (f, g) \longmapsto \sum_{n=0}^{k+\ell-1} z^n \pi_{k+\ell-n}\big(\mathfrak{q}^{-1}(\mathfrak{q}(f) \odot \mathfrak{q}(g))\big). \tag{3.2.3}$$

This is the original way in which Gutt defined the star product in [48], but there are two more ways to do it. Define

$$\mathfrak{I}_z = \langle \xi \otimes \eta - \eta \otimes \xi - z[\xi, \eta] \rangle$$

for $z \in \mathbb{C}$. Then we set

$$\mathscr{U}(\mathfrak{g}_z) = \frac{\mathrm{T}^\bullet(\mathfrak{g})}{\mathfrak{I}_z}, \tag{3.2.4}$$

and get the map

$$\mathfrak{q}_{z,n} \colon \mathrm{S}^n(\mathfrak{g}) \longrightarrow \mathscr{U}(\mathfrak{g}_z), \quad \beta_{i_1} \cdots \beta_{i_n} \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \beta_{i_{\sigma(1)}} \odot \cdots \odot \beta_{i_{\sigma(n)}}, \quad \mathfrak{q}_z = \sum_{n=0}^{\infty} \mathfrak{q}_{z,n}. \tag{3.2.5}$$

This way, we also get a star product:

$$\widehat{\star}_z \colon \mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad (f, g) \longmapsto \mathfrak{q}_z^{-1}(\mathfrak{q}_z(f) \odot \mathfrak{q}_z(g)). \tag{3.2.6}$$

In [37], Drinfel'd also constructed a star product using the Baker-Campbell-Hausdorff series: take $\xi, \eta \in \mathfrak{g}$ and set

$$\exp(\xi) *_z \exp(\eta) = \exp\left(\frac{1}{z} \mathrm{BCH}(z\xi, z\eta)\right), \tag{3.2.7}$$

where the exponential series is understood a formal power series in ξ and η. By formally differentiating, one gets the star product on all polynomials.

Of course, our aim is to show that these three maps are in fact identical and that they define a star product. Since this is a long way to go, we postpone the proof to the end of this chapter. It will be useful to learn something about the Baker-Campbell-Hausdorff series and the Bernoulli number first.

3.3 The Baker-Campbell-Hausdorff Series

Since we have a formula for $\star_z$ which involves the Baker-Campbell-Hausdorff series, we want to give a short overview about it and introduce some results that will be helpful later on. Note however, that there is not *the BCH formula*, since one can always rearrange terms using antisymmetry or Jacobi identity, but for $\xi, \eta \in \mathfrak{g}$, we can always write it as

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \mathrm{BCH}_n(\xi, \eta) = \sum_{a,b=0}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta), \tag{3.3.1}$$

where $\mathrm{BCH}_n(\xi,\eta)$ denotes all expressions having n letters and $\mathrm{BCH}_{a,b}(\xi,\eta)$ denotes all expressions with a times the letter ξ and b times the letter η. We have $\mathrm{BCH}_{0,0}(\xi,\eta) = 0$, $\mathrm{BCH}_{1,0}(\xi,\eta) = \xi$ and $\mathrm{BCH}_{0,1}(\xi,\eta) = \eta$. Clearly this gives

$$\mathrm{BCH}_n(\xi,\eta) = \sum_{a+b=n} \mathrm{BCH}_{a,b}(\xi,\eta).$$

Of course, this only moves the problem of non-uniqueness to a later point when we will have to discuss the partial expressions. Yet, in the beginning, this will be helpful.

3.3.1 Some General and Historical Remarks

Assume $\mathfrak{g}$ to be the Lie algebra of a finite-dimensional Lie group G. From the geometric point of view, the BCH formula is the infinitesimal counterpart of the multiplication law in G. Since the multiplication is smooth and the exponential function locally diffeomorphic around the unit element e, we would expect that there is a Lie algebraic analogon to the group multiplication, at least near the origin, which depends somehow "smoothly on the arguments". Finding this expression is, however, a different task.

One approach to this would be the following: consider an algebra $\mathscr{A}$ with the noncommuting elements ξ, η. We want to study the identity of formal power series

$$\exp(\chi) = \exp(\xi)\exp(\eta).$$

There should be a $\chi \in \mathscr{A}$, which fulfils this relation. We can rewrite the right hand side as

$$\exp(\xi)\exp(\eta) = \sum_{n,m=0}^{\infty} \frac{\xi^n \eta^m}{n!m!}$$

and use the formal power series for the logarithm

$$\log(\chi) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (\chi - 1)^k$$

in order to get an expression for χ. This yields

$$\chi = \log(\exp(\xi)\exp(\eta)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{i\in\{1,\dots,k\} \\ n_i,m_i\geq 0 \\ n_i+m_i\geq 1}} \frac{\xi^{n_1}\eta^{m_1}\cdots\xi^{n_k}\eta^{m_k}}{n_1!m_1!\cdots n_k!m_k!}. \tag{3.3.2}$$

It is far from trivial, if and how this can be expressed using Lie brackets. The first one who found a general way for this was Dynkin in the 1950's [38, 39]. Of course, the question of convergence still remains, although we would expect the expression to converge at least in a neighbourhood of 0.

A different approach works via differential equations. We can consider flows on the Lie group. This gives also an expression of the group multiplication in logarithmic coordinates just using Lie brackets. One gets recursive relations for the $\mathrm{BCH}_n(\xi,\eta)$ and the first formulas due to Baker [4], Campbell [24, 25] and Hausdorff [53] were of this kind. For the first terms, one finds

$$\begin{aligned}\mathrm{BCH}(\xi,\eta) &= \log(\exp(\xi)\exp(\eta)) \\ &= \xi + \eta + \frac{1}{2}[\xi,\eta] + \frac{1}{12}([[\eta,\xi],\xi] + [[\xi,\eta],\eta]) + \frac{1}{24}[[[\eta,\xi],\xi],\eta]\end{aligned}$$

$$+\frac{1}{120}([[[[\eta,\xi],\eta],\xi],\eta]+[[[[\xi,\eta],\xi],\eta],\xi])+\frac{1}{360}([[[[\eta,\xi],\xi],\xi],\eta]+[[[[\xi,\eta],\eta],\eta],\xi])$$
$$-\frac{1}{720}([[[[\eta,\xi],\xi],\xi],\xi]+[[[[\xi,\eta],\eta],\eta],\eta])+\cdots \tag{3.3.3}$$

which coincides of course with the result from Dynkin.

3.3.2 Forms of the BCH Series

As already mentioned, there are different forms of stating the BCH formula and depending on the problem one wants to solve, not every one is equally well suited. One can classify them roughly into four groups.

i.) There are recursive formulas, which calculate each term from the previous one. The first expressions due to Baker, Campbell and Hausdorff were of this kind. Though the idea is old, this approach is still much in use and allows powerful applications: Casas and Murua found an efficient algorithm [27] for calculating a form of BCH series without redundancies based on a recursive formula, which was given by Varadarajan in his textbook [92]. For such a non-redundant formula one needs a notion of basis of the free Lie algebra. There are approaches to such (Hall or Hall-Viennot) basis, which can e.g. be found in [84].

ii.) Most textbooks prove an integral form of the series, like [51] and [54]. Since we will use it, too, we want to introduce it briefly. Take the function

$$g\colon \mathbb{C}\longrightarrow\mathbb{C},\quad z\longmapsto\frac{z\log(z)}{z-1} \tag{3.3.4}$$

and denote for $\xi\in\mathfrak{g}$ by

$$\mathrm{ad}_\xi\colon\mathfrak{g}\longrightarrow\mathfrak{g},\quad \eta\longmapsto[\xi,\eta]$$

the usual adjoint operator. Then one has for $\xi,\eta\in\mathfrak{g}$

$$\mathrm{BCH}(\xi,\eta)=\xi+\int_0^1 g(\exp(\mathrm{ad}_\xi)\exp(t\,\mathrm{ad}_\eta))(\eta)dt. \tag{3.3.5}$$

iii.) As already mentioned, Dynkin found a closed form for (3.3.2), which is the only one of this kind known so far. A proof can be found in [56], for example. It reads

$$\mathrm{BCH}(\xi,\eta)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sum_{\substack{i\in\{1,\dots,k\}\\ n_i,m_i\geq 0\\ n_i+m_i\geq 1}}\frac{1}{\sum_{i=1}^{k}(n_i+m_i)}\frac{[\xi^{n_1}\eta^{m_1}\cdots\xi^{n_k}\eta^{m_k}]}{n_1!m_1!\cdots n_k!m_k!}, \tag{3.3.6}$$

where the expression $[\dots]$ denotes Lie brackets nested to the left: for instance, we have

$$[\xi\eta\eta\xi]=[[[\xi,\eta],\eta],\xi].$$

Unfortunately, the combinatorics get extremely complicated for higher degrees and increasingly many terms belong to the same Lie bracket expression.

iv.) Goldberg gave a form of the series which is based on words in two letters:

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{|w|=n} g_w w. \tag{3.3.7}$$

The g_w are coefficients, which can be calculated using the recursively defined Goldberg polynomials (see [45]). It was put into commutator form by Thompson in [89]:

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=1}^{\infty} \sum_{|w|=n} \frac{g_w}{n}[w]. \tag{3.3.8}$$

Again, the $[w]$ are Lie brackets nested to the left. Of course, this formula will also have redundancies, but its combinatorial aspect is much easier than the one of Equation (3.3.6). Since there are estimates for the coefficients g_w, we will use this form for our Main Theorem.

3.3.3 The Goldberg-Thompson Formula and some Results

Goldberg's Theorems

We now introduce the results of Goldberg: he denoted a word in the letters ξ and η as

$$w_\xi(s_1, s_2, \ldots, s_m) = \xi^{s_1} \eta^{s_2} \cdots (\xi \vee \eta)^{s_m},$$

with $m \in \mathbb{N}$ and the last letter will be ξ if m is odd and η if m is even. The index ξ of w_ξ means that the word starts with a ξ. Now we can assign to each word $w_{\xi\vee\eta}(s_1, \ldots, s_m)$ a coefficient $c_{\xi\vee\eta}(s_1, \ldots, s_m)$. This is done by the following formula:

$$c_\xi(s_1, \ldots, s_m) = \int_0^1 t^{m'}(t-1)^{m''} G_{s_1}(t) \cdots G_{s_m}(t) dt, \tag{3.3.9}$$

where we have $m' = \lfloor \frac{m}{2} \rfloor$, $m'' = \lfloor \frac{m-1}{2} \rfloor$ with $\lfloor \cdot \rfloor$ denoting the entire part of a real number and we have $n = \sum_{i=1}^{m} s_i$. The G_s are the recursively defined Goldberg polynomials

$$G_s(t) = \frac{1}{s} \frac{d}{dt} t(t-1) G_{s-1}(t), \tag{3.3.10}$$

for $s > 1$ and $G_1(t) = 1$. For c_η we have

$$c_\eta(s_1, \ldots, s_m) = (-1)^{n-1} c_\xi(s_1, \ldots, s_m) \tag{3.3.11}$$

and furthermore

$$c_\eta(s_1, \ldots, s_m) = c_\xi(s_1, \ldots, s_m)$$

if m is odd. This yields immediately

$$c_\xi(s_1, \ldots, s_m) = c_\eta(s_1, \ldots, s_m) = 0$$

if m is odd and n is even. Of course, Goldberg found interesting identities which are fulfilled by the coefficients. A very remarkable one is that for all permutations $\sigma \in S_m$ one has

$$c_\xi(s_1, \ldots, s_m) = c_\xi\big(s_{\sigma(1)}, \ldots, s_{\sigma(m)}\big),$$

since (3.3.9) obviously does not see the ordering of the s_i and m', m'' and n are not affected by reordering. For words with $m = 2$, an easier formula can be found:

$$c_\xi(s_1, s_2) = \frac{(-1)^{s_1}}{s_1! s_2!} \sum_{n=1}^{s_2} \binom{s_2}{n} B_{s_1+s_2-n},$$

where the B_s denote the Bernoulli numbers, which will be explained more precisely in the next paragraph. First, we note that the only case which matters to us is of course $s_1 = 1$, since for $s_1, s_2 > 1$ we will find something like $[[\xi, \xi], \ldots] = 0$. For simplicity, let's set $s_2 = 1$ and to permute $s_1 \leftrightarrow s_2$:

$$c_\xi(1, s) = \frac{(-1)^s}{s!} B_s. \tag{3.3.12}$$

Bernoulli numbers

We have seen the Bernoulli numbers B_n showing up and we will encounter them very often in the following. Hence it is useful to learn a few important things about them. They are defined by the series expansion of

$$g\colon \mathbb{C} \longrightarrow \mathbb{C}, \quad z \longmapsto \frac{z}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \tag{3.3.13}$$

Clearly, g has poles at $z = 2k\pi\mathrm{i}$, $k \in \mathbb{Z}\backslash\{0\}$. Moreover, one can easily show that all odd Bernoulli numbers are zero, except for $B_1 = -\frac{1}{2}$ and since in some applications one wants B_1 to be positive, there is a different convention for naming them: one often encounters $B_n^* = (-1)^n B_n$ (which only differs for $n = 1$). The nonzero Bernoulli numbers alternate in sign. For their absolute value, one can show the asymptotic behaviour (see [85, 86])

$$|B_{2n}| \sim (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}}.$$

This is not surprising, since we know that the generating function g had poles at $\pm 2\pi\mathrm{i}$. The Bernoulli numbers can also be calculated by the recursion formula

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k, \tag{3.3.14}$$

which is well-known in the literature (e.g. [2]). Since we will deal with them, we want to give the first numbers of this series here.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$	0	$\frac{691}{2730}$	0	$\frac{7}{6}$	0	$\frac{3617}{510}$

BCH up to first order

Proposition 3.3.1 *Let $\mathfrak{g}$ be a Lie algebra and the Bernoulli numbers as defined before. Then we have for $\xi, \eta \in \mathfrak{g}$*

$$\mathrm{BCH}(\xi, \eta) = \sum_{n=0}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2) \tag{3.3.15}$$

$$= \sum_{n=0}^{\infty} \frac{B_n}{n!} (\mathrm{ad}_\eta)^n(\xi) + \mathcal{O}(\xi^2) \tag{3.3.16}$$

PROOF: We want to can calculate this using the Goldberg coefficients. Remind that we put words to Lie brackets, and for computing the coefficients we need the words $\eta\xi^n$ *and* $\xi\eta\xi^{n-1}$ because of antisymmetry and words of the form $\xi^k\eta\xi^{n-k}$ with $k > 1$ give vanishing expressions. Now let $n \in \mathbb{N}$. We have

$$c_\eta(1,n) = (-1)^n c_\xi(1,n) = (-1)^n \frac{(-1)^n}{n!} B_n = \frac{B_n}{n!}.$$

By n-fold skew-symmetry and (3.3.8), we get the contribution

$$\frac{(-1)^n}{(n+1)!} B_n (\mathrm{ad}_\xi)^n(\eta) = \frac{1}{(n+1)!} B_n^* (\mathrm{ad}_\xi)^n(\eta)$$

Now we need $c_\xi(1,1,n-1)$: let $n > 1$, then

$$\begin{aligned}
c_\xi(1,1,n-1) &= \int_0^1 t(t-1)G_{n-1}(t)dt \\
&= -\int_0^1 t\frac{d}{dt}(t(t-1)G_{n-1}(t))dt \\
&= -\int_0^1 ntG_n(t)dt \\
&= -nc_\xi(1,n) \\
&= -n\frac{(-1)^n}{n!}B_n \\
&= (-1)^{n+1}\frac{1}{(n-1)!}B_n,
\end{aligned}$$

where we have done an integration by parts in the third step. So by using $n-1$ times the skew-symmetry of the Lie bracket, we get

$$\frac{1}{n+1} \cdot (-1)^{n+1} \frac{1}{(n-1)!} B_n[\ldots[\xi,\eta],\xi]\ldots],\xi] = \frac{n}{(n+1)!} B_n (\mathrm{ad}_\xi)^n(\eta).$$

For $n > 1$ we add up those two and use the fact that $B_n = B_n^*$ and find the result we want. For $n = 1$, there is just the first contribution and $c_\xi(1,1) = -B_1$, which gives

$$B_1^* \,\mathrm{ad}_\xi(\eta)$$

in total. For $n = 0$, we get $c_\xi(1) = c_\eta(1) = 1$ and finally get (3.3.15). For (3.3.16), note that we need $c_\xi(1,n)$ and $c_\eta(1,1,n-1)$. We have $c_\xi(1,n) = (-1)^n c_\eta(1,n)$ and $c_\eta(1,1,n-1) = (-1)^n c_\xi(1,1,n-1$. This gives $(-1)^n$ and switches B_n to B_n^*. □

Remark 3.3.2 (Alternative Proof) Note that we could also have used the integral formula (3.3.5) to prove this. We want to sketch an alternative proof here: if we write the second of the two exponential functions as a series, we see that it can be cut after the constant term, since we are looking for contributions which are linear in η. The function left to integrate is then just $(g \circ \log)(z)$. Since we insert $\exp(\mathrm{ad}_\xi)$, we get

$$\mathrm{BCH}(\xi,\eta) = \xi + \int_0^1 g(\mathrm{ad}_\xi)(\eta)dt + \mathcal{O}(\eta^2) = \xi + \sum_{n=1}^{\infty} \frac{B_n^*}{n!}(\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2),$$

since there is no dependence on t left and we get the same result.

Thompson's estimates on the coefficients

We know that we can put the BCH series into the form of Equation (3.3.8):

$$\mathrm{BCH}(\xi,\eta)=\sum_{n=1}^{\infty}\sum_{|w|=n}\frac{g_w}{n}[w]. \tag{3.3.17}$$

Later, we will need estimates on the coefficients g_w in order to show the continuity of the Gutt star product. The first simple estimate (together with a table of the first Goldberg coefficients and good explanation of Goldberg polynomials) was given by Newman and Thompson in [71]. The idea behind it was an analysis of the structure of the polynomials an its roots. This allows to put tight bounds on the values of $|g_w|$. We will need a bit more than that, but luckily, Thompson [90] gave an estimate in exactly the form we will need.

Proposition 3.3.3 *Let $n\in\mathbb{N}$ and g_w denotes the Goldberg coefficient of a word in two letters. Then we have the estimate*

$$\sum_{|w|=n}|g_w|\leq 2 \tag{3.3.18}$$

Proof: We want to sketch the proof here for convenience. One can see from the recursion formula (3.3.10), that the $G_s(t)$ are symmetric around $z=\frac{1}{2}$ (maybe up to a factor $t-\frac{1}{2}$): doing a shift $t\mapsto t\frac{1}{2}$, they are of the form

$$G_s(t)=\frac{1}{s!}\frac{d}{dt}(t^2-\frac{1}{4})\cdots\frac{d}{dt}(t^2-\frac{1}{4}).$$

Their roots lie all in the interval $(0,1)$ and the polynomials are normed (which can also be seen from the recursion formula). This means that they can be written (in the shifted form) as

$$G_s(t)=t^{s_0}(t+t_{s_1})(t-t_{s_1})\cdots(t+t_{s_r})(t-t_{s_r})$$

with $r=\lfloor\frac{s-1}{2}\rfloor$, $t_{s_i}\in(0,\frac{1}{2})$ and $s_0=0$ if s is odd and $s_0=1$ if s is even. The symmetric, quadratic terms are bounded in their absolute value by $\frac{1}{4}$ and the linear term by $\frac{1}{2}$, since $t\in(-\frac{1}{2},\frac{1}{2})$. Hence we have the estimate $|G_s(t)|\leq 2^{-s+1}$ for the integral domain. Now note that integration by parts yields

$$\int_0^1 t^a(t-1)^b dt=\frac{a!b!}{(a+b+1)!}.$$

We can put this together to get an estimate for $g_w=c_{\xi,\eta}(s_1,\ldots,s_m)$. Note again by $n=s_1+\cdots+s_m$. A slight rearranging of the factor from integration by parts using the fact that one of the numbers $m'=\frac{m}{2}$ and $m''=\frac{m-1}{2}$ is not an integer and will therefore be rounded downwards gives

$$\begin{aligned}|g_w|&\leq 2^{-n+m}\frac{m'!m''!}{(m'+m''+1)!}\\&=2^{-n+m}\frac{1}{m}\binom{m-1}{m'}^{-1}.\end{aligned}$$

Now we just need to sum up all the expressions corresponding to words of the length n. Note, that the words can start with ξ or η and we therefore get a factor 2 in front. The number of possible arrangements $(s_1, \dots, s_m)$ is due to a combinatorial argument ($n-m$ balls into m buckets, since every bucket must contains at least one ball) given by $\binom{n-1}{m-1}$ and we have to sum over all possible m. We get

$$\begin{aligned}
\sum_{|w|=n} |g_w| &\leq \sum_{m=1}^{n} 2 \binom{n-1}{m-1} 2^{-n+m} \frac{1}{m} \binom{m-1}{m'}^{-1} \\
&= 2^{-n+1} \sum_{m=1}^{n} \binom{n-1}{m-1} \underbrace{2^m \frac{1}{m} \binom{m-1}{m'}^{-1}}_{\leq 2 \ (*)} \\
&\leq 2^{-n+2} \sum_{m=1}^{n} \binom{n-1}{m-1} \\
&= 2.
\end{aligned}$$

In $(*)$ we used the fact that $\binom{m-1}{m'}$ is the biggest term (or one of the two) biggest terms in the binomial expansion of $(1+1)^{m-1}$ and hence we have $m\binom{m-1}{m'} \geq 2^{m-1}$. $\square$

3.4 The Equality of the Star Products

We want to prove the equality of the three star products. For a general and possibly infinite-dimensional Lie algebra, this is quite tedious. As a first step, it will be helpful to show their associativity.

Remark 3.4.1 In the finite-dimensional case, there are different proofs for Theorem 3.4.6, the main theorem of this section, which mostly rely on geometric arguments, like the one in [20]. Unluckily, these techniques are not at hand in infinite dimensions and one has to find an algebraic proof instead. Since in the community of deformation quantization, this statement is somehow folklore and believed for any Lie algebra, it strongly seems like such a proof already exists. However, the we were not able to trace it down in literature and therefore give an own proof.

Proposition 3.4.2 *The three maps $\star_z$, $\widehat{\star}_z$ and $*_z$ from* (3.2.3), (3.2.6) *and* (3.2.7) *respectively define associative multiplications.*

Proof: All maps are defined as $\mathrm{S}^\bullet(\mathfrak{g}) \times \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g})$, so we have to show bilinearity and associativity.

i.) For $\widehat{\star}_z$, associativity and bilinearity are clear from the construction, since we just pull-back the multiplication in $\mathscr{U}(\mathfrak{g}_z)$.

ii.) For $\star_z$, bilinearity follows from the fact that all maps, that are used in its definition, are (bi-)linear. For associativity, we have to interchange sums and shift projections. Recall that $\pi_n(f \star_z g) = 0$, if $n > \deg(f) + \deg(g)$. Take homogeneous tensors $f, g, h \in \mathrm{S}^\bullet(\mathfrak{g})$ of degree $k, \ell, m \in \mathbb{N}$ respectively. Then we have

$$\begin{aligned}
&\big(f \star_z g\big) \star_z h \\
&\quad = \sum_{j=0}^{k+\ell-1} \sum_{i=0}^{k+\ell+m-j-1} z^i \big(\pi_{k+\ell+m-j-i} \circ \mathfrak{q}^{-1}\big) \big(\mathfrak{q}\big(z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g))\big) \odot \mathfrak{q}(h)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k+\ell-1} \sum_{i=0}^{k+\ell+m-1} z^{i-j} \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)\big(\mathfrak{q}\big(z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g))\big) \odot \mathfrak{q}(h)\big) \\
&= \sum_{i=0}^{k+\ell+m-1} z^i \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)\left(\mathfrak{q}\left(\sum_{j=0}^{k+\ell-1} z^{-j} z^j \big(\pi_{k+\ell-j} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g))\right) \odot \mathfrak{q}(h)\right) \\
&= \sum_{i=0}^{k+\ell+m-1} z^i \big(\pi_{k+\ell+m-i} \circ \mathfrak{q}^{-1}\big)(\mathfrak{q}(f) \odot \mathfrak{q}(g) \odot \mathfrak{q}(h)),
\end{aligned}
$$

and we just need to do the reversed process on the right hand side to get the wanted result.

iii.) For $*_z$, we get associativity using the exponential function and the logarithm. We have

$$
\begin{aligned}
(\exp(\xi) *_z \exp(\eta)) *_z \exp(\chi) &= \exp\left(\frac{1}{z}\mathrm{BCH}\left(\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right), z\chi\right)\right) \\
&= \exp\left(\frac{1}{z}\mathrm{BCH}\left(z\xi, \left(\frac{1}{z}\mathrm{BCH}(z\eta, z\chi)\right)\right)\right) \\
&= \exp(\xi) *_z (\exp(\eta) *_z \exp(\chi)),
\end{aligned}
$$

since

$$
\begin{aligned}
\mathrm{BCH}\left(\left(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\right), z\chi\right) &= \log\left(\exp\left(\log\left(\frac{1}{z}\exp(z\xi)\exp(z\eta)\right)\right)\exp(z\chi)\right) \\
&= \log\left(\frac{1}{z}\exp(z\xi)\exp(z\eta)\exp(z\chi)\right) \\
&= \log\left(\exp(z\xi)\log\left(\left(\frac{1}{z}\exp(z\eta)\exp(z\chi)\right)\right)\right) \\
&= \mathrm{BCH}\left(z\xi, \left(\frac{1}{z}\mathrm{BCH}(z\eta, z\chi)\right)\right).
\end{aligned}
$$

Bilinearity follows from differentiating. $\square$

Note that star products must fulfil the classical and the semi-classical limit. We will do this in Corollary 4.2.1 and so just the equality is left to show. It is enough to prove the coincidence for terms of the form $\xi^k \star \eta$ with $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$, because $\mathrm{S}^\bullet(\mathfrak{g})$ is a commutative algebra and hence we get the coincidence on arbitrary monomials by polarization. The equality for the product of two monomials then follows by iteration, which is possible due to associativity. The next lemma presents a first big step.

Lemma 3.4.3 *Let $\xi, \eta \in \mathfrak{g}$, then we have*

$$
\xi^k \widehat{\star}_z \eta = \sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta). \tag{3.4.1}
$$

Proof: This proof is divided into the two following lemmata:

Lemma 3.4.4 *Let $\xi, \eta \in \mathfrak{g}$ and $k \in \mathbb{N}$. Then we have*

$$
\mathfrak{q}_z\left(\sum_{n=0}^{k} z^n \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^n(\eta)\right) = \sum_{s=0}^{k} \mathcal{K}(k, s) \xi^{k-s} \odot \eta \odot \xi^s
$$

with

$$\mathcal{K}(k,s) = \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j}.$$

PROOF: Since the map $\mathfrak{q}_z$ is linear, we can pull out the constants and get

$$\mathfrak{q}_z\left(\sum_{n=0}^{k} z^n\binom{k}{n}B_n^*\xi^{k-n}(\mathrm{ad}_\xi)^n(\eta)\right) = \sum_{n=0}^{k}\binom{k}{n}B_n^*\mathfrak{q}_z\Big(z^n\xi^{k-n}(\mathrm{ad}_\xi)^n(\eta)\Big).$$

Now we need the two equalities

$$\mathfrak{q}_z(\xi^n\eta) = \frac{1}{n+1}\sum_{\ell=0}^{n}\xi^{k-\ell}\odot\eta\odot\xi^\ell$$

and

$$\mathfrak{q}_z((z^n\,\mathrm{ad}_\xi)^n(\eta)) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi^{n-j}\odot\eta\odot\xi^j$$

which can easily be shown by induction. They give

$$\begin{aligned}
&\sum_{n=0}^{k}\binom{k}{n}B_n^*\mathfrak{q}_z\Big(z^n\xi^{k-n}(\mathrm{ad}_\xi)^n(\eta)\Big)\\
&\qquad = \sum_{n=0}^{k}\binom{k}{n}\frac{B_n^*}{k-n+1}\sum_{\ell=0}^{k-n}\xi^{k-n-\ell}\odot\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi^{n-j}\odot\eta\odot\xi^j\right)\odot\xi^\ell\\
&\qquad = \sum_{n=0}^{k}\binom{k}{n}\frac{B_n^*}{k-n+1}\sum_{\ell=0}^{k-n}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\xi^{k-\ell-j}\odot\eta\odot\xi^{\ell+j}\\
&\qquad = \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\xi^{k-\ell-j}\odot\eta\odot\xi^{\ell+j}
\end{aligned}$$

We just need to collect those terms for which we have $\ell+j=s$ for all $s=0,\dots,k$. If we do this with a Kronecker-delta, we will get exactly the $\mathcal{K}(k,s)$. ▽

For the second lemma, we need some statements on Bernoulli numbers and binomial coefficients. Let $k,m,n\in\mathbb{N}$. Then we have the following identities:

$$\sum_{j=0}^{k}\binom{k+1}{n}B_j^* = k+1 \tag{3.4.2}$$

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{m+j} = (-1)^m\sum_{i=0}^{m}\binom{m}{i}B_{k+i} \tag{3.4.3}$$

$$\sum_{j=0}^{m}(-1)^j\binom{n}{j} = (-1)^m\binom{n-1}{m} \tag{3.4.4}$$

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}. \tag{3.4.5}$$

The first one can easily be proven using the recursive definition of the Bernoulli numbers (3.3.14). Equation (3.4.4) and (3.4.5) are standard identities in combinatorics and can be found in the textbook of Aigner [1]. Finally, Equation (3.4.3) is a theorem due to Carlitz [26]. With them, we can show the next lemma which will finish this proof.

Lemma 3.4.5 *Let $\mathcal{K}(k,s)$ be defined as in Lemma 3.4.4, then we have for all $k \in \mathbb{N}$*

$$\mathcal{K}(k,s) = \begin{cases} 1 & s = 0 \\ 0 & \textit{else.} \end{cases}$$

PROOF: This is divided into three parts. First, we show the statement for $s = 0$, then we show it for $s = 1$ and then proceed by induction.

(i) $s = 0$: The Kronecker-delta will always be zero unless $l = j = 0$. So we get

$$\mathcal{K}(k,0) = \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^* = \frac{k+1}{k+1} = 1,$$

where we have used (3.4.2).

(ii) $s = 1$: To get a contribution from the δ, we must have $(j,\ell) = (1,0)$ or $(0,1)$. Except for $n = 0$ and $n = k$, both cases are possible. We split them off:

$$\begin{aligned}
\mathcal{K}(k,1) &= \underbrace{\frac{1}{k+1}}_{n=0} - \underbrace{kB_k^*}_{n=k} + \frac{1}{k+1}\sum_{n=1}^{k-1}\binom{k+1}{n}B_n^*\left(1+(-1)\binom{n}{1}\right) \\
&= \frac{1}{k+1} + \frac{1}{k+1}\sum_{n=1}^{k-1}\binom{k+1}{n}B_n^* - \frac{1}{k+1}\sum_{n=1}^{k-1}\binom{k+1}{n}nB_n^* - kB_k^* \\
&= \underbrace{\frac{1}{k+1}\sum_{n=0}^{k-1}\binom{k+1}{n}B_n^*}_{=1-B_k^*} - \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}nB_n^* \\
&= 1 - B_k^* - \underbrace{\frac{k+1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*}_{=k+1} + \sum_{n=0}^{k}\underbrace{\frac{k+1-n}{k+1}\binom{k+1}{n}}_{\binom{k}{n}}B_n^* \\
&= 1 - B_k^* - k - 1 + \sum_{n=0}^{k-1}\binom{k}{n}B_n^* + B_k^* \\
&= -k + \sum_{n}^{k-1}\binom{k}{n}B_n^* \\
&= 0.
\end{aligned}$$

(iii) $s \mapsto s+1$: Due to the induction, it is sufficient to prove $\mathcal{K}(k,s+1) - \mathcal{K}(k,s) = 0$. In order to do that, we must get rid of the δ's and therefore rewrite $\mathcal{K}(k,s)$:

$$\mathcal{K}(k,s) = \frac{1}{k+1}\sum_{n=0}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j}$$

$$
\begin{aligned}
&= \frac{1}{k+1}\sum_{n=0}^{s}\binom{k+1}{n}B_n^*\sum_{j=0}^{n}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j} \\
&\quad + \frac{1}{k+1}\sum_{n=s+1}^{k}\binom{k+1}{n}B_n^*\sum_{j=0}^{s}(-1)^j\binom{n}{j}\sum_{\ell=0}^{k-n}\delta_{s,\ell+j} \\
&= \frac{1}{k+1}\sum_{n=0}^{s}\binom{k+1}{n}B_n^*\sum_{j=\max\{0,s+n-k\}}^{n}(-1)^j\binom{n}{j} \\
&\quad + \frac{1}{k+1}\sum_{n=s+1}^{k}\binom{k+1}{n}B_n^*\sum_{j=\max\{0,s+n-k\}}^{s}(-1)^j\binom{n}{j}.
\end{aligned}
$$

As long as $\max\{0, s+n-k\} = 0$, the first sum over j will be zero as it is just the binomial expansion of $(1-1)^n$, except for $n=0$. Hence we get a special case and a shorter first sum over n. In the sums over j we use again the binomial expansion of $(1-1)^n$ and get

$$
\begin{aligned}
\mathcal{K}(k,s) = \frac{1}{k+1}\Bigg[1 &+ \sum_{k+1-s}^{s}\binom{k+1}{n}B_n^*\left(-\sum_{j=0}^{s+n-k-1}(-1)^j\binom{n}{j}\right) \\
&+ \sum_{n=s+1}^{k}\binom{k+1}{n}B_n^*\left(-\sum_{j=0}^{s+n-k-1}(-1)^j\binom{n}{j} - \sum_{j=s+1}^{n}(-1)^j\binom{n}{j}\right)\Bigg].
\end{aligned}
$$

Now it is helpful to use (3.4.4) and $\binom{k}{n-k} = \binom{k}{n}$. We also get $(-1)^n$-terms which we can put together with the B_n^* to get B_n:

$$
\begin{aligned}
\mathcal{K}(k,s) = \frac{1}{k+1}\Bigg[1 &+ \sum_{n=k+1-s}^{s}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s} \\
&+ \sum_{n=s+1}^{k}\binom{k+1}{n}B_n\left((-1)^{k-s}\binom{n-1}{k-s} + (-1)^{n+s}\binom{n-1}{s}\right)\Bigg].
\end{aligned}
$$

We finally made the δ disappear. Hence we must compute $\mathcal{K}(k,s+1) - \mathcal{K}(k,s)$. Since we want to show that it is 0, we can multiply it with $k+1$ in order to get rid of the factor in front:

$$
\begin{aligned}
&(k+1)(\mathcal{K}(k,s+1) - \mathcal{K}(k,s)) \\
&= \sum_{n=k-s}^{s+1}\binom{k+1}{n}B_n(-1)^{k-s-1}\binom{n-1}{k-s-1} - \sum_{n=k+1-s}^{s}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s} \\
&\quad + \sum_{n=s+2}^{k}\binom{k+1}{n}B_n\left((-1)^{k-s-1}\binom{n-1}{k-s-1} + (-1)^{n+s+1}\binom{n-1}{s+1}\right) \\
&\quad - \sum_{n=s+1}^{k}\binom{k+1}{n}B_n\left((-1)^{k-s}\binom{n-1}{k-s} + (-1)^{n+s}\binom{n-1}{s}\right) \\
&= -\sum_{n=k-s}^{k}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s-1} - \sum_{n=k-s+1}^{k}\binom{k+1}{n}B_n(-1)^{k-s}\binom{n-1}{k-s} \\
&\quad - \sum_{n=s+2}^{k}\binom{k+1}{n}B_n(-1)^{n+s}\binom{n-1}{s+1} - \sum_{n=s+1}^{k}\binom{k+1}{n}B_n(-1)^{n+s}\binom{n-1}{s}
\end{aligned}
$$

$$\begin{aligned}
&= -\sum_{n=k-s}^{k} \binom{k+1}{n} B_n (-1)^{k-s}\left(\binom{n-1}{k-s-1} + \binom{n-1}{k-s}\right) \\
&\quad - \sum_{n=s+1}^{k} \binom{k+1}{n} B_n (-1)^{n+s}\left(\binom{n-1}{s+1} + \binom{n-1}{s}\right).
\end{aligned}$$

We have rearranged the sums, added some zeros and shortened the expression. Now we will use the recursion formula for the binomial coefficients

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

and our binomial multiplication equality (3.4.5):

$$\begin{aligned}
&= -\sum_{n=k-s}^{k} \binom{k+1}{n} B_n (-1)^{k-s}\binom{n}{k-s} - \sum_{n=s+1}^{k} \binom{k+1}{n} B_n (-1)^{n+s}\binom{n}{s+1} \\
&= -\sum_{n=k-s}^{k} \binom{k+1}{s+1}\binom{s+1}{n+s-k} B_n (-1)^{k-s} - \sum_{n=s+1}^{k} \binom{k+1}{s+1}\binom{k-s}{n-s-1} B_n (-1)^{n+s}.
\end{aligned}$$

Since we want to show that this is 0, we can divide by $\binom{k+1}{s+1}$ which will never be zero because $s \in \{0, 1, \ldots, k\}$. After doing so, can use $n > 1$ in the second sum and thus only even n will show up, because for odd n the Bernoulli numbers are zero. For this reason we have $(-1)^n = 1$. Then we rewrite these sums by shifting the indices and we add two zeros:

$$\begin{aligned}
&- \sum_{n=k-s}^{k} \binom{s+1}{n+s-k} B_n (-1)^{k-s} + \sum_{n=s+1}^{k} \binom{k-s}{n-s-1} B_n (-1)^{s+1} \\
&= (-1)^{s+1} \sum_{\ell=0}^{k-s-1} \binom{k-s}{\ell} B_{\ell+s+1} - (-1)^{k-s} \sum_{\ell=0}^{s} \binom{s+1}{\ell} B_{\ell+k-s} \\
&= (-1)^{s+1} \sum_{\ell=0}^{k-s} \binom{k-s}{\ell} B_{\ell+s+1} - (-1)^{k-s} \sum_{\ell=0}^{s+1} \binom{s+1}{\ell} B_{\ell+k-s} \\
&\quad - (-1)^{s+1}\binom{k-s}{k-s} B_{k+1} + (-1)^{k-s}\binom{s+1}{s+1} B_{k+1}.
\end{aligned}$$

The first two terms give the Carlitz-identity (3.4.3) and vanish. So we are left with the last two terms and get

$$-(-1)^{s+1} B_{k+1} + (-1)^{k-s} B_{k+1} = (-1)^s B_{k+1}\left(1 + (-1)^k\right) = 0,$$

since the bracket will be zero if k is odd and $B_{k+1} = 0$ if k is even. $\triangledown$

Finally, Lemma 3.4.3 is proven. $\square$

In Lemma 4.1.1, we will see that also $*_z$ fulfils this identity. Hence $*_z = \widehat{\star}_z$. We only need to show $\widehat{\star}_z = \star_z$. For $z = 1$, the two maps are clearly identical and therefore we find

$$\xi^k \star_1 \eta = \sum_{n=0}^{k} \binom{k}{n} B_n^* \xi^{k-n} (\mathrm{ad}_\xi)^k(\eta).$$

But now $\widehat{\star}_z = \star_z$ follows from the definition of $\star_z$: we just have to plug in powers of z and find (3.4.1). So with the proofs in Chapter 4, we will have proven the following theorem:

Theorem 3.4.6 *The three maps $\star_z$, $\widehat{\star}_z$ and $*_z$ coincide on $\mathrm{S}^\bullet(\mathfrak{g})$ and define star products.*

Chapter 4

Formulas for the Gutt star product

We have seen some results on the Baker-Campbell-Hausdorff series and an identity for the Gutt star product. The latter one, stated in Theorem 3.4.6, will be a very useful tool in the following, since we want to get explicit formulas for $\star_z$. There is still a part of the proof missing, but this will be caught up at the beginning of the first section of this chapter. From there, we will come to a first easy formula for $\star_z$. Afterwards, we will use the same procedure to find two more formulas for it: the first is a rather involved one for the n-fold star product of vectors. It will not be helpful for algebraic computations, but very useful for estimates. The second one is a more explicit formula for the product of two monomials.

From those formulas, we will be able to draw some easy, but nice conclusion in the second section and we will prove the classical and the semi-classical limit. Then, we will show how to calculate the Gutt star product explicitly by computing two easy examples.

4.1 Formulas for the Gutt Star Product

4.1.1 A Monomial with a Linear Term

The easiest case for which we will develop a formula is surely the following one: for a given Lie algebra $\mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}$ we would like to compute

$$\xi^k \star_z \eta = \sum_{n=0}^{k} z^n C_n(\xi^k, \eta)$$

We have already done this for $\star_z$ and $\widehat{\star}_z$ in Lemma 3.4.3, and want to do the same for $*_z$ now. This will finish the proof of the equality of the star products from Theorem 3.4.6. We will use that

$$\xi^k = \frac{\partial^k}{\partial t^k}\Big|_{t=0} \exp(t\xi). \tag{4.1.1}$$

Now we have all the ingredients to prove the following lemma:

Lemma 4.1.1 *Let $\mathfrak{g}$ be a Lie algebra and $\xi, \eta \in \mathfrak{g}$. We have the following identity for $*_z$:*

$$\xi^k *_z \eta = \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j(\eta). \tag{4.1.2}$$

PROOF: We start from the simplified form for the Baker-Campbell-Hausdorff series from Equation (3.3.15) in Proposition 3.3.1:

$$\mathrm{BCH}(\xi,\eta) = \xi + \sum_{n=0}^{\infty} \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta) + \mathcal{O}(\eta^2).$$

If we insert this into the definition of the Drinfel'd star product and use Equation (4.1.1) we get

$$\begin{aligned}
\xi^k *_z \eta &= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(\frac{1}{z}\mathrm{BCH}(zt\xi, zs\eta)\right) \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \exp\left(t\xi + \sum_{j=0}^{\infty} z^j \frac{B_j^*}{j!}(\mathrm{ad}_{t\xi})^j(s\eta) + \mathcal{O}(\eta^2)\right).
\end{aligned}$$

We see that only terms which have exactly k of the ξ's in them and which are linear in η will contribute. This means we can cut off the sum at $j = k$ and omit higher orders in η. We now use the exponential series, cut it at k for the same reason and get

$$\begin{aligned}
\xi^k *_z \eta &= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k} \frac{1}{n!}\left(t\xi + \sum_{j=0}^{k}(zt)^j \frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^n \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \sum_{n=0}^{k} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} (t\xi)^{n-m} \left(\sum_{j=0}^{k}(zt)^j \frac{B_j^*}{j!}(\mathrm{ad}_\xi)^j(s\eta)\right)^m \\
&= \frac{\partial^k}{\partial t^k}\frac{\partial}{\partial s}\Big|_{t=0,s=0} \left(\sum_{n=0}^{k}\frac{1}{n!}(t\xi)^n + \sum_{n=0}^{k}\sum_{j=0}^{k} \frac{1}{(n-1)!} t^{n+j-1} z^j \frac{B_j^*}{j!} \xi^{n-1} (\mathrm{ad}_\xi)^j(s\eta)\right).
\end{aligned}$$

In the last step we set $m = 1$ since the other term have either too many or not enough η's and will vanish because of the differentiation with respect to s. We can finally differentiate to get the formula

$$\begin{aligned}
\xi^k *_z \eta &= \sum_{n=0}^{k}\sum_{j=0}^{k} \delta_{k,n+j-1} \frac{k!}{j!(n-1)!} z^j B_j^* \xi^{n-1} (\mathrm{ad}_\xi)^j(\eta) \\
&= \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \xi^{k-j} (\mathrm{ad}_\xi)^j(\eta),
\end{aligned}$$

which is the wanted result. □

Remark 4.1.2 We have proven the equality of the star products $\widehat{\star}_z$ $*_z$ by deriving an easy formula for both of them. From now on, we will get all other formulas from $*_z$, since this is the one which is easier to compute.

Now it is actually easy to get the formula for monomials of the form $\xi_1 \dots \xi_k$ with $\eta \in \mathfrak{g}$:

Proposition 4.1.3 *Let $\mathfrak{g}$ be a Lie algebra and $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$. We have*

$$\xi_1 \cdots \xi_k \star_z \eta = \sum_{j=0}^{k} \frac{1}{k!}\binom{k}{j} z^j B_j^* \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\dots[\xi_{\sigma(j)}, \eta]\dots]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)} \text{ and} \tag{4.1.3}$$

$$\eta \star_z \xi_1 \cdots \xi_k = \sum_{j=0}^{k} \frac{1}{k!}\binom{k}{j} z^j B_j \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\dots[\xi_{\sigma(j)}, \eta]\dots]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)}. \tag{4.1.4}$$

PROOF: We get the result by just polarizing the formula from Lemma 4.1.1. Let $\xi_1, \ldots, \xi_k \in \mathfrak{g}$, then we introduce the parameters t_i for $i = 1, \ldots, k$ and set

$$\Xi = \Xi(t_1, \ldots, t_k) = \sum_{i=1}^{k} t_i \xi^i.$$

Then we see that

$$\xi_1 \cdots \xi_k = \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1, \cdots, t_k = 0} \Xi^k$$

since for every $i = 1, \ldots, k$ we have

$$\frac{\partial}{\partial t_i}\Big|_{t_i = 0} \Xi = \xi_i. \tag{4.1.5}$$

By writing out the Ξ's and using multilinearity, we find

$$\begin{aligned}
\xi_1 \cdots \xi_k \star_z \eta &= \frac{1}{k!} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1, \cdots, t_k = 0} \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \Xi^{k-j} (\mathrm{ad}_\Xi)^j (\eta) \\
&= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z^j B_j^* \sum_{\{i_1, \ldots, i_k\} \in \{1, \ldots, k\}^k} \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\Big|_{t_1, \cdots, t_k = 0} t_{i_1} \cdots t_{i_k} \\
&\quad \cdot \xi_{i_1} \cdots \xi_{i_{k-j}} \mathrm{ad}_{\xi_{i_{k-j+1}}} \circ \cdots \circ \mathrm{ad}_{\xi_{i_k}} (\eta) \\
&= \sum_{j=0}^{k} \frac{1}{k!} \binom{k}{j} z^j B_j^* \sum_{\sigma \in S_k} [\xi_{\sigma(1)}, [\ldots [\xi_{\sigma(j)}, \eta] \ldots]] \xi_{\sigma(j+1)} \cdots \xi_{\sigma(k)}.
\end{aligned}$$

In the last step, all expression which did not contain each ξ_i exactly once disappeared due to the differentiation. The proof of Equation (4.1.4) works analogously. □

Remark 4.1.4 This formula is actually not a new result: Gutt already gave it in her paper [48, Prop. 1] and referred to Dixmier [35, part 2.8.12 (c)], who already gave it in his textbook. It can also be found in the diploma thesis of Neumaier [70, Rem. 5.2.8] and a work due to Kathotia [58, Eq. 2.23]. Probably the first one to mention it was Berezin in [10, Eq. 30].

4.1.2 An Iterated Formula for the General Case

Proposition 4.1.3 allows theoretically to get a formula for the case of $\xi_1, \ldots, \xi_k \in \mathfrak{g}$

$$\xi_1 \star_z \ldots \star_z \xi_k = \sum_{j=0}^{k} C_{z,j}(\xi_1, \ldots, \xi_k)$$

which we will need to prove the functoriality of our later construction. This could also be used to give an alternative proof for our main theorem. Unluckily, this approach has a problem: iterating this formula, we get strangely nested Lie brackets, which would be very difficult to bring into a nice form with Jacobi identity. So this is not a good way to find a handy formula for the usual star product of two monomials. Nevertheless, we want to pursue it for a moment, since we will get an equality which will be, although rather involved looking, very useful in the following: for analytic observations, it will be enough to put (even rough) estimates on it and the exact nature of the combinatorics in the formula will not be important. Hence we rewrite Equation (4.1.3) in order to cook up such a formula.

Take $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$, then we have

$$\xi_1 \cdots \xi_k \star_z \eta = \sum_{n=0}^{k} C_n(\xi_1 \ldots \xi_k, \eta)$$

with the C_n being as bilinear operators which are given explicitly on monomials by

$$C_n^k \colon \mathrm{S}^k(\mathfrak{g}) \times \mathfrak{g} \longrightarrow \mathrm{S}^{k-n+1}(\mathfrak{g}) \tag{4.1.6}$$

$$(\xi_1 \cdots \xi_k, \eta) \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{j} B_j^* z^j \xi_{\sigma(1)} \cdots \xi_{\sigma(k-j)} [\xi_{\sigma(k-j+1)}, [\ldots, [\xi_{\sigma(k)}, \eta]]] \tag{4.1.7}$$

with

$$C_n = \sum_{k=0}^{\infty} C_n^k.$$

This gives us a good way of writing the n-fold star product of vectors:

Proposition 4.1.5 *Let $\mathfrak{g}$, $2 \leq k \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k \in \mathfrak{g}$. Then we have*

$$\xi_1 \star_z \ldots \star_z \xi_k = \sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \ldots + i_{k-1}} C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k). \tag{4.1.8}$$

PROOF: This is an easy prof by induction over k. For $k = 2$ the statement is clearly true. For the step $k \to k+1$ we get

$$\begin{aligned}
\xi_1 \star_z \ldots \star_z \xi_{k+1} &= \left(\sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \cdots + i_{k-1}} C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k) \right) \star_z \xi_{k+1} \\
&= \sum_{i_k=0}^{k} z^{i_k} C_{i_k} \left(\sum_{\substack{1 \leq j \leq k-1 \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \cdots + i_{k-1}} C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k), \xi_{k+1} \right) \\
&= \sum_{\substack{1 \leq j \leq k \\ i_j \in \{0, \ldots, j\}}} z^{i_1 + \cdots + i_k} \big(C_{i_{k-1}}(\ldots C_{i_2}(C_{i_1}(\xi_1, \xi_2), \xi_3) \ldots, \xi_k), \xi_{k+1} \big)
\end{aligned}$$ □

Remark 4.1.6

i.) Of course, Proposition 4.1.5 is an easy consequence from Proposition 4.1.3. It's value, however, is that we know how the C_n's look like and what the summation range in (4.1.8) is. This will allow us to put estimates on things like iterated star products.

ii.) As already mentioned, we would get an identity for the star product of two monomials via

$$\xi_1 \cdots \xi_k \star_z \eta_1 \cdots \eta_\ell = \frac{1}{k!\ell!} \sum_{\sigma \in S_k} \sum_{\tau \in S_\ell} \xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(k)} \star_z \eta_{\tau(1)} \star_z \cdots \star_z \eta_{\tau(\ell)}. \tag{4.1.9}$$

This can be proven from the definition of the map $\mathfrak{q}_z$. Unfortunately, this would give a very clumsy formula to deal with.

4.1.3 A Formula for two Monomials

If we want to get an identity for the star product of two monomials, we have to go back to Equation (3.2.7). This will not give a simple looking formula either, but we will at least be able to do some computations with concrete examples. As a first step, we must introduce a bit of notation:

Definition 4.1.7 (G-Index) *Let $k, \ell, n \in \mathbb{N}$ and $r = k + \ell - n$. Then we call an r-tuple J*

$$J = (J_1, \dots, J_r) = ((a_1, b_1), \dots, (a_r, b_r))$$

a G-index if it fulfils the following properties:

(i) $J_i \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$

(ii) $|J_i| = a_i + b_i \geq 1 \quad \forall_{i=1,\dots,r}$

(iii) $\sum\limits_{i=1}^{r} a_i = k$ *and* $\sum\limits_{i=1}^{r} b_i = \ell$

(iv) The tuple is ordered in the following sense: $i > j \Rightarrow |J_i| \geq |J_j| \quad \forall_{i,j=1,\dots,r}$ *and* $|a_i| \geq |a_j|$ *if* $|J_i| = |J_j|$

(v) If $a_i = 0$ *[or* $b_i = 0$*] for some i, then* $b_i = 1$ *[or* $a_i = 1$*].*

We call the set of all such G-indices $\mathcal{G}_r(k, \ell)$.

Definition 4.1.8 (G-Factorial) *Let* $J = ((a_1, b_1), \dots, (a_r, b_r)) \in \mathcal{G}_r(k, \ell)$ *be a G-Index. We set for a given tuple* $(a, b) \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$

$$\#_J(a, b) = \text{ number of times that } (a, b) \text{ appears in } J.$$

Then we define the G-factorial of $J \in \{0, 1, \dots, k\} \times \{0, 1, \dots, \ell\}$ *as*

$$J! = \prod_{(a,b) \in \{0,1,\dots,k\} \times \{0,1,\dots,\ell\}} (\#_J(a, b))!$$

Each pair (a, b) will later correspond to $\mathrm{BCH}_{a,b}(\xi, \eta)$. Now we can state a good formula for the Gutt star product:

Lemma 4.1.9 *Let $\mathfrak{g}$ be a Lie algebra, $\xi, \eta \in \mathfrak{g}$ and $k, \ell \in \mathbb{N}$. Then we have the following identity for the Gutt star product:*

$$\xi^k \star_z \eta^\ell = \sum_{n=0}^{k+\ell-1} z^n C_n\left(\xi^k, \eta^\ell\right),$$

where the C_n are given by

$$C_n\left(\xi^k, \eta^\ell\right) = \frac{k!\ell!}{(k+\ell-n)!} \sum_{\substack{a_1, b_1, \dots, a_r, b_r \geq 0 \\ a_i + b_i \geq 1 \\ a_1 + \dots + a_r = k \\ b_1 + \dots + b_r = \ell}} \mathrm{BCH}_{a_i, b_i}(\xi, \eta) \cdots \mathrm{BCH}_{a_r, b_r}(\xi, \eta) \tag{4.1.10}$$

$$= \sum_{J \in \mathcal{G}_{k+\ell-n}(k,\ell)} \frac{k!\ell!}{J!} \prod_{i=1}^{k+\ell-n} \mathrm{BCH}_{a_i, b_i}(\xi, \eta) \tag{4.1.11}$$

and the product is taken in the symmetric tensor algebra.

Proof: We want to see what the C_n look like. Let's denote $r = k + \ell - n$ for brevity. Then we have

$$C_n\Big(\xi^k, \eta^\ell\Big) \in \mathrm{S}^r(\mathfrak{g}).$$

Of course, the only part of the series

$$\exp\bigg(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\bigg) = \sum_{n=0}^{k+\ell}\bigg(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\bigg)^n + \mathcal{O}(\xi^{k+1}, \eta^{\ell+1})$$

which lies in $\mathrm{S}^r(\mathfrak{g})$ is the summand for $n = r$. We introduce the formal parameters t and s. Since we differentiate with respect to them, we can omit terms of higher orders in ξ and η than k and ℓ respectively.

$$\begin{aligned}
z^n C_n\Big(\xi^k, \eta^\ell\Big) &= \frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0}\frac{1}{z^r}\frac{\mathrm{BCH}(zt\xi, zs\eta)^r}{r!} \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0}\left(\sum_{j=1}^{k+\ell}\mathrm{BCH}_j(zt\xi, zs\eta)\right)^r \\
&= \frac{1}{z^r}\frac{1}{r!}\frac{\partial^k}{\partial t^k}\frac{\partial^\ell}{\partial s^\ell}\Big|_{t,s=0}\sum_{\substack{j_1,\ldots,j_r\geq 1\\ j_1+\ldots+j_r=k+\ell}}\mathrm{BCH}_{j_1}(zt\xi, zs\eta)\cdots\mathrm{BCH}_{j_r}(zt\xi, zs\eta) \\
&= z^n\frac{k!\ell!}{r!}\sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\cdots+a_r=k\\ b_1+\cdots+b_r=\ell}}\mathrm{BCH}_{a_i,b_i}(\xi,\eta)\cdots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)
\end{aligned}$$

We sum over all possible arrangements of the (a_i, b_i). In order to find an easier summation range, we put the ordering from Definition 4.1.7 on these multi-indices and avoid therefore double counting. We loose the freedom of arranging the (a_i, b_i) and need to count the number of multi-indices $((a_1, b_1), \ldots, (a_r, b_r))$ which belong to the same G-index J. This number will be $\frac{r!}{J!}$, since we can not interchange the (a_i, b_i) any more (therefore $r!$), unless they are equal (therefore $J!^{-1}$). Since the ranges of the (a_i, b_i) in Equation (4.1.10) and of the elements in $\mathcal{G}_r(k, \ell)$ are the same, we can change the summation there to $J \in \mathcal{G}_r(k, \ell)$ and need to multiply by $\frac{r!}{J!}$. This gives

$$z^n C_n\Big(\xi^k, \eta^\ell\Big) = z^n\frac{k!\ell!}{J!}\sum_{J\in\mathcal{G}_r(k,\ell)}\mathrm{BCH}_{a_i,b_i}(\xi,\eta)\cdots\mathrm{BCH}_{a_r,b_r}(\xi,\eta)$$

which is equivalent to Equation (4.1.11). □

Now we need to generalize this to factorizing tensors. To do so, we need a last definition:

Definition 4.1.10 *Let $a, b \in \mathbb{N}$ and $\xi_1, \ldots, \xi_a, \eta_1, \ldots, \eta_b \in \mathfrak{g}$. Then we define by*

$$\widetilde{\mathrm{BCH}}_{a,b}\colon \mathfrak{g}^{a+b} \longrightarrow \mathfrak{g}$$

the map which we get when we replace in $\mathrm{BCH}_{a,b}(\xi, \eta)$ the i-th ξ by ξ_i and the j-th η by η_j for $i = 1, \ldots, a$ and $j = 1, \ldots, b$.

Proposition 4.1.11 *Let $\mathfrak{g}$ be a Lie algebra, $k, \ell \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. Then we have the following identity for the Gutt star product:*

$$\xi_1 \ldots \xi_k \star_z \eta_1 \ldots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell),$$

where the C_n are given by

$$C_n(\xi_1\cdots\xi_k,\eta_1\cdots\eta_\ell) = \sum_{J\in\mathcal{G}_{k+\ell-n}(k,\ell)} \frac{1}{J!}\sum_{\sigma\in S_k}\sum_{\tau\in S_\ell}\prod_{i=1}^{l+\ell-n} \widetilde{\mathrm{BCH}}_{a_i,b_i}\big(\xi_{\sigma(a_1+\cdots+a_{i-1}+1)},\ldots,$$
$$\ldots,\xi_{\sigma(a_1+\cdots+a_i)}\big)\big(\eta_{\tau(b_1+\cdots+b_{i-1}+1)},\ldots,\eta_{\tau(b_1+\cdots+b_i)}\big). \tag{4.1.12}$$
$$= \frac{1}{(k+\ell-n)!}\sum_{\sigma\in S_k,\tau\in S_\ell}\sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\cdots+a_r=k\\ b_1+\cdots+b_r=\ell}}$$
$$\widetilde{\mathrm{BCH}}_{a_i,b_i}\big(\xi_{\sigma(1)},\ldots,\xi_{\sigma(a_1)};\eta_{\tau(1)},\ldots,\eta_{\tau(b_1)}\big)\cdots$$
$$\widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(k-a_r+1)},\ldots,\xi_{\sigma(k)};\eta_{\tau(\ell-b_r+1)},\ldots,\eta_{\tau(\ell)}\big). \tag{4.1.13}$$

PROOF: The proof relies on polarization again and is completely analogous to the one of Proposition 4.1.3. We set

$$\Xi = \sum_{i=1}^{k} t_i\xi^i \quad\text{and}\quad \mathrm{H} = \sum_{i=1}^{\ell} t_j\eta^j.$$

Then it is easy to see that we will get rid of the factorials in Equation (4.1.11) since

$$\xi_1\cdots\xi_k\star_z\eta_1\cdots\eta_\ell = \frac{1}{k!\ell!}\frac{\partial^{k+\ell}}{\partial_{t_1}\cdots\partial_{s_\ell}}\Big|_{t_1,\ldots,s_\ell=0}\Xi^k\star_z\mathrm{H}^\ell\,.$$

Instead of the factorials, we get symmetrizations over the ξ_i and the η_j as we did in Proposition 4.1.3, which gives the wanted result. □

Remark 4.1.12 This formula was in some sense already found by Cortiñas in [31]. However, his formula is somewhat less transparent, since he uses an explicit form of the Baker-Campbell-Hausdorff series, namely the one due to Dynkin (3.3.6). As already mentioned, this formula is very complicated from a combinatorial point of view and hence also less adapted to put continuity estimates on it, as we will have to do. This is why we derived another formula for it.

4.2 Consequences and Examples

4.2.1 Some Consequences

Proposition 4.1.11 allows us to get some algebraic results. For example, we would like to see that the Gutt star product fulfils the classical and the semi-classical limit from Definition 2.3.1. We can prove this using Proposition 4.1.3. This will finish the proof of Theorem 3.4.6.

Corollary 4.2.1 *Let $\mathfrak{g}$ be a Lie algebra and $\mathrm{S}^\bullet(\mathfrak{g})$ endowed with the Gutt star product*

$$x\star_z y = \sum_{n=0}^{\infty} z^n C_n(x,y).$$

i.) On factorizing tensors $\xi_1\ldots\xi_k$ and $\eta_1\ldots\eta_\ell$, C_0 and C_1 give

$$C_0(\xi_1\cdots\xi_k,\eta_1\cdots\eta_\ell) = \xi_1\cdots\xi_k\eta_1\cdots\eta_\ell \tag{4.2.1}$$

$$C_1(\xi_1\cdots\xi_k,\eta_1\cdots\eta_\ell) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\xi_1\cdots\widehat{\xi_i}\cdots\xi_k\eta_1\cdots\widehat{\eta_j}\cdots\eta_\ell[\xi_i\eta_j], \tag{4.2.2}$$

where the hat denotes elements which are left out.

ii.) For $\mathfrak{g}$ finite-dimensional and the canonical isomorphism $\mathcal{J}\colon \mathrm{S}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{Pol}^\bullet(\mathfrak{g}^*)$ from Proposition 3.1.3, we have for $f, g \in \mathrm{Pol}^\bullet(\mathfrak{g}^*)$

$$C_1\big(\mathcal{J}^{-1}(f), \mathcal{J}^{-1}(g)\big) - C_1\big(\mathcal{J}^{-1}(f), \mathcal{J}^{-1}(g)\big) = \mathcal{J}^{-1}(\{f,g\}_{KKS})$$

where $\{\cdot,\cdot\}_{KKS}$ is the Kirillov-Kostant-Souriau bracket.

iii.) The map $\star_z$ fulfils the classical and the semi-classical limit and is therefore a star product.

PROOF: We take $\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell \in \mathrm{S}^\bullet(\mathfrak{g})$ and consider the G-indices in $\mathcal{G}_{k+\ell}(k,\ell)$ first. This is easy, since there is just one element inside:

$$\mathcal{G}_{k+\ell}(k,\ell) = \Big\{\big(\underbrace{(0,1),\dots,(0,1)}_{\ell \text{ times}}, \underbrace{(1,0),\dots,(1,0)}_{k \text{ times}}\big)\Big\}.$$

So we find

$$\begin{aligned} C_0(\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell) &= \sum_{\sigma\in S_k}\sum_{\tau\in S_\ell} \frac{1}{J!}\mathrm{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(1)}\big)\cdots \mathrm{BCH}_{0,1}\big(\varnothing, \xi_{\sigma(k)}\big) \\ &\quad \cdot \mathrm{BCH}_{1,0}\big(\eta_{\tau(1)},\varnothing\big)\cdots\mathrm{BCH}_{1,0}\big(\eta_{\tau(\ell)},\varnothing\big) \\ &= \sum_{\sigma\in S_k}\sum_{\tau\in S_\ell}\frac{1}{k!\ell!}\xi_{\sigma(1)}\cdots\xi_{\sigma(k)}\eta_{\tau(1)}\cdots\eta_{\tau(\ell)} \\ &= \xi_1\cdots\xi_k\eta_1\cdots\eta_\ell \end{aligned}$$

where we used $J! = k!\ell!$ according to Definition 4.1.8. We do the same for C_1. Also here, we have just one element in $\mathcal{G}_{k+\ell-1}(k,\ell)$:

$$\mathcal{G}_{k+\ell}(k,\ell) = \Big\{\big(\underbrace{(0,1),\dots,(0,1)}_{\ell-1 \text{ times}}, \underbrace{(1,0),\dots,(1,0)}_{k-1 \text{ times}}, (1,1)\big)\Big\}.$$

Using

$$\mathrm{BCH}_{1,1}(\xi,\eta) = \frac{1}{2}[\xi,\eta]$$

and $J! = (k-1)!(\ell-1)!$, we find

$$\begin{aligned} C_1(\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell) &= \frac{1}{2}\sum_{\sigma\in S_k}\sum_{\tau\in S_\ell}\frac{1}{(k-1)!(\ell-1)!}\xi_{\sigma(1)}\cdots\xi_{\sigma(k-1)}\eta_{\tau(1)}\cdots\eta_{\tau(\ell-1)}[\xi_{\sigma(k)},\eta_{\tau(\ell)}] \\ &= \frac{1}{2}\sum_{i=0}^{k}\sum_{j=0}^{\ell}\xi_1\cdots\widehat{\xi_i}\cdots\xi_k\eta_1\cdots\widehat{\eta_j}\cdots\eta_\ell[\xi_i,\eta_j]. \end{aligned}$$

This finishes part one. From this, the anti-symmetry of the Lie bracket yields

$$C_1(\xi_1\cdots\xi_k, \eta_1\cdots\eta_\ell) - C_1(\eta_1\cdots\eta_\ell, \xi_1\cdots\xi_k) = \sum_{i=0}^{k}\sum_{j=0}^{\ell}\xi_1\cdots\widehat{\xi_i}\cdots\xi_k\eta_1\cdots\widehat{\eta_j}\cdots\eta_\ell[\xi_i,\eta_j].$$

We now need to compute the KKS brackets on polynomials. Because of the linearity in both arguments, it is sufficient to check it on monomials of coordinates. Let $e_1,\dots,e_n$ be a basis of $\mathfrak{g}$ with linear coordinates $x_1,\dots,x_n$ on $\mathfrak{g}^*$. Now take $\mu_1,\dots,\mu_n,\nu_1,\dots\nu_n \in \mathbb{N}$ and consider the monomials $f = x_1^{\mu_1}\cdots x_n^{\mu_n}$ and $g = x_1^{\nu_1}\cdots x_n^{\nu_n}$. We use the notation from Proposition 3.1.1 and find for $x \in \mathfrak{g}^*$

$$\{f,g\}_{KKS}(x) = x_k c_{ij}^k \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}$$

$$= \mu_i \nu_j c_{ij}^k x_k x_1^{\mu_1} \cdots x_i^{\mu_i - 1} \cdots x_n^{\mu_n} x_1^{\nu_1} \cdots x_j^{\nu_j - 1} \cdots x_n^{\nu_n}.$$

Applying $\mathcal{J}^{-1}$ to it gives

$$\mathcal{J}^{-1}(\{f,g\}_{KKS}) = \sum_{i=0}^{n} \sum_{j=0}^{n} \mu_i \nu_j e_1^{\mu_1} \cdots e_i^{\mu_i - 1} \cdots e_n^{\mu_n} e_1^{\nu_1} \cdots e_j^{\nu_j - 1} \cdots e_n^{\nu_n} [e_i, e_j]. \tag{4.2.3}$$

On the other hand, we have

$$\mathcal{J}^{-1}(f) = e_1^{\mu_1} \cdots e_n^{\mu_n} \quad \text{and} \quad \mathcal{J}^{-1}(g) = e_1^{\nu_1} \cdots e_n^{\nu_n}.$$

Together with (4.2.2) this gives (4.2.3) and proves part two. Due to the bilinearity of the C_n, the third part follows. □

It is clear, that the formulas from Proposition 4.1.11 and Proposition 4.1.3 should coincide. However, we want to check it, to have the evidence that everything works as we wanted.

Corollary 4.2.2 *Given $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$, the results of the Equations* (4.1.12) *and* (4.1.3) *are compatible.*

PROOF: We have to compute sets of G-indices for $\xi_1, \dots, \xi_k, \eta_\ell \in \mathfrak{g}$. Again, they only have one element:

$$\mathcal{G}_{k+1-n}(k,1) = \Big\{ (\underbrace{(1,0), \dots, (1,0)}_{k-n \text{ times}}, (n,1)) \Big\}.$$

So we have with $J! = (k-n)!$ and $\mathrm{BCH}_{n,1}(\xi,\eta) = \frac{B_n^*}{n!} (\mathrm{ad}_\xi)^n(\eta)$

$$\begin{aligned} z^n C_n(\xi_1 \dots \xi_k, \eta_\ell) &= z^n \sum_{\sigma \in S_k} \frac{1}{(k-n)!} \frac{B_n^*}{n!} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} [\xi_{\sigma(k-n+1)}, [\dots, [\xi_{\sigma(k), \eta}] \dots]] \\ &= z^n \frac{1}{k!} \sum_{\sigma \in S_k} \binom{k}{n} B_n^* \xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} [\xi_{\sigma(k-n+1)}, [\dots, [\xi_{\sigma(k), \eta}] \dots]] \end{aligned}$$

Summing up over all n gives Equation (4.1.3). □

4.2.2 Two Examples

Equation (4.1.12) is useful if one wants to do real computations with the star product, but it is maybe not intuitive to apply. This is why we will give two examples here. The easiest one which is not covered by the simpler formula (4.1.3) will be the star product of two quadratic terms. The second one should be the a bit more complex case of a cubic term with a quadratic term.

Two quadratic terms

Let $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathfrak{g}$. We want to compute

$$\xi_1 \xi_2 \star_z \eta_1 \eta_2 = C_0(\xi_1\xi_2, \eta_1\eta_2) + z C_1(\xi_1\xi_2, \eta_1\eta_2) + z^2 C_2(\xi_1\xi_2, \eta_1\eta_2) + z^3 C_3(\xi_1\xi_2, \eta_1\eta_2).$$

The very first thing we have to do is computing the set of G-indices. Then we calculate the G-factorial and finally go through the permutations.

C_0: We already did this in Corollary 4.2.1, and know that the zeroth order in z is just the symmetric product. Therefore we have

$$C_0(\xi_1\xi_2, \eta_1\eta_2) = \xi_1\xi_2\eta_1\eta_2$$

C_1: We also did this one in Corollary 4.2.1: There is just one G-index and we finally get

$$C_1(\xi_1\xi_2, \eta_1\eta_2) = \frac{1}{2}(\xi_2\eta_2[\xi_1, \eta_1] + \xi_2\eta_1[\xi_1, \eta_2] + \xi_1\eta_2[\xi_2, \eta_1] + \xi_1\eta_1[\xi_2, \eta_2]).$$

C_2: This is the first time, something interesting happens. We have three G-indices:

$$\mathcal{G}_2(2,2) = \{J_1, J_2, J_3\} = \big\{\big((0,1),(2,1)\big), \big((1,0),(1,2)\big), \big((1,1),(1,1)\big)\big\}.$$

The G-factorials give $J_1! = J_2! = 1$ and $J_3! = 2$, since the index $(1,1)$ appears twice in J_3. We take $\mathrm{BCH}_{a,b}(\xi, \eta)$ from Equation (3.3.3) for $(a, b) \in \{(1, 2), (2, 1)\}$:

$$\mathrm{BCH}_{1,2}(\xi,\eta) = \frac{1}{12}[[\xi,\eta],\eta] \quad \text{and} \quad \mathrm{BCH}_{2,1}(\xi,\eta) = \frac{1}{12}[[\eta,\xi],\xi].$$

So we have to insert the ξ_i and the η_j into $\frac{1}{12}\xi[[\xi, \eta], \eta]$ and $\frac{1}{12}\eta[[\eta, \xi], \xi]$ respectively and then we go on with the last one, which is

$$\frac{1}{2}\mathrm{BCH}_{1,1}(\xi,\eta)\mathrm{BCH}_{1,1}(\xi,\eta) = \frac{1}{8}[\xi,\eta][\xi,\eta].$$

We hence get

$$\begin{aligned} C_2(\xi_1,\xi_2,\eta_1,\eta_2) = \frac{1}{12}\big(&\eta_1[[\eta_2,\xi_1],\xi_2] + \eta_1[[\eta_2,\xi_2],\xi_1] + \eta_2[[\eta_1,\xi_1],\xi_2] + \eta_2[[\eta_1,\xi_2],\xi_1] + \\ &\xi_1[[\xi_2,\eta_1],\eta_2] + \xi_1[[\xi_2,\eta_2],\eta_1] + \xi_2[[\xi_1,\eta_1],\eta_2] + \xi_2[[\xi_1,\eta_2],\eta_1]\big) + \\ &\frac{1}{4}\big([\xi_1,\eta_1][\xi_2,\eta_2] + [\xi_1,\eta_2][\xi_2,\eta_1]\big) \end{aligned}$$

C_3: Here, we only have one G-index:

$$\mathcal{G}_1(2,2) = \big\{\big((2,2)\big)\big\}$$

The G-factorial is 1. We take again Equation (3.3.3) and see

$$\mathrm{BCH}_{2,2}(\xi,\eta) = \frac{1}{24}[[[\eta,\xi],\xi],\eta].$$

This gives

$$C_3(\xi_1,\xi_2,\eta_1,\eta_2) = \frac{1}{24}\big([[[\eta_1,\xi_1],\xi_2],\eta_2] + [[[\eta_1,\xi_2],\xi_1],\eta_2] + [[[\eta_2,\xi_1],\xi_2],\eta_1] + [[[\eta_2,\xi_2],\xi_1],\eta_1]\big)$$

We just have to put all the four terms together and have the star product.

A cubic and a quadratic term

Let $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2 \in \mathfrak{g}$. We compute

$$\xi_1\xi_2\xi_3 \star_G \eta_1\eta_2 = \sum_{n=0}^{4} z^n C_n(\xi_1\xi_2\xi_3, \eta_1\eta_2)$$

C_0: The first part is again just the symmetric product:

$$C_0(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \xi_1\xi_2\xi_3\eta_1\eta_2.$$

C_1: Here we have again the term from Corollary 4.2.1:

$$\begin{aligned} C_1(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \frac{1}{2}\big(&\xi_2\xi_3\eta_2[\xi_1, \eta_1] + \xi_2\xi_3\eta_1[\xi_1, \eta_2] + \xi_1\xi_3\eta_2[\xi_2, \eta_1]+ \\ &\xi_1\xi_3\eta_1[\xi_2, \eta_2] + \xi_1\xi_2\eta_2[\xi_3, \eta_1] + \xi_1\xi_2\eta_1[\xi_3, \eta_2]\big) \end{aligned}$$

C_2: Here the calculation is very similar to the one of C_2 in the example before. We have three G-indices:

$$\mathcal{G}_3(3,2) = \{J_1, J_2, J_3\} = \big\{\big((0,1),(1,0),(2,1)\big), \big((1,0),(1,0),(1,2)\big), \big((1,0),(1,1),(1,1)\big)\big\}.$$

The G-factorials are now $J_1! = 1$ and $J_2! = J_3! = 2$. Again, we take the BCH terms from Equation (3.3.3) and see, that we must insert the ξ_i and the η_j into

$$\frac{1}{12}\xi\eta[[\eta,\xi],\xi] + \frac{1}{24}\xi\xi[[\xi,\eta],\eta] + \frac{1}{8}\xi[\xi,\eta][\xi,\eta].$$

Now we go through all the possible permutations and get

$$\begin{aligned} C_2(\xi_1\xi_2\xi_3, \eta_1\eta_2) = \;&\frac{1}{12}\big(\xi_1\xi_2[[\xi_3,\eta_1],\eta_2] + \xi_1\xi_2[[\xi_3,\eta_2],\eta_1] + \xi_1\xi_3[[\xi_2,\eta_1],\eta_2]+ \\ &\xi_1\xi_3[[\xi_2,\eta_2],\eta_1] + \xi_2\xi_3[[\xi_1,\eta_1],\eta_2] + \xi_2\xi_3[[\xi_1,\eta_2],\eta_1]\big)+ \\ &\frac{1}{12}\big(\xi_1\eta_1[[\eta_2,\xi_2],\xi_3] + \xi_1\eta_2[[\eta_1,\xi_2],\xi_3] + \xi_1\eta_1[[\eta_2,\xi_3],\xi_2]+ \\ &\xi_1\eta_2[[\eta_1,\xi_3],\xi_2] + \xi_2\eta_1[[\eta_2,\xi_1],\xi_3] + \xi_2\eta_2[[\eta_1,\xi_1],\xi_3]+ \\ &\xi_2\eta_1[[\eta_2,\xi_3],\xi_1] + \xi_2\eta_2[[\eta_1,\xi_3],\xi_1] + \xi_3\eta_1[[\eta_2,\xi_2],\xi_1]+ \\ &\xi_3\eta_2[[\eta_1,\xi_2],\xi_1] + \xi_3\eta_1[[\eta_2,\xi_1],\xi_2] + \xi_3\eta_2[[\eta_1,\xi_1],\xi_2]\big)+ \\ &\frac{1}{4}\big(\xi_1[\xi_2,\eta_1][\xi_3,\eta_2] + \xi_1[\xi_3,\eta_1][\xi_2,\eta_2] + \xi_2[\xi_1,\eta_1][\xi_3,\eta_2]+ \\ &\xi_2[\xi_3,\eta_1][\xi_1,\eta_2] + \xi_3[\xi_1,\eta_1][\xi_2,\eta_2] + \xi_3[\xi_2,\eta_1][\xi_1,\eta_2]\big). \end{aligned}$$

C_3: We first calculate the G-indices:

$$\mathcal{G}_2(3,2) = \{J_1, J_2, J_3\} = \big\{\big((0,1),(3,1)\big), \big((1,0),(2,2)\big), \big((1,1),(2,1)\big)\big\}.$$

We can omit J_1, since $\mathrm{BCH}_{3,1}(\xi,\eta) = 0$. The G-factorials for the other two indices are 1. The BCH terms have been computed before. So we have to fill in the expression

$$\frac{1}{24}\xi[[[\eta,\xi],\xi],\eta] + \frac{1}{2\cdot 12}[\xi,\eta][[\eta,\xi],\xi].$$

Going through the permutations we get

$$\begin{aligned}
C_3(\xi_1\xi_2\xi_3,\eta_1\eta_2) = \frac{1}{24}\big(&\xi_1[[[\eta_1,\xi_2],\xi_3],\eta_2] + \xi_1[[[\eta_2,\xi_2],\xi_3],\eta_1] + \xi_1[[[\eta_1,\xi_3],\xi_2],\eta_2]+\\
&\xi_1[[[\eta_2,\xi_3],\xi_2],\eta_1] + \xi_2[[[\eta_1,\xi_1],\xi_3],\eta_2] + \xi_2[[[\eta_2,\xi_1],\xi_3],\eta_1]+\\
&\xi_2[[[\eta_1,\xi_3],\xi_1],\eta_2] + \xi_2[[[\eta_2,\xi_3],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_2],\xi_1],\eta_2]+\\
&\xi_3[[[\eta_2,\xi_2],\xi_1],\eta_1] + \xi_3[[[\eta_1,\xi_1],\xi_2],\eta_2] + \xi_3[[[\eta_2,\xi_1],\xi_2],\eta_1]\big)+\\
&\frac{1}{24}\big([\xi_1,\eta_1][[\eta_2,\xi_2],\xi_3] + [\xi_1,\eta_2][[\eta_1,\xi_2],\xi_3] + [\xi_1,\eta_1][[\eta_2,\xi_3],\xi_2]+\\
&[\xi_1,\eta_2][[\eta_1,\xi_3],\xi_2] + [\xi_2,\eta_1][[\eta_2,\xi_1],\xi_3] + [\xi_2,\eta_2][[\eta_1,\xi_1],\xi_3]+\\
&[\xi_2,\eta_1][[\eta_2,\xi_3],\xi_1] + [\xi_2,\eta_2][[\eta_1,\xi_3],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_2],\xi_1]+\\
&[\xi_3,\eta_2][[\eta_1,\xi_2],\xi_1] + [\xi_3,\eta_1][[\eta_2,\xi_1],\xi_2] + [\xi_3,\eta_2][[\eta_1,\xi_1],\xi_2]\big).
\end{aligned}$$

C_4: Now there is only C_4 left. We have one G-index:

$$\mathcal{G}_1(3,2) = \big\{\big((3,2)\big)\big\},$$

but there are more terms which belong to it. We have to go through

$$\mathrm{BCH}_{3,2}(\xi,\eta) = \frac{1}{120}[[[[\xi,\eta],\xi],\eta],\xi] + \frac{1}{360}[[[[\eta,\xi],\xi],\xi],\eta].$$

So we permute and get

$$\begin{aligned}
C_4(\xi_1\xi_2\xi_3,\eta_1\eta_2) = \frac{1}{120}\big(&[[[[\xi_1,\eta_1],\xi_2],\eta_2],\xi_3] + [[[[\xi_1,\eta_2],\xi_2],\eta_1],\xi_3] + [[[[\xi_1,\eta_1],\xi_3],\eta_2],\xi_2]+\\
&[[[[\xi_1,\eta_2],\xi_3],\eta_1],\xi_2] + [[[[\xi_2,\eta_1],\xi_1],\eta_2],\xi_3] + [[[[\xi_2,\eta_2],\xi_1],\eta_1],\xi_3]+\\
&[[[[\xi_2,\eta_1],\xi_3],\eta_2],\xi_1] + [[[[\xi_2,\eta_2],\xi_3],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_2],\eta_2],\xi_1]+\\
&[[[[\xi_3,\eta_2],\xi_2],\eta_1],\xi_1] + [[[[\xi_3,\eta_1],\xi_1],\eta_2],\xi_2] + [[[[\xi_3,\eta_2],\xi_1],\eta_1],\xi_2]\big)+\\
&\frac{1}{360}\big([[[[\eta_1,\xi_1],\xi_2],\xi_3],\eta_2] + [[[[\eta_2,\xi_1],\xi_2],\xi_3],\eta_1] + [[[[\eta_1,\xi_1],\xi_3],\xi_2],\eta_2]+\\
&[[[[\eta_2,\xi_1],\xi_3],\xi_2],\eta_1] + [[[[\eta_1,\xi_2],\xi_1],\xi_3],\eta_2] + [[[[\eta_2,\xi_2],\xi_1],\xi_3],\eta_1]+\\
&[[[[\eta_1,\xi_2],\xi_3],\xi_1],\eta_2] + [[[[\eta_2,\xi_2],\xi_3],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_2],\xi_1],\eta_2]+\\
&[[[[\eta_2,\xi_3],\xi_2],\xi_1],\eta_1] + [[[[\eta_1,\xi_3],\xi_1],\xi_2],\eta_2] + [[[[\eta_2,\xi_3],\xi_1],\xi_2],\eta_1]\big).
\end{aligned}$$

Now we only have to add up all those terms and we have finally computed the star product.

Chapter 5

A Locally Convex Topology for the Gutt Star Product

We have finished the algebraic part of this work, except for one little lemma concerning the Hopf theoretic chapter. Our next goal is setting up a locally convex topology on the symmetric tensor algebra, in which the Gutt star product will be continuous. At the beginning of this chapter, we will first give a motivation why the setting of locally convex algebras is convenient and necessary. In the second section, we will briefly recall the most important things on locally convex algebras and introduce the topology which we will work with. In the third section, the core of this chapter, the continuity of the star product and the analytic dependence on the formal parameter are proven. We also show that continuous representations of Lie algebras lift to those of the deformed symmetric algebra and that our construction is in fact functorial. Part four treats the case when the formal parameter $z = 1$ and hence talks about a locally convex topology on universal enveloping algebras. We will show, that our topology is "optimal" in a specific sense.

5.1 Why Locally Convex?

There are very different types of topologies on vector spaces and algebras. On one hand, we want a topology on the symmetric tensor algebra – which will after its completion be the algebra of observables – that has as many convenient features as possible, of course. On the other hand, if we choose a too good topology, we will risk to have no representations of our algebra as (unbounded) operators on a Hilbert space, which means something like the $\hat{q}$ and $\hat{p}$ operators. Thus we have to think of the possible options first. Assume that $\mathfrak{g}$ is a complex Lie algebra (if not, we can always complexify it), then the complex conjugation plays the role of a *star involution* on the algebra of observables $\mathscr{A}$, that means it is an automorphism

$$^*\colon \mathscr{A} \longrightarrow \mathscr{A}, \quad a \longmapsto a^*,$$

which fulfils for all $a \in \mathscr{A}$ the identity $(a^*)^* = a$. We want to summarize briefly the options, which we would have in this case.

i.) We can try a C^*-algebra. This means that we have a norm $\|\cdot\|$ on our algebra, which fulfils the two properties

$$\|a \cdot b\| \leq \|a\|\|b\|, \quad \forall_{a,b\in\mathscr{A}} \tag{5.1.1}$$

and

$$\|a^* \cdot a\| = \|a\|^2, \quad \forall_{a,b\in\mathscr{A}}. \tag{5.1.2}$$

The bounded operators on a Hilbert space form a C^*-algebra, for example. In particular, we can define continuous functions of algebra elements in this case, see e.g. [17] or [97].

ii.) A Banach-*-algebra is a bit less: we still have a norm, which fulfils (5.1.1), but not (5.1.2) any more. This stil allows holomorphic functions of algebra elements (like it can be seen in [81]).

iii.) A weaker choice is a *locally multiplicatively convex *-algebra*. We do not have a norm any more in this case, but a set of submultiplicative *seminorms* $\{p_i\}_{i\in I}$. A seminorm p is almost the same as a norm, except that we may have $p(x) = 0$ for some $x \in \mathscr{A}$ with $x \neq 0$. Submultiplicativity means that every seminorm p fulfils

$$p(a \cdot b) \leq p(a)p(b), \quad \forall_{a,b\in\mathscr{A}}. \tag{5.1.3}$$

Examples for such algebras are the entire functions on $\mathbb{C}$ or the continuous functions on $\mathbb{R}$. In this case, entire functions of algebra elements can still be defined (see e.g. [64]).

iv.) As a last possibility, we can have a *locally convex *-algebra*. Here again, we have seminorms, but without (5.1.3). Instead, we have to use another seminorm q to control p of the product of two elements. We will make this statement more precise later. Examples for such structures are distributions or vector fields on vector space. Good books on this kind of structure are [55], [81] or [57].

So in principle, we have a lot of possibilities. Unfortunately, the canonical commutation relations, which should be fulfilled, do not allow submultiplicative structures: this is precisely what we show in the next proposition (and what can also be found in some textbooks, like [75], for example). So all the nice features like a continuous or an entire calculus will simply not be there, and therefore we have to aim at an honestly locally convex topology on the symmetric tensor algebra.

Proposition 5.1.1 *Let $\mathscr{A}$ be a unital associative algebra which contains the quantum mechanical observables $\hat{q}$ and $\hat{p}$ and in which the canonical commutation relation*

$$[\hat{q}, \hat{p}] = i\hbar\mathbb{1}$$

is fulfilled. Then the only submultiplicative seminorm on it is 0.

PROOF: First, we need to show a little lemma:

Lemma 5.1.2 *In the algebra given above, we have for $n \in \mathbb{N}$*

$$(\mathrm{ad}_{\hat{q}})^n(\hat{p}^n) = (i\hbar)^n n!\mathbb{1}. \tag{5.1.4}$$

PROOF: To show it, we use the fact that for $a \in \mathscr{A}$ the operator ad_a is a derivation. This is always true for the commutator Lie algebra of an associative algebra, since for $a, b, c \in \mathscr{A}$ we have

$$[a, bc] = abc - bca = abc - bac + bac - bca = [a, b]c + b[a, c].$$

For $n = 1$, Equation (5.1.4) is certainly true. So let's look at the step $n \to n + 1$. We make use of the derivation property and have

$$(\mathrm{ad}_{\hat{q}})^{n+1}\big(\hat{p}^{n+1}\big) = (\mathrm{ad}_{\hat{q}})^n(i\hbar\hat{p}^n + \hat{p}\,\mathrm{ad}_{\hat{q}}(\hat{p}^n))$$

$$
\begin{aligned}
&= (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^n (\hat{p}\, \mathrm{ad}_{\hat{q}}(\hat{p}^n)) \\
&= (i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n-1} \Big([\hat{q}, \hat{p}]\, \mathrm{ad}_{\hat{q}}(\hat{p}^n) + \hat{p} (\mathrm{ad}_{\hat{q}})^2 (\hat{p}^n) \Big) \\
&= (i\hbar)^{n+1} n! + i\hbar (\mathrm{ad}_{\hat{q}})^n (\hat{p}^n) + (\mathrm{ad}_{\hat{q}})^{n-1} \Big(\hat{p} (\mathrm{ad}_{\hat{q}})^2 (\hat{p}^n) \Big) \\
&= 2(i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n-1} \Big(\hat{p} (\mathrm{ad}_{\hat{q}})^2 (\hat{p}^n) \Big) \\
&\stackrel{(*)}{=} \ \vdots \\
&= n(i\hbar)^{n+1} n! + \mathrm{ad}_{\hat{q}} (\hat{p} (\mathrm{ad}_{\hat{q}})^n (\hat{p}^n)) \\
&= n(i\hbar)^{n+1} n! + i\hbar (i\hbar)^n n! \\
&= (i\hbar)^{n+1} (n+1)!.
\end{aligned}
$$

At $(*)$, we actually used another statement which is to be proven by induction over k and says

$$
(\mathrm{ad}_{\hat{q}})^{n+1} \big(\hat{p}^{n+1} \big) = k(i\hbar)^{n+1} n! + (\mathrm{ad}_{\hat{q}})^{n+1-k} \Big(\hat{p} (\mathrm{ad}_{\hat{q}})^k (\hat{p}^n) \Big).
$$

Since this proof is analogous to the first lines of the computation before, we omit it here and the lemma is proven. $\triangledown$

Now we can go on with the actual proof. Let $\|\cdot\|$ be a submultiplicative seminorm. Then we see from Equation (5.1.4) that

$$
\|(\mathrm{ad}_{\hat{q}})^n (\hat{p}^n)\| = |\hbar|^n n! \|\mathbb{1}\|.
$$

On the other hand, we have

$$
\begin{aligned}
\|(\mathrm{ad}_{\hat{q}})^n (\hat{p}^n)\| &= \|\hat{q} (\mathrm{ad}_{\hat{q}})^{n-1} (\hat{p}^n) - (\mathrm{ad}_{\hat{q}})^{n-1} (\hat{p}^n) \hat{q}\| \\
&\leq 2\|\hat{q}\| \|(\mathrm{ad}_{\hat{q}})^{n-1} (\hat{p}^n)\| \\
&\leq \ \vdots \\
&\leq 2^n \|\hat{q}\|^n \|\hat{p}^n\| \\
&\leq 2^n \|\hat{q}\|^n \|\hat{p}\|^n
\end{aligned}
$$

So in the end we get

$$
|\hbar|^n n! \|\mathbb{1}\| \leq c^n
$$

for $1\|\hat{q}\|\|\hat{p}\| = c \in \mathbb{R}$. This cannot be fulfilled for all $n \in \mathbb{N}$ unless $\|\mathbb{1}\| = 0$. But then, by submultiplicativity, the seminorm itself must be equal to 0. $\square$

Remark 5.1.3 The so called Weyl algebra, which fulfils the properties of the foregoing proposition, can be constructed from a Poisson algebra with constant a Poisson tensor. On one hand, it is a fair to ask the question, why this restriction of not being locally m-convex should also be apply to linear Poisson systems. On the other hand, there is no reason to expect that things become easier when we make the Poisson structure more complex. Moreover, the Weyl algebra is actually nothing but a quotient of the universal enveloping algebra of the so called Heisenberg algebra, which is a particular Lie algebra. There is no reason why the original algebra should have a "better" analytical structure than its quotient, since the ideal, which is divided out by this procedure, is a closed one.

There is a second good reason why we should avoid our topology to be locally m-convex. The topology we set up on $\mathrm{S}^\bullet(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ will also give a topology on $\mathscr{U}(\mathfrak{g})$. In Proposition 5.4.1, we will show that, under weak (but for our purpose necessary) additional assumptions, there can be no topology on $\mathscr{U}(\mathfrak{g})$, which allows an entire holomorphic calculus. This underlines the results from Proposition 5.1.1, since locally m-convex algebras always have such a calculus.

In this sense, we have good reasons to think that $\mathrm{S}^\bullet(\mathfrak{g})$ will not allow a better setting than the one of a locally convex algebra if we want the Gutt star product to be continuous. Before we go to this task, we have to recall some technology from locally convex analysis.

5.2 Locally Convex Algebras

5.2.1 Locally Convex Spaces and Algebras

Every locally convex algebra is also a locally convex space which is, of course, a topological vector space. To make clear what we talk about, we first give a definition which is taken from [81, Definition 1.6].

Definition 5.2.1 (Topological vector space) *Let V be a vector space endowed with a topology τ. Then we call (V, τ) (or just V, if there is no confusion possible) a topological vector space, if the two following things hold:*

i.) for every point in $x \in V$ the set $\{x\}$ is closed and

ii.) the vector space operations (addition, scalar multiplication) are continuous.

Not all books require axiom (i) for a topological vector space. It is, however, useful, since it assures that our topological vector space is Hausdorff – a feature which we always want to have. The proof for this is not difficult, but since we do not want to go too much into details here, we refer to [81] again, where it can be found in Theorem 1.10 and 1.12.

The most important class of topological vector spaces are, at least from a physical point of view, locally convex ones. Almost all interesting physical examples belong to this class: Finite-dimensional spaces, inner product (or pre-Hilbert) spaces, Banach spaces, Fréchet spaces, nuclear spaces and many more. There are at least two equivalent definitions of what is a locally convex space. While the first is more geometrical, the second is better suited for our analytic purpose.

Theorem 5.2.2 *For a topological vector space V, the following things are equivalent.*

i.) V has a local base $\mathscr{B}$ of the topology whose members are convex, i.e. for all $U in \mathscr{B}$ we have for all $x, y \in U$ and all $\lambda \in [0, 1]$: $\lambda x + (1 - \lambda)y \in U$.

ii.) The topology on V is generated by a separating family of seminorms $\mathcal{P}$.

PROOF: This theorem is a very well-known result and can be found in standard literature, such as [81] again, where it is divided into two Theorems (namely 1.36 and 1.37). □

Definition 5.2.3 (Locally convex space) *A locally convex space is a topological vector space in which one (and thus all) of the properties from Theorem 5.2.2 are fulfilled.*

The first property explains the term "locally convex". For our concrete computations, the second property is more helpful, since in this setting proving continuity just means putting estimates on seminorms. For this purpose, one often extends the set of seminorms $\mathcal{P}$ to the set of all continuous seminorms $\mathscr{P}$ which contains all seminorms that are compatible with the topology:

$$p \in \mathscr{P} \iff \exists_{n \in \mathbb{N}} \exists_{c_1,\ldots,c_n > 0} \exists_{p_1,\ldots,p_n \in \mathcal{P}} : p \leq \sum_{i=1}^{n} c_i p_i.$$

In particular, one can take maxima of finitely many continuous seminorms to get again a continuous seminorm. From here, we can start looking at locally convex algebras.

Definition 5.2.4 (Locally convex algebra) *A locally convex algebra is a locally convex vector space with an additional algebra structure which is continuous.*

More precisely, let $\mathscr{A}$ be a locally convex algebra and $\mathscr{P}$ the set of all continuous seminorms, then for all $p \in \mathscr{P}$ there exists a $q \in \mathscr{P}$ such that for all $x, y \in \mathscr{A}$ one has

$$p(a \cdot b) \leq q(a)q(b). \tag{5.2.1}$$

Remind that we did not require our algebras to be associative. The product in this equation could also be a Lie bracket. If we talk about associative algebras, we will always say it explicitly.

5.2.2 A Special Class of Locally Convex Algebras

For our study of the Gutt star product, the usual continuity estimate (5.2.1) will not be enough, since there will be an arbitrarily high number of nested brackets to control. We will need an estimate which does not depend on the number of Lie brackets involved. Since Lie algebras are just one type of algebras, we can define the property we need also for other locally convex algebras.

Definition 5.2.5 (Asymptotic estimate algebra) *Let $\mathscr{A}$ be a locally convex algebra (not necessarily associative) with $\cdot$ denoting the multiplication and $\mathscr{P}$ the set of all continuous seminorms. Let $p \in \mathscr{P}$.*

i.) We call $q \in \mathscr{P}$ an asymptotic estimate for p, if

$$p(w_n(x_1, \ldots, x_n)) \leq q(x_1) \cdots q(x_n) \tag{5.2.2}$$

for all words $w_n(x_1, \ldots, x_n)$ made out of $n-1$ products of the elements $x_1, \ldots, x_n \in \mathscr{A}$ with arbitrary position of placing brackets.

ii.) A locally convex algebra is said to be an asymptotic estimate algebra (AE algebra), if every $p \in \mathscr{P}$ has an asymptotic estimate.

Remark 5.2.6 (The notion "asymptotic estimate")

i.) The term asymptotic estimate has, to the best of our knowledge, first been used by Boseck, Czichowski and Rudolph in [22]. They gave a seemingly weaker definition of asymptotic estimates, which is in fact equivalent. They wanted that for every seminorm $p \in \mathscr{P}$, there is a $m \in \mathbb{N}$ and a $q \in \mathscr{P}$ such that for all $n \geq m$ one has

$$p(x_1 \cdots x_n) \leq q(x_1) \cdots q(x_n) \quad \forall_{x_1, \ldots, x_n \in \mathscr{A}}. \tag{5.2.3}$$

But here we can set $m = 1$, since we just need to take the maximum of a finite number of continuous seminorms. Let q satisfy (5.2.3). By Continuity, we have for all $i = 2, \ldots, m-1$

$$p(x_1 \cdots x_i) \leq q^{(i)}(x_1) \cdots q^{(i)}(x_i) \quad \forall_{x_1, \ldots, x_i \in \mathscr{A}}$$

Now we get an asymptotic estimate q' for p in our sense by setting

$$q' = \max\{p, q^{(2)}, \ldots, q^{(m-1)}, q\}.$$

Also AE algebras were defined by Boseck et al. in [22], but here the notion is really different from ours: for them, in an AE algebra every continuous seminorm admits a sequence of asymptotic estimates. This sequence must fulfil two additional properties, which actually make the algebra locally m-convex. Our definition is weaker, since it does not imply, a priori, the existence of an topologically equivalent set of submultiplicative seminorms.

ii.) In [44], Glöckner and Neeb used a property to which they referred as $(*)$ for associative algebras. It was then used in [18] by Bogfjellmo, Dahmen and Schmedig, who called it the GN-property. It is rather easy to see that it is equivalent to our AE condition.

There are, of course, a lot of examples of AE (Lie) algebras. All finite dimensional and Banach (Lie) algebras fulfil Inequality (5.2.2), just as locally m-convex (Lie) algebras do. The same is true for nilpotent locally convex Lie algebras, since here again one just has to take the maximum of a finite number of semi-norms, analogously to the procedure in Remark 5.2.6. On the other hand, it is not clear what the AE property implies exactly. Are there examples for associative algebras which are AE but not locally m-convex, for example? Are there Lie algebras which are truly AE and not locally m-convex or nilpotent? We don't have an answer to this questions, but we can make some simple observations, which allow us to give an answer for special cases.

Proposition 5.2.7 (Entire calculus) *Let $\mathscr{A}$ be an associative AE algebra. Then it admits an entire calculus.*

Proof: The proof is the same as for locally m-convex algebras: let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and p a continuous seminorm with an asymptotic estimate q. Then one has $\forall_{x \in \mathcal{A}}$

$$p(f(x)) = p\left(\sum_{n=0}^{\infty} a_n x^n\right) \leq \sum_{n=0}^{\infty} |a_n| p(x^n) \leq \sum_{n=0}^{\infty} |a_n| q(x)^n < \infty. \qquad \square$$

Remark 5.2.8 (Entire Calculus, AE and LMC algebras) The fact that associative AE algebras have an entire calculus makes them very similar to locally m-convex ones. Now there is something we can say about associative algebras which have an entire calculus: if such an algebra is additionally commutative and Fréchet, then must be even locally m-convex. This statement was proven in [65] by Mitiagin, Rolewicz and Żelazko. Oudadess and El kinani extended this result to commutative, associative algebras, in which the Baire category theorem holds [40]. For non-commutative algebras, the situation is different. There are associative "Baire algebras" having an entire calculus, which are not locally m-convex. Żelazko gave an example for such an algebra in [101]. Unfortunately, his example is also not AE. It seems to be is an interesting (and non-trivial) question, if a non locally m-convex but AE algebra exists at all and if yes, how an example could look like.

5.2.3 The Projective Tensor Product

We want to set up a topology on $\mathrm{S}^\bullet(\mathfrak{g})$. Therefore, we first construct a topology on the tensor algebra $\mathrm{T}^\bullet(\mathfrak{g})$. Since the construction is possible starting from any locally convex vector space, we want to go back to this more general setting for a moment. V will always denote a locally convex vector space and $\mathscr{P}$ the set of its continuous seminorms. In this situation, we can use the projective tensor product $\otimes_\pi$ to get a locally convex topology on each tensor power $V^{\otimes_\pi n}$. The precise construction of the projective tensor product can be found in good textbooks on locally convex analysis like [57, Chapter 15] or in the lecture notes [97, Lemma 4.1.4]. Recall that for $p_1, \ldots, p_n \in \mathscr{P}$ we have a continuous seminorm on $V^{\otimes_\pi n}$ via

$$(p_1 \otimes_\pi \cdots \otimes_\pi p_n)(x) = \inf\left\{ \sum_i p_1\left(x_i^{(1)}\right) \cdots p_n\left(x_i^{(n)}\right) \;\middle|\; x = \sum_i x_i^{(1)} \otimes \cdots \otimes x_i^{(n)} \right\}.$$

On factorizing tensors, we moreover have the property

$$(p_1 \otimes_\pi \cdots \otimes_\pi p_n)(x_1 \otimes_\pi \cdots \otimes_\pi x_n) = p_1(x_1) \cdots p_n(x_n) \tag{5.2.4}$$

which will be extremely useful in the following and which can be proven by the Hahn-Banach theorem. We also have

$$(p_1 \otimes \cdots \otimes p_n) \otimes (q_1 \otimes \cdots \otimes q_m) = p_1 \otimes \cdots \otimes p_n \otimes q_1 \otimes \cdots \otimes q_m.$$

For a given $p \in \mathscr{P}$ we will denote $p^n = p^{\otimes_\pi n}$ and p^0 is just the absolute value on the field $\mathbb{K}$. The π-topology on $V^{\otimes_\pi n}$ is set up by all the projective tensor products of continuous seminorms, or, equivalently, by all the p^n for $p \in \mathscr{P}$.

The projective tensor product has a very helpful feature: if we want to show a (continuity) estimate on the tensor algebra, it is enough to do so on factorizing tensors. We will use this very often and just refer to it as the "infimum argument".

Lemma 5.2.9 (Infimum argument for the projective tensor product) *Let $V_1, \ldots, V_n$, W be locally convex vector spaces and*

$$\phi\colon V_1 \times \cdots \times V_n \longrightarrow W$$

a n-linear map, from which we get the linear map $\Phi\colon V_1 \otimes_\pi \cdots \otimes_\pi V_n \longrightarrow W$. Then Φ is continuous if and only if this is true for ϕ. If the estimate

$$p(\Phi(x_1 \otimes \cdots \otimes x_n)) \leq q_i(x_1) \cdots q_i(x_n) \tag{5.2.5}$$

is fulfilled for $p \in \mathscr{P}_V$, $q_i \in \mathscr{P}_{W_i}$ and all $x_i \in V_i, i = 1, \ldots, n$ then we have

$$p(\Phi(x)) \leq (q_1 \otimes \cdots \otimes q_n)(x) \tag{5.2.6}$$

for all $x \in V_1 \otimes \cdots \otimes V_n$.

Proof: If Φ is continuous, the continuity of ϕ is clear. The other implication is more interesting. Continuity for ϕ means, that for every continuous seminorm p on W we have continuous seminorms q_i on V_i with $i = 1, \ldots, n$ such that for all $x^{(i)} \in V_i$ the estimate

$$p\left(\phi\left(x^{(1)}, \ldots, x^{(n)}\right)\right) \leq q_1\left(x^{(1)}\right) \cdots q_n\left(x^{(n)}\right) \tag{5.2.7}$$

holds. Let $x \in V_1 \otimes_\pi \ldots \otimes_\pi V_n$, then it has a representation in terms of factorizing tensors like

$$x = \sum_j x_j^{(1)} \otimes_\pi \cdots \otimes_\pi x_j^{(n)}.$$

We thus have

$$p(\Phi(x)) = p\left(\sum_j \Phi\left(x_j^{(1)} \otimes_\pi \cdots \otimes_\pi x_j^{(n)}\right) \right)$$

$$
\begin{aligned}
&\leq \sum_j p\Big(\phi\Big(x_j^{(1)}, \cdots, x_j^{(n)}\Big)\Big) \\
&\leq \sum_j q_1\Big(x_j^{(1)}\Big) \cdots q_n\Big(x_j^{(n)}\Big).
\end{aligned}
$$

Now we take the infimum over all possibilities of writing x as a sum of factorizing tensors on both sides. While nothing happens on the left hand side, on the right hand side we find $(q_1 \otimes_\pi \ldots \otimes_\pi q_n)(x)$:

$$
p(\Phi(x)) \leq \inf\left\{ \sum_j q_1\Big(x_j^{(1)}\Big) \cdots q_n\Big(x_j^{(n)}\Big) \;\middle|\; x = \sum_j x_j^{(1)} \cdots x_j^{(n)} \right\}.
$$

This is the estimate we wanted and the Lemma is proven. □

Most of the time, we will deal with the symmetric tensor algebra. Therefore, we want to recall some basic facts about $\mathrm{S}^n(V)$, when it inherits the π-topology from the $V^{\otimes_\pi n}$. We will call it $\mathrm{S}^n_\pi(V)$ when we endow it with this topology.

Lemma 5.2.10 *Let V be a locally convex vector space, p a continuous seminorm and $n, m \in \mathbb{N}$.*

i.) The symmetrization map

$$
\mathscr{S}_n \colon V^{\otimes_\pi n} \longrightarrow V^{\otimes_\pi n}, \quad (x_1 \otimes \ldots \otimes x_n) \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}
$$

is continuous and we have for all $x \in V^{\otimes_\pi n}$ the estimate

$$
p^n(\mathscr{S}_n(x)) \leq p^n(x). \tag{5.2.8}
$$

ii.) Each symmetric tensor power $\mathrm{S}^n_\pi(V) \subseteq V^{\otimes_\pi n}$ is a closed subspace.

iii.) For $x \in \mathrm{S}^n_\pi(V)$ and $y \in \mathrm{S}^m_\pi(V)$ we have

$$
p^{n+m}(xy) \leq p^n(x) p^m(y).
$$

Proof: The first part is very easy to see and uses most of the tools which are typical for the projective tensor product. We have the estimate for factorizing tensors $x_1 \otimes \ldots \otimes x_n$

$$
\begin{aligned}
p^n(\mathscr{S}(x_1 \otimes \ldots \otimes x_n)) &= p^n\left(\frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\right) \\
&\leq \frac{1}{n!} \sum_{\sigma \in S_n} p^n\big(x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}\big) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} p\big(x_{\sigma(1)}\big) \ldots p\big(x_{\sigma(n)}\big) \\
&= p(x_1) \ldots p(x_n) \\
&= p^n(x_1 \otimes \ldots \otimes x_n).
\end{aligned}
$$

Then we use the infimum argument from Lemma 5.2.9 and we are done. The second part is also easy since the kernel of a continuous map is always a closed subspace of the initial space and we have

$$
\mathrm{S}^n_\pi = \ker(\mathsf{id} - \mathscr{S}_n).
$$

The third part is a consequence from the first. □

One could maybe think that the inequality in the first part of this lemma is just an artefact which is due to the infimum argument and should actually be an equality, if one looked at it more closely. It is very interesting to see, that this is *not* the case, since it may happen that this inequality is strict. The following example illustrates this.

Example 5.2.11 We take $V = \mathbb{R}^2$ with the standard basis e_1, e_2 and V is endowed with the maximum norm. Hence we have $\|e_1\| = \|e_2\| = 1$. Now look at $e_1 \otimes e_2$, which has the norm

$$\|e_1 \otimes e_2\| = \|e_1\| \otimes \|e_2\| = 1.$$

We now evaluate the symmetrization map on $V \otimes_\pi V$:

$$\mathscr{S}(e_1 \otimes e_2) = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1).$$

Our aim is to show, that the projective tensor product of the norm of this symmetrized vector is not 1. Therefore we need to find another way of writing it which has a norm of less than 1. Observe that

$$\frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) = \frac{1}{4}((e_1 + e_2) \otimes (e_1 + e_2) + (-e_1 + e_2) \otimes (e_1 - e_2))$$

and we have

$$\begin{aligned}
&\frac{1}{4}\|(e_1 + e_2) \otimes (e_1 + e_2) + (-e_1 + e_2) \otimes (e_1 - e_2)\| \\
&\leq \frac{1}{4}(\|(e_1 + e_2) \otimes (e_1 + e_2)\| + \|(-e_1 + e_2) \otimes (e_1 - e_2)\|) \\
&\leq \frac{1}{4}(\|e_1 + e_2\| \cdot \|e_1 + e_2\| + \|-e_1 + e_2\| \cdot \|e_1 - e_2\|) \\
&= \frac{1}{4}(1 \cdot 1 + 1 \cdot 1) \\
&= \frac{1}{2}.
\end{aligned}$$

So we have $\|\mathscr{S}(e_1 \otimes e_2)\| \leq \frac{1}{2} < 1$.

5.2.4 A Topology for the Gutt Star Product

The next step is to set up a topology on $\mathrm{T}^\bullet(V)$ which has the π-topology on each component. A priori, there are a lot of such topologies and at least two natural ones: the direct sum topology which is very fine and has a very small closure, and the cartesian product topology which is very coarse and therefore has a very big closure. We need something in between, which we can adjust in a convenient way.

Definition 5.2.12 (T_R-topology) *Let p be a continuous seminorm on a locally convex vector space V and $R \in \mathbb{R}$. We define by*

$$p_R = \sum_{n=0}^{\infty} n!^R p^n$$

a seminorm on the tensor algebra $\mathrm{T}^\bullet(V)$. We write for the tensor or the symmetric algebra endowed with all such seminorms $\mathrm{T}^\bullet_R(V)$ or $\mathrm{S}^\bullet_R(V)$ respectively.

Now we want to collect the most important results on the locally convex algebras $(\mathrm{T}^\bullet_R(V), \otimes)$ and $(\mathrm{S}^\bullet_R(V), \vee)$.

Lemma 5.2.13 (The T_R-topology) *Let $R' \geq R \geq 0$ and q, p are continuous semi-norms on V.*

i.) If $q \geq p$ then $q_R \geq p_R$ and $p_{R'} \geq p_R$.

ii.) The tensor product is continuous and satisfies the following inequality:

$$p_R(x \otimes y) \leq \left(2^R p\right)_R(x)\left(2^R p\right)_R(y)$$

iii.) For all $n \in \mathbb{N}$ the induced topology on $\mathrm{T}^n(V) \subset \mathrm{T}^\bullet_R(V)$ and on $\mathrm{S}^n(V) \subset \mathrm{S}^\bullet_R(V)$ is the π-topology.

iv.) For all $n \in \mathbb{N}$ the projection and the inclusion maps

$$\begin{array}{ccccc} \mathrm{T}^\bullet_R(V) & \xrightarrow{\pi_n} & V^{\otimes_\pi n} & \xrightarrow{\iota_n} & \mathrm{T}^\bullet(V) \\ \mathrm{S}^\bullet_R(V) & \xrightarrow{\pi_n} & \mathrm{S}^n_\pi(V) & \xrightarrow{\iota_n} & \mathrm{S}^\bullet_R(V) \end{array}$$

are continuous.

v.) The completions $\widehat{\mathrm{T}}^\bullet_R(V)$ of $\mathrm{T}^\bullet_R(V)$ and $\widehat{\mathrm{S}}^\bullet_R(V)$ of $\mathrm{S}^\bullet_R(V)$ can be described explicitly as

$$\widehat{\mathrm{T}}^\bullet_R(V) = \left\{ x = \sum_{n=0}^\infty x_n \;\middle|\; p_R(x) < \infty, \text{ for all } p \right\} \subseteq \prod_{n=0}^\infty V^{\hat{\otimes}_\pi n}$$

$$\widehat{\mathrm{S}}^\bullet_R(V) = \left\{ x = \sum_{n=0}^\infty x_n \;\middle|\; p_R(x) < \infty, \text{ for all } p \right\} \subseteq \prod_{n=0}^\infty \mathrm{S}^n_{\hat{\otimes}_\pi}$$

with p running through all continuous semi-norms on V and the p_R are extended to the Cartesian product allowing the value $+\infty$.

vi.) If $R' > R$, then the topology on $\mathrm{T}^\bullet_{R'}(V)$ is strictly finer than the one on $\mathrm{T}^\bullet_R(V)$, the same holds for $\mathrm{S}^\bullet_{R'}(V)$ and $\mathrm{S}^\bullet_R(V)$. Therefore the completions get smaller for bigger R.

vii.) The inclusion maps $\widehat{\mathrm{T}}^\bullet_{R'}(V) \longrightarrow \widehat{\mathrm{T}}^\bullet_R(V)$ and $\widehat{\mathrm{S}}^\bullet_{R'}(\mathfrak{g}) \longrightarrow \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ are continuous.

viii.) The topology on $\mathrm{T}^\bullet_R(V)$ with the tensor product and on $\mathrm{S}^\bullet_R(V)$ with the symmetric product is locally m-convex if and only if $R = 0$.

ix.) The algebras $\mathrm{T}^\bullet_R(V)$ and $\mathrm{S}^\bullet_R(V)$ are first countable if and only if this is true for V.

x.) The evaluation functionals $\delta_\varphi \colon \mathrm{S}^\bullet_R(V) \longrightarrow \mathbb{K}$ for $\varphi \in \mathfrak{g}'$ are continuous.

PROOF: The first part is clear on factorizing tensors and extends to the whole tensor algebra via the infimum argument. For part *(ii)*, take two factorizing tensors

$$x = x^{(1)} \otimes \cdots \otimes x^{(n)} \quad \text{and } y = y^{(1)} \otimes \cdots \otimes y^{(m)}$$

and compute:

$$\begin{aligned} p_R(x \otimes y) &= (n+m)!^R p^{n+m}\left(x^{(1)} \otimes \cdots \otimes x^{(n)} \otimes y^{(1)} \otimes \cdots \otimes y^{(m)}\right) \\ &= (n+m)!^R p^n\left(x^{(1)} \otimes \cdots \otimes x^{(n)}\right) p^m\left(y^{(1)} \otimes \cdots \otimes y^{(m)}\right) \\ &= \binom{n+m}{n}^R n!^R m!^R p^n\left(x^{(1)} \otimes \cdots \otimes x^{(n)}\right) p^m\left(y^{(1)} \otimes \cdots \otimes y^{(m)}\right) \\ &\leq 2^{(n+m)R} p_R\left(x^{(1)} \otimes \cdots \otimes x^{(n)}\right) p_R\left(y^{(1)} \otimes \cdots \otimes y^{(m)}\right) \\ &= \left(2^R p\right)_R\left(x^{(1)} \otimes \cdots \otimes x^{(n)}\right)\left(2^R p\right)_R\left(y^{(1)} \otimes \cdots \otimes y^{(m)}\right). \end{aligned}$$

The parts (*iii*) and (*iv*) are clear from the construction of the R- topology. In part (*v*) we used the completion of the tensor product $\hat{\otimes}$, the statement itself is clear and implies (*vi*) directly, since we have really more elements in the completion for $R < R'$, like the series over $x^n \frac{1}{n!^t}$ for $t \in (R, R')$ and $0 \neq x \in V$. Statement (*vii*) follows from the first. For (*viii*), it is easy to see that $\mathrm{T}_0^\bullet(V)$ and $\mathrm{S}_0^\bullet(V)$ are locally m-convex. For every $R > 0$ we have

$$p_R(x^n) = n!^R p(x)^n$$

for all $n \in \mathbb{N}$ and all $x \in V$. If we had a submultiplicative seminorm φ from an equivalent topology, then we would have some $x \in V$, and a continuous seminorm p with $p(x) \neq 0$ such that $p_R \leq \varphi$, and hence

$$n!^R p(x)^R \leq \varphi(x^n) \leq \varphi(x)^n.$$

Since this is valid for all $n \in \mathbb{N}$, we get a contradiction. For the ninth part, the tensor algebras can not be first countable if this is not true for V itself. On the other hand, if V has a finite base of the topology, then $\mathrm{T}_R^\bullet(V)$ and $\mathrm{S}_R^\bullet(V)$ are just a countable multiple of V and stay therefore first countable. The last part finally assures, that every symmetric tensor really gives a continuous function and comes from the continuity of φ. □

The projective tensor product obviously keeps a lot of important and strong properties of the original vector space V. But Lemma 5.2.13 still leaves out some important things. We will not make use of them in the following, but it is worth naming them for completeness. To do this in full generality, we need one more definition, which will be also very important in Chapter 6.

Definition 5.2.14 *For a locally convex vector space V and $R \geq 0$ we set*

$$\mathrm{S}_{R^-}^\bullet(V) = \operatorname*{proj\,lim}_{\epsilon \longrightarrow 0} \mathrm{S}_{1-\epsilon}^\bullet(V)$$

and call its completion $\widehat{\mathrm{S}}_{R^-}^\bullet(V)$.

The projective limit is the intersection of all the algebras $\mathrm{S}_R^\bullet(V)$ for $R < 1$. Its completion can be understood as all those series in the cartesian product, which converge for all p_R for $R < 1$ and p a continuous seminorm on V. We hence see that the completion $\widehat{\mathrm{S}}_{R^-}^\bullet(V)$ is bigger than $\widehat{\mathrm{S}}_R^\bullet(V)$ without the projective limit. This will become important in Chapter 6. Now we can state two more propositions. Since we won't use them, we omit the proofs here. They can be found in [96].

Proposition 5.2.15 (Schauder bases) *Let $R \geq 0$ and V a locally convex vector space. If $\{e_i\}_{i \in I}$ is an absolute Schauder basis of V with coefficient functionals $\{\varphi^i\}_{i \in I}$, i.e. for every $x \in V$ we have*

$$x = \sum_{i \in I} \varphi^i(x) e_i$$

such that for every $p \in \mathscr{P}$ there is a $q \in \mathscr{P}$ such that

$$\sum_{i \in I} |\varphi^i(x)| p(e_i) \leq q(x), \tag{5.2.9}$$

then the set $\{e_{i_1} \otimes \ldots \otimes e_{i_n}\}_{i_1, \ldots, i_n \in I}$ defines an absolute Schauder basis of $\mathrm{T}_R^\bullet(V)$ together with the linear functionals $\{\varphi^{i_1} \otimes \ldots \otimes \varphi^{i_n}\}_{i_1, \ldots, i_n \in I}$ which satisfy

$$\sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n \in I} \left| \left(\varphi^{i_1} \otimes \ldots \otimes \varphi^{i_n}\right)(x) \right| p_R(e_{i_1} \otimes \ldots \otimes e_{i_n}) \leq q_R(x)$$

for every $x \in \mathrm{T}^\bullet_R(V)$ whenever p and q satisfy (5.2.9). *The same statement is true for $\mathrm{S}^\bullet_R(V)$ and for $\mathrm{S}^\bullet_{R^-}(V)$ (for $R > 0$) when we choose a maximal linearly independent subset out of the set $\{e_{i_1} \dots e_{i_n}\}_{i_1,\dots,i_n \in I}$.*

Proposition 5.2.16 (Nuclearity) *Let V be a locally convex space. For $R \geq 0$ the following statements are equivalent:*

i.) V is nuclear.

ii.) $\mathrm{T}^\bullet_R(V)$ is nuclear.

iii.) $\mathrm{S}^\bullet_R(V)$ is nuclear.

If moreover $R > 0$, then the following statements are equivalent:

i.) V is strongly nuclear.

ii.) $\mathrm{T}^\bullet_R(V)$ is strongly nuclear.

iii.) $\mathrm{S}^\bullet_R(V)$ is strongly nuclear.

Nuclearity is an important and natural property in physics. Actually every example of a locally convex vector space, which play a role in physics, is either a normed space or nuclear. Therefore we want this property to be conserved.

Remark 5.2.17 (The T_R-topology in the literature) It seems that this topology is actually not a new construction, but it was already found and refound in the past. A very similar construction is due to Goodman [46], who looked at finite-dimensional Lie algebras. He studied differential operators on the Lie group, that means the universal enveloping algebra, which he topologised via the tensor algebra using basically the same weights. He needed a basis to do so and proved afterwards that the topology he found is independent of the choice of the basis. He also showed the continuity of the multiplication, but by going a different way: he proved that certain ideals are closed in this topology and divided them out. In this sense his approach is less explicit than ours and also does not apply to infinite-dimensional Lie algebras, where the proofs are more involved. Goodman was more interested in the representation theory which arises from his construction and he also knew about its functoriality. However, he did not know about the deformation aspect behind it, since the ideas of deformation quantization did not exist at that time. It seems a bit like his work has gone more or less unnoticed, we could not find a work which used this topology later for studies in Lie theory. A very different way of constructing a topology which coincides with ours for $R = 1$ was moreover given before by Raševskiĭ in [74], who also restricted to the finite-dimensional case and used bases of his Lie algebra to construct the topology. For this reason, also his ideas are bounded to finite-dimensional Lie algebras.

5.3 Continuity Results for the Gutt Star Product

From now on, we start with an AE Lie algebra $\mathfrak{g}$ rather than with a general locally convex space V. We have most of the tools at hand to show the continuity of the Gutt star product, except for precise estimates on the $\widetilde{\mathrm{BCH}}_{a,b}$-terms. Therefore, we state the following lemma.

Lemma 5.3.1 *Let $\mathfrak{g}$ be a AE-Lie algebra, $\xi, \eta \in \mathfrak{g}$, p a continuous seminorm, q an asymptotic estimate for it and $a, b, n \in \mathbb{N}$ with $a + b = n$. Then, using the Goldberg-Thompson form of the Baker-Campbell-Hausdorff series, we have the following estimates:*

i.) The coefficients g_w from (3.3.8) *fulfil the estimate*

$$\sum_{|w|=n} \left| \frac{g_w}{n} \right| \leq \frac{2}{n}. \tag{5.3.1}$$

Recall that $|w|$ denotes the length of a word w and $[w]$ is the word put in Lie brackets nested to the left.

ii.) *For every word w, which consists of a ξ and b η, we have*

$$p([w]) \leq q(\xi)^a q(\eta)^b. \tag{5.3.2}$$

iii.) *We have the estimate*

$$p\Big(\widetilde{\mathrm{BCH}}_{a,b}(\xi_1,\ldots,\xi_a;\eta_1,\ldots,\eta_b)\Big) \leq \frac{2}{a+b} q(\xi_1)\cdots q(\xi_a) q(\eta_1)\cdots q(\eta_b) \tag{5.3.3}$$

PROOF: We already showed part (i) in Proposition 3.3.3 and put in the factor $\frac{1}{n}$ because these factors will appear later. The next estimate (5.3.2) is due to the AE property which does not see the way how brackets are set but just counts the number of ξ_i and η_j in the whole expression. Let us use the notation $|w|_\xi$ for the number of ξ's appearing in a word w and $|w|_\eta$ for the number of η's. Clearly, $|w| = |w|_\xi + |w|_\eta$. With (5.3.1) and the AE property of $\mathfrak{g}$, we get

$$\begin{aligned}
p\Big(\widetilde{\mathrm{BCH}}_{a,b}(\xi_1,\ldots,\xi_a;\eta_1,\ldots,\eta_b)\Big) &\leq \sum_{\substack{|w|_\xi = a \\ |w|_\eta = b}} p^{a+b}\left(\frac{g_w}{a+b}[w]\right) \\
&\leq \sum_{\substack{|w|_\xi = a \\ |w|_\eta = b}} \frac{|g_w|}{a+b} p([w]) \\
&\leq \sum_{|w| = a+b} \frac{|g_w|}{a+b} q(\xi_1)\cdots q(\xi_a) q(\eta_1)\cdots q(\eta_b) \\
&\leq \frac{2}{a+b} q(\xi_1)\cdots q(\xi_a) q(\eta_1)\cdots q(\eta_b). \qquad \square
\end{aligned}$$

For showing the continuity of the star product we can either proceed via the big formula (4.1.12) for two monomials or via the smaller one (4.1.3) for a monomial with a vector and iterate it. While the results are a bit better for the first approach, the second makes the proof easier. Nevertheless, both approaches give strong results, and this is why we will to give both proofs here.

There will be a very general way how most of the proofs will work and which tools will be used in the following. If we want to show the continuity of a map $f\colon \mathrm{S}^\bullet_R(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet_R(\mathfrak{g})$, we will proceed most of the time like this:

i.) First, we extend the map to the whole tensor algebra by symmetrizing beforehand: $f = f \circ \mathscr{S}$. This is a real extension since the symmetrization does not affect symmetric tensors. We do this here for the Gutt star product

$$\star_z\colon \mathrm{T}^\bullet(\mathfrak{g}) \otimes \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{T}^\bullet(\mathfrak{g}), \quad \star_z = \star_z \circ (\mathscr{S} \otimes)$$

and for the C_n-operators analogously.

ii.) Then we start with an estimate which we prove only on factorizing tensors in order to use the infimum argument (Lemma 5.2.9).

iii.) During the estimation process, we find products of Lie brackets. Those will be split up by the continuity of the symmetric product (5.2.8) from Lemma 5.2.10 the AE property (5.2.2).

iv.) Finally, we rearrange the split up seminorms to the seminorm of a factorizing tensor by (5.2.4).

5.3.1 Continuity of the Product

In the first proof, we want to give an estimate via the formula

$$\xi_1 \cdots \xi_k \star_z \eta_1 \cdots \eta_\ell = \sum_{n=0}^{k+\ell-1} z^n C_n(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell). \tag{5.3.4}$$

To shorten the very long expression from Equation (4.1.13), we abbreviate the summations by

$$C_n(\xi_1 \cdots \xi_k, \eta_1 \cdots \eta_\ell) = \frac{1}{r!} \sum_{\sigma,\tau} \sum_{a_i,b_j} \widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big) \cdots \widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big), \tag{5.3.5}$$

meaning the summations as given in Proposition 4.1.11 and using $r = k + \ell - n$.

Theorem 5.3.2 (Continuity of the star product) *Let $\mathfrak{g}$ be an AE-Lie algebra, $R \geq 0$, p a continuous seminorm with an asymptotic estimate q and $z \in \mathbb{C}$.*

i.) For $n \in \mathbb{N}$, the operator C_n is continuous and for all $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ we have the estimate:

$$p_R(C_n(x,y)) \leq \frac{n!^{1-R}}{2 \cdot 8^n} (16q)_R(x)(16q)_R(y) \tag{5.3.6}$$

ii.) For $R \geq 1$, the Gutt star product is continuous and for all $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ we have the estimate:

$$p_R(x \star_z y) \leq (cq)_R(x)(cq)_R(y) \tag{5.3.7}$$

with $c = 16(|z| + 1)$. Hence, the Gutt star product is continuous and the estimate (5.3.7) holds on $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ for all $z \in \mathbb{C}$

Proof: Let us use $r = k + \ell - n$ as before and recall that the products are taken in the symmetric algebra. Then we can use Equation (5.3.5) and put estimates on it. Let p be a continuous seminorm and let q be an asymptotic estimate for it. Then we get

$$\begin{aligned}
&p_R\big(C_n\big(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell\big)\big) \\
&\quad = p_R\Bigg(\frac{1}{r!} \sum_{\sigma,\tau} \sum_{a_i,b_j} \widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big) \cdots \widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big)\Bigg) \\
&\quad \overset{(a)}{\leq} \frac{1}{r!} r!^R \sum_{\sigma,\tau} \sum_{a_i,b_j} p\Big(\widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big)\Big) \cdots p\Big(\widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)}; \eta_{\tau(j)}\big)\Big) \\
&\quad \overset{(b)}{\leq} \frac{1}{r!^{1-R}} \sum_{\sigma,\tau} \sum_{a_i,b_j} \frac{2}{a_1 + b_1} \cdots \frac{2}{a_r + b_r} q(\xi_1) \cdots q(\xi_k) q(\eta_1) \cdots q(\eta_\ell) \\
&\quad \overset{(c)}{\leq} q(\xi_1) \cdots q(\xi_k) q(\eta_1) \cdots q(\eta_\ell) 2^r \frac{k!\ell!}{r!^{1-R}} \sum_{a_i,b_j} 1,
\end{aligned}$$

where we just used the continuity estimate for the symmetric tensor product in (a), Lemma 5.3.1, *iii.)*, in (b) and $\frac{2}{a_i+b_i} \leq 2$ in (c). We estimate the number of terms in the sum and get

$$\sum_{\substack{a_1,b_1,\ldots,a_r,b_r \geq 0 \\ a_i + b_i \geq 1 \\ a_1 + \cdots + a_r = k \\ b_1 + \cdots + b_r = \ell}} 1 \leq \sum_{\substack{a_1,b_1,\ldots,a_r,b_r \geq 0 \\ a_1 + b_1 + \cdots + a_r + b_r = k + \ell}} 1 = \binom{k + \ell + 2r - 1}{k + \ell} \leq 2^{3(k+\ell)-2n-1}.$$

Using this estimate, we get

$$
\begin{aligned}
&p_R\big(C_n\big(\xi_1\otimes\cdots\otimes\xi_k,\eta_1\otimes\cdots\otimes\eta_\ell\big)\big)\\
&\quad\le q(\xi_1)\cdots q(\xi_k)q(\eta_1)\cdots q(\eta_\ell)2^{k+\ell-n}\frac{k!\ell!}{(k+\ell-n)!^{1-R}}2^{3(k+\ell)-2n-1}\\
&\quad= q_R(\xi_1\otimes\cdots\otimes\xi_k)q_R(\eta_1\otimes\cdots\otimes\eta_\ell)2^{4(k+\ell)-3n-1}\left(\frac{k!\ell!n!}{(k+\ell-n)!n!}\right)^{1-R}\\
&\quad\le q_R(\xi_1\otimes\cdots\otimes\xi_k)q_R(\eta_1\otimes\cdots\otimes\eta_\ell)2^{4(k+\ell)-3n-1}2^{(1-R)(k+\ell)}n!^{1-R}\\
&\quad=\frac{n!^{1-R}}{2\cdot 8^n}(16q)_R(\xi_1\otimes\cdots\otimes\xi_k)(16q)_R(\eta_1\otimes\cdots\otimes\eta_\ell).
\end{aligned}
$$

The estimate (5.3.6) is now proven on factorizing tensors. For general tensors $x, y \in \mathrm{T}^{\bullet}_R(\mathfrak{g})$, we use the infimum argument from Lemma 5.2.9 and the first part is done. For the second part, let x and y be tensors of degree at most k and ℓ respectively. We use (5.3.6) in (a), the fact that $R \geq 1$ in (b) and have

$$
\begin{aligned}
p_R(x\star_z y) &= p_R\left(\sum_{n=0}^{l+\ell-1} z^n C_n(x,y)\right)\\
&\le \sum_{n=0}^{k+\ell-1} p_R(z^n C_n(x,y))\\
&\overset{(a)}{\le} \sum_{n=0}^{k+\ell-1}\frac{|z|^n}{2\cdot 8^n}n!^{1-R}(16q)_R(x)(16q)_R(y)\\
&\overset{(b)}{\le} \frac{(|z|+1)^{k+\ell}}{2}(16q)_R(x)(16q)_R(y)\sum_{n=0}^{\infty}\frac{1}{8^n}\\
&\le (16(|z|+1)q)_R(x)(16(|z|+1)q)_R(y).
\end{aligned}
\tag{5.3.8}
$$

Since estimates on $\mathrm{S}^{\bullet}_R(\mathfrak{g})$ also hold for the completion, the theorem is proven. □

Remark 5.3.3 (Uniform continuity) Part (*i*) of the theorem makes clear, why exactly continuity will only hold if $R \geq 1$: the estimate in (5.3.6) shows, that all the C_n are indeed continuous for any $R \geq 0$, but only for $R \geq 1$ there is something like a uniform continuity. When R decreases, the continuity of the C_n "gets worse" and the uniform continuity finally breaks down when the threshold $R = 1$ is trespassed. But we need this uniform estimate, since we have to control the operators up to an arbitrarily high order if we want to guarantee the continuity of the whole star product. Continuity up to a previously fixed order n does not suffice.

Now, we want to give a second proof, which relies on (4.1.3). Approaching like this, we do not account for the fact that we will encounter terms like $[\eta, \eta]$ which vanish, but we estimate more brutally. During this procedure, we also count the formal parameter z more often than it is actually there. This is why we now have to make assumptions on R and z which are a bit stronger than before. Moreover, we split up tensor products and put them together again various times, which is the reason why an AE Lie algebra does not suffice any more: we need $\mathfrak{g}$ to be locally m-convex. But if we make these assumptions, we get the following lemma, which finally simplifies the proof.

Lemma 5.3.4 *Let $\mathfrak{g}$ be a locally m-convex Lie algebra and $R \geq 1$. Then if $|z| < 2\pi$ or $R > 1$ there exists, for $x \in \mathrm{T}^\bullet(\mathfrak{g})$ of degree at most k, $\eta \in \mathfrak{g}$ and each continuous submultiplicative seminorm p, a constant $c_{z,R}$ only depending on z and R such that the following estimate holds:*

$$p_R(x \star_z \eta) \leq c_{z,R}(k+1)^R p_R(x) q(\eta) \tag{5.3.9}$$

Proof: We have for $\xi_1, \ldots, \xi_k, \eta \in \mathfrak{g}$

$$\begin{aligned}
p_R(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta) &= p_R\left(\sum_{n=0}^{k} \frac{B_n^* z^n}{n!(k-n)!} \sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} \left(\mathrm{ad}_{\xi_{\sigma(k-n+1)}} \circ \cdots \circ \mathrm{ad}_{\xi_{\sigma(k)}}\right)(\eta)\right) \\
&= \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!(k-n)!} \sum_{\sigma \in S_k} (k+1-n)!^R p^{k+1-n}\left(\xi_{\sigma(1)} \cdots \xi_{\sigma(k-n)} \left(\mathrm{ad}_{\xi_{\sigma(k-n+1)}} \circ \cdots \circ \mathrm{ad}_{\xi_{\sigma(k)}}\right)(\eta)\right) \\
&\leq (k+1)^R \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!} (k-n)!^{R-1} k! p(\xi_1) \cdots p(\xi_k) p(\eta) \\
&= (k+1)^R \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R} \left(\frac{(k-n)! n!}{k!}\right)^{R-1} p_R(\xi_1 \otimes \cdots \otimes \xi_k) p(\eta) \\
&\leq (k+1)^R p_R(\xi_1 \otimes \cdots \otimes \xi_k) p(\eta) \sum_{n=0}^{k} \frac{|B_n^*||z|^n}{n!^R}.
\end{aligned}$$

Now if $|z| < 2\pi$ the sum can be estimated by extending it to a series which converges. So we get a constant $c_{z,R}$ depending on R and on z such that

$$p_R(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta) \leq (k+1)^R c_{z,R} p_R(\xi_1 \otimes \cdots \otimes \xi_k) p(\eta).$$

On the other hand, if $|z| \geq 2\pi$ and $R > 1$ we can estimate

$$p_R(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta) \leq (k+1)^R p_R(\xi_1 \otimes \cdots \otimes \xi_k) p(\eta) \left(\sum_{n=0}^{k} \frac{|B_n^*|}{n!}\right) \left(\sum_{n=0}^{k} \frac{|z|^n}{n!^{R-1}}\right).$$

Again, both series will converge and give constants depending only on z and R. Hence, we have the estimate on factorizing tensors and can extend this to generic tensors of degree at most k by using the infimum argument. □

In the following, we assume again that either $R > 1$ or $R \geq 1$ and $|z| < 2\pi$ in order the use Lemma 5.3.4. Now we can give a simpler proof of Theorem 5.3.2 for the case of a locally m-convex Lie algebra:

Proof (Alternative Proof of Theorem 5.3.2): Assume that $\mathfrak{g}$ is now even locally m-convex. We want to replace η in the foregoing lemma by an arbitrary tensor y of degree at most ℓ. Again, we do that on factorizing tensors first and get

$$\begin{aligned}
p_R(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta_1 \otimes \cdots \otimes \eta_\ell) &= p_R\left(\frac{1}{\ell!} \sum_{\tau \in S_\ell} \xi_1 \otimes \cdots \otimes \xi_k \star_z \eta_{\tau(1)} \otimes \cdots \otimes \eta_{\tau(\ell)}\right) \\
&\leq \frac{1}{\ell!} \sum_{\tau \in S_\ell} p_R\left(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta_{\tau(1)} \otimes \cdots \otimes \eta_{\tau(\ell)}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq c_{z,R}(k+\ell)^R \frac{1}{\ell!} \sum_{\tau \in S_\ell} p_R\big(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta_{\tau(1)} \star \cdots \star_z \eta_{\tau(\ell-1)}\big) p\big(\eta_{\tau(\ell)}\big) \\
&\leq \ \vdots \\
&\leq c_{z,R}^\ell((k+\ell)\cdots(k+1))^R \frac{1}{\ell!} \sum_{\tau \in S_\ell} p_R(\xi_1 \otimes \cdots \otimes \xi_k) p\big(\eta_{\tau(1)}\big) \cdots p\big(\eta_{\tau(\ell)}\big) \\
&= c_{z,R}^\ell \left(\frac{(k+\ell)!}{k!\ell!} \right)^R p_R(\xi_1 \otimes \cdots \otimes \xi_k) p_R(\eta_1 \otimes \cdots \otimes \eta_\ell) \\
&\leq (2^R p)_R(\xi_1 \otimes \cdots \otimes \xi_k)(2^R c_{z,R}\, p)_R(\eta_1 \otimes \cdots \otimes \eta_\ell).
\end{aligned}
$$

Once again, we extend the estimate via the infimum argument to the whole tensor algebra, since the estimate depends no longer on the degree of the tensors. □

Using this approach for continuity, it is easy to see that nilpotency of the Lie algebra would change the estimate substantially: if we knew that we had at most N brackets because $N+1$ brackets vanish, then the sum in the proof of Lemma 5.3.4 would end at N instead of k and would therefore be independent of the degree of x.

In both proofs, it is easy to see that we need at least $R \geq 1$ to get rid of the factorials which come up because of the combinatorics of the star product. It is nevertheless interesting to see that this result is sharp, that means the Gutt star product really fails continuity for $R < 1$:

Example 5.3.5 (A counter-example) Let $0 \leq R < 1$ and $\mathfrak{g}$ be the Heisenberg algebra in three dimensions, i.e. the Lie algebra generated by the elements P, Q and E with the bracket $[P, Q] = E$ and all other brackets vanishing. This is a very simple example for a non-abelian Lie algebra and if continuity of the star product fails for this one, then we can not expect it to hold for more complex ones. We impose on $\mathfrak{g}$ the ℓ^1-topology with the norm n and $n(P) = n(Q) = n(E) = 1$. This will be helpful, since here we really have the equality

$$
n^{k+\ell}\left(X^k Y^\ell\right) = n^k\left(X^k\right) n^\ell\left(Y^\ell\right)
$$

for the symmetric product of vectors X, Y. Then we consider the sequences

$$
a_k = \frac{P^k}{k!^{R+\epsilon}} \quad \text{and} \quad b_k = \frac{Q^k}{k!^{R+\epsilon}}
$$

with $2\epsilon < 1 - R$. It is easy to see that

$$
n_R(a_k) = n_R(b_k) = k!^{-\epsilon}
$$

and hence we get the limit for any $c > 0$ by

$$
\lim_{n \longrightarrow \infty} (cn)_R(a_k) = \lim_{n \longrightarrow \infty} (cn)_R(a_k) = 0
$$

We want to show that there is no $c > 0$ such that

$$
n_R(a_k \star_z b_k) \leq (cn)_R(a_k)(cn)_R(b_k).
$$

In other words, $n_R(a_k \star_z b_k)$ grows faster than exponentially. But this is the case, since we can calculate the star product explicitly and see

$$
n_R(a_k \star_z b_k) = n_R\left(\sum_{j=0}^k \binom{k}{j}\binom{k}{j} j! \frac{1}{k!^{2R+2\epsilon}} P^{k-j} Q^{k-j} E^j \right)
$$

$$\begin{aligned}
&= \sum_{j=0}^{k} \frac{k!^2 j!(2k-j)!^R}{(k-j)!^2 j!^2 k!^{2R+2\epsilon}} \underbrace{n^{2k-j}(P^{k-j}Q^{k-j}E^j)}_{=1} \\
&= \sum_{j=0}^{k} \underbrace{\binom{k}{j}^2 \binom{2k}{k} \binom{2k}{j}^{-1}}_{\geq 1} \frac{j!^{1-R}}{k!^{2\epsilon}} \\
&\geq \sum_{j=0}^{k} \frac{j!^{1-R}}{k!^{2\epsilon}} \\
&\geq k!^{1-R-2\epsilon}.
\end{aligned}$$

Finally, we get a contradiction to the continuity of the star product since $a_k \star_z b_k$ is unbounded in the topology of $\mathrm{S}_R(\mathfrak{g})$ although the sequences themselves go to zero.

5.3.2 Dependence on the Formal Parameter

Now we want to analyse the completion $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ of the symmetric algebra with $\star_z$. Concerning exponential functions, we get the following negative result:

Proposition 5.3.6 *Let $\xi \in \mathfrak{g}$ and $R \geq 1$, then $\exp(\xi) \notin \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$, where $\exp(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{n!}$.*

Proof: Take a seminorm p such that $1 \geq p(\xi) \neq 0$. Then set $c = p(\xi)^{-1} \geq 1$. For ξ^n the powers in the sense of either the usual tensor product, or the symmetric product or the star product are the same. So we have for $N \in \mathbb{N}$

$$(cp)_R\left(\sum_{n=0}^{N} \frac{1}{n!}\xi^n\right) = \sum_{n=0}^{N} \frac{n!^R}{n!} c^n p(\xi)^n = \sum_{n=0}^{N} n!^{R-1} \geq N,$$

and hence $\exp(\xi)$ does not converge for the seminorm $(cp)_R$. □

When we go back to Theorem 5.3.2, we see that we have actually proven slightly more than we stated: we showed that the star product converges absolutely and locally uniform in z. This means that the star product is *not only continuous*, but also that *the formal series converges to the star product in the completion*. We can take a closer look at this proof in order to get a new result for the dependence on the formal parameter z:

Proposition 5.3.7 (Dependence on z) *Let $R \geq 1$, then for all $x, y \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ the map*

$$\mathbb{K} \ni z \longmapsto x \star_z y \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}) \tag{5.3.10}$$

is analytic with (absolutely convergent) Taylor expansion at $z = 0$ given by Equation (4.1.11). For $\mathbb{K} = \mathbb{C}$, the collection of algebras $\left\{\left(\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}), \star_z\right)\right\}_{z\in\mathbb{C}}$ is an entire holomorphic deformation of the completed symmetric tensor algebra $\left(\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g}), \vee\right)$.

Proof: The crucial point is that for $x, y \in \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ and every continuous seminorm p we have an asymptotic estimate q such that

$$p_R(x \star_z y) = p_R\left(\sum_{n=0}^{\infty} z^n C_n(x, y)\right)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} |z|^n p_R(C_n(x,y)) \\
&\leq (16q)_R(x)(16q)_R(y) \sum_{n=0}^{\infty} \frac{|z|^n n!^{1-R}}{2 \cdot 8^n},
\end{aligned}
$$

where we used the fact that the estimate (5.3.6) extends to the completion. For $R > 1$, this map is clearly analytic and absolutely convergent for all $z \in \mathbb{K}$. If $R = 1$, then for every $M \geq 1$ we go back to homogeneous, factorizing tensors $x^{(k)}$ and $y^{(\ell)}$ of degree k and ℓ respectively, and have

$$
\begin{aligned}
M^n p_R\Big(C_n\Big(x^{(k)}, y^{(\ell)}\Big)\Big) &\leq \frac{M^n}{2 \cdot 8^n} (16q)_R\Big(x^{(k)}\Big)(16q)_R\Big(y^{(\ell)}\Big) \\
&\leq M^{k+\ell} (16q)_R\Big(x^{(k)}\Big)(16q)_R\Big(y^{(\ell)}\Big) \\
&= (16Mq)_R\Big(x^{(k)}\Big)(16Mq)_R\Big(y^{(\ell)}\Big),
\end{aligned}
$$

where we used that $0 \leq n \leq k + \ell - 1$. The infimum argument gives the estimate on all tensors $x, y \in \mathrm{T}^\bullet_R(\mathfrak{g})$ and it extends to the completion such that

$$
p_R(z^n C_n(x,y)) \leq (16Mq)_R(x)(16Mq)_R(y) \frac{|z|^n}{2 \cdot (8M)^n}
$$

and hence

$$
p_R(x \star_z y) \leq (16Mq)_R(x)(16Mq)_R(y) \sum_{n=0}^{\infty} \frac{|z|^n}{2 \cdot (8M)^n}.
$$

So the power series converges for all $z \in \mathbb{C}$ with $|z| < 8M$ and converges uniformly if $|z| \leq cM$ for $c < 8$. But then, the map (5.3.10) converges on all open discs centered around $z = 0$, and it must therefore be entire. □

Remark 5.3.8 ((Weakly) holomorphic maps with values in locally convex spaces) One could argue that the term “holomorphic” in a locally convex space V does not necessarily mean that a map has a absolutely convergent Taylor expansion, but one should rather see the map (5.3.10) as a collection of paths $\mathbb{C} \longrightarrow \widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ and ask for their complex differentiability in the sense of a differential quotient. Of course, this is also a valid formulation of the word “holomorphic” and it is actually even the standard definition for holomorphic maps with values in locally convex spaces. According to this definition one calls a map “weakly holomorphic”, if every continuous linear form $\lambda \colon V \longrightarrow \mathbb{C}$ applied to it gives a holomorphic map $\mathbb{C} \longrightarrow \mathbb{C}$. Using this terminology, we would just have proven the map (5.3.10) to be *weakly holomorphic*. Yet, we have also proven “strong” holomorphicity, since in [81, Theorem 3.31] Rudin proved that the two notions of holomorphicity coincide in locally convex spaces.

5.3.3 Representations

We now want to identify the deformed symmetric algebra $(\mathrm{S}^\bullet_R(\mathfrak{g}), \star_z)$ with the the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_z)$ with the scaled product.

Definition 5.3.9 (The $\mathscr{U}_R$-Topology) *We denote by $\mathscr{U}_R(\mathfrak{g}_z)$ the locally convex algebra which we get from $\mathscr{U}(\mathfrak{g}_z)$ by pulling back the $\mathrm{S}^\bullet_R(\mathfrak{g})$-topology via the Poincaré-Birkholl-Witt isomophism* $\mathfrak{q}$.

We know this algebra has the universal property that Lie algebra homomorphisms into associative algebras can be lifted to unital homomorphisms of associative algebras. As a commutative diagram, this reads

$$\begin{tikzcd}
\mathscr{U}_R(\mathfrak{g}) \arrow[r, "\mathfrak{q}^{-1}"] \arrow[dr, "\Phi"] & \mathrm{S}^\bullet_R(\mathfrak{g}) \arrow[d, "\widetilde{\Phi}"] \\
\mathfrak{g} \arrow[u, "\iota"] \arrow[r, "\phi"'] & \mathscr{A}
\end{tikzcd}$$

where $\mathscr{A}$ is the mentioned associative algebra. Since we endowed $\mathscr{U}_R(\mathfrak{g}_z)$ and $\mathrm{S}^\bullet_R(\mathfrak{g})$ with a topology, we can now ask if the homomorphisms Φ and $\widetilde{\Phi}$ are continuous. This question is partly answered by the following result:

Proposition 5.3.10 (Universal property) *Let $\mathfrak{g}$ be an AE-Lie algebra, $\mathscr{A}$ an associative AE algebra and $\phi\colon \mathfrak{g} \longrightarrow \mathscr{A}$ is a continuous Lie algebra homomorphism. If $R \geq 1$, then the induced algebra homomorphisms Φ and $\widetilde{\Phi}$ are continuous.*

PROOF: We define an extension of Φ on the whole tensor algebra again:

$$\Psi\colon \mathrm{T}^\bullet_R(\mathfrak{g}) \longrightarrow \mathscr{A}, \quad \Psi = \widetilde{\Phi} \circ \mathscr{S}$$

It is clear that if Ψ is continuous on factorizing tensors, we will get the continuity of $\widetilde{\Phi}$ and of Φ via the infimum argument. So let p be a continuous seminorm on $\mathscr{A}$ with an asymptotic estimate q and $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. Since ϕ is continuous, we find a continuous seminorm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$ we have $q(\phi(\xi)) \leq r(\xi)$. Then we have

$$\begin{aligned}
p(\Psi(\xi_1 \otimes \cdots \otimes \xi_n)) &= p\Big(\widetilde{\Phi}(\xi_1 \star_z \cdots \star_z \xi_n)\Big) \\
&= p(\phi(\xi_1) \cdots \phi(\xi_n)) \\
&\leq q(\phi(\xi_1)) \cdots q(\phi(\xi_n)) \\
&\leq r(\xi_1) \cdots r(\xi_n) \\
&\leq r_R(\xi_1 \otimes \cdots \otimes \xi_n),
\end{aligned}$$

where the last inequality is true for all $R \geq 0$ and hence for all $R \geq 1$. □

Although this is a nice result, our construction fails to be a universal object in the category of associative AE algebras, since the universal enveloping algebra endowed with our topology is *not* AE in general. This is even very easy to see directly.

Example 5.3.11 Take $\xi \in \mathfrak{g}$, then we know that $\xi^{\otimes n} = \xi^{\star_z n} = \xi^n$ for $n \in \mathbb{N}$. Let $R > 0$ and p and q be a continuous seminorms on $\mathfrak{g}$ with $q(\xi) \neq 0$, then we find

$$p_R(\xi^n) = n!^R p(\xi)^n = \frac{n!^R}{c^n} q(\xi)^n \tag{5.3.11}$$

for $c = \frac{p(\xi)}{q(\xi)}$. But since $\frac{n!^R}{c^n}$ always diverges for $n \longrightarrow \infty$, we never get an asymptotic estimate for p.

Another argument would be that we know that we can not have exponential functions in $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$. If $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ (and hence $\mathscr{U}_R(\mathfrak{g}_z)$) were AE, then we would have an entire calculus which clearly includes exponential functions. Although the construction is not universal, we can draw a strong conclusion from Proposition 5.3.10:

Corollary 5.3.12 (Continuous Representations) *Let $R \geq 1$ and $\mathscr{U}_R(\mathfrak{g})$ the universal enveloping algebra of an AE-Lie algebra $\mathfrak{g}$, then for every continuous representation ϕ of $\mathfrak{g}$ into the bounded linear operators $\mathfrak{B}(V)$ on a Banach space V the induced homomorphism of associative algebras $\Phi\colon \mathcal{U}(\mathfrak{g}) \longrightarrow \mathfrak{B}(V)$ is continuous.*

PROOF: This follows directly from Proposition 5.3.10 and $\mathfrak{B}(V)$ being a Banach algebra. □

Remark 5.3.13 From this, we get the special case that all finite-dimensional representations of an AE-Lie algebra can be lifted to continuous representations of $\mathscr{U}_R(\mathfrak{g}_z)$. For infinite-dimensional Lie algebras, this statement will not be very important, since there, one rather has strongly continuous representations and no norm-continuous ones. Nevertheless, our topology may help to think about continuous linear functionals on $\mathscr{U}_R(\mathfrak{g}_z)$. We could then do GNS-representation theory with it. This procedure would yield a representation of $\mathscr{U}_R(\mathfrak{g}_z)$ on a (Pre-)Hilbert space and we could talk about what strongly continuous representations of the universal enveloping algebra should be. In this sense, the results of this chapter may also open a door towards some new approaches in this field.

5.3.4 Functoriality

Now let $\mathfrak{g}, \mathfrak{h}$ be two AE-Lie algebras. We know that a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathscr{U}(\mathfrak{h}_z)$ would lift to a homomorphism $\mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathscr{U}(\mathfrak{h}_z)$, if the latter one was AE, but this is not the case. Yet, we would like to have this result and prove that our construction is functorial, but it will be more complicated than that. We draw the commutative diagram to clarify the situation.

$$\begin{tikzcd}
\mathrm{S}^\bullet_R(\mathfrak{g}) \arrow[r, "\widetilde{\Phi}_z"] & \mathrm{S}^\bullet_R(\mathfrak{h}) \\
\mathscr{U}_R(\mathfrak{g}_z) \arrow[u, "\mathfrak{q}_z^{-1}"] \arrow[r, "\Phi_z"] & \mathscr{U}_R(\mathfrak{h}_z) \arrow[u, "\mathfrak{q}_z^{-1}"'] \\
\mathfrak{g} \arrow[u, "\iota_z"] \arrow[r, "\phi"'] & \mathfrak{h} \arrow[u, "\iota_z"']
\end{tikzcd}$$

Assume that ϕ is a continuous Lie algebra homomorphism. We want to know if Φ_z and $\widetilde{\Phi}_z$ will be continuous, too. Luckily, the answer is yes and our construction is functorial. For the proof, we need the next two lemmas.

Lemma 5.3.14 *Let $\mathfrak{g}$ be an AE-Lie algebra, $n \in \mathbb{N}$, $\xi_1, \dots, \xi_n \in \mathfrak{g}$, $i_j \in \{0, \dots, j\}$, $\forall_{j=1,\dots,n-1}$ and denote $I = \sum_j i_j$. Then we have the formula*

$$z^{i_{n-1}} C_{i_{n-1}}\big(\dots z^{i_2} C_{i_2}\big(z^{i_1} c_{i_1}(\xi_1, \xi_2), \xi_3\big) \dots, \xi_n\big) = z^I B^*_{i_{n-1}} \cdots B^*_{i_1}$$

$$\cdot \frac{\binom{1}{i_1}}{1!} \frac{\binom{2-i_1}{i_2}}{(2-i_1)!} \cdots \frac{\binom{n-1-i_1-\cdots-i_{n-2}}{i_{n-1}}}{(n-1-i_1-\cdots-i_{n-2})!} \sum_{\substack{\sigma_1\in S_{2-i_1}\\ \cdots \\ \sigma_{n-1}\in S_{n-1-i_1-\ldots-i_{n-2}}}} [w_1]\cdots[w_{n-I}], \quad (5.3.12)$$

where the expressions $[w_i]$ denote nested Lie brackets in the ξ_i.

PROOF: The statement follows from the bilinearity of the C_{i_j}-operators. They are always applied to homogeneous symmetric tensors. The proof itself is an easy induction and follows very directly from Formula (4.1.3). □

Lemma 5.3.15 *Let $\mathfrak{g}$ be an AE-Lie algebra, $R \geq 1$, $z \in \mathbb{C}$ and p a continuous seminorm with an asymptotic estimate q. Then for every $n \in \mathbb{N}$ and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ the estimate*

$$p_R(\xi_1 \star_z \cdots \star_z \xi_n) \leq c^n n!^R q^n(\xi_1 \otimes \cdots \otimes \xi_n) \quad (5.3.13)$$

holds with $c = 8\mathrm{e}(|z|+1)$.

PROOF: We start with a continuous seminorm p:

$$\begin{aligned}
p_R\big(\xi_1 \star_z \cdots \star_z \xi_n\big) &= p_R\left(\sum_{\ell=0}^{n-1} \sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} z^{i_{n-1}} C_{i_{n-1}}\big(\ldots z^{i_2} C_{i_2}\big(z^{i_1} C_{i_1}(\xi_1,\xi_2),\xi_3\big)\ldots,\xi_n\big)\right) \\
&\leq \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} p^{n-\ell}\Big(z^{i_{n-1}} C_{i_{n-1}}\big(\ldots z^{i_2} C_{i_2}\big(z^{i_1} C_{i_1}(\xi_1,\xi_2),\xi_3\big)\ldots,\xi_n\big)\Big) \\
&\leq \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |B^*_{i_{n-1}}|\cdots|B^*_{i_1}||z|^{i_{n-1}}\cdots|z|^{i_1} \frac{\binom{1}{i_1}}{1!} \frac{\binom{2-i_1}{i_2}}{(2-i_1)!} \cdots \\
&\quad \cdot \frac{\binom{n-1-i_1-\cdots i_{n-2}}{i_{n-1}}}{(n-1-i_1-\cdots-i_{n-2})!} \sum_{\substack{\sigma_1\in S_{2-i_1}\\ \cdots \\ \sigma_{n-1}\in S_{n-1-i_1-\ldots-i_{n-2}}}} p^{n-\ell}([w_1]\cdots[w_{n-\ell}]). \quad (5.3.14)
\end{aligned}$$

In the last step, we used Lemma 5.3.14. Now we can apply the AE property to it and get $q(\xi_1)\cdots q(\xi_n)$. Thus the sum over the σ_j vanishes with the inverse factorials. By using the fact that $|B^*_m| \leq m!$ for all $m \in \mathbb{N}$ and grouping together the powers of $|z|$, we find

$$\begin{aligned}
&p_R(\xi_1 \star_z \cdots \star_z \xi_n) \\
&\overset{\text{(a)}}{\leq} \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell \frac{1!(2-i_1)!\cdots(n-1-i_1-\cdots-i_{n-2})!}{(1-i_1)!\cdots(n-1-i_1-\cdots i_{n-1})!} q(\xi_1)\cdots q(\xi_n) \\
&\overset{\text{(b)}}{\leq} \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1\leq j\leq n-1\\ i_j\in\{0,\ldots,j\}\\ \sum_{j=1}^{n-1} i_j=\ell}} |z|^\ell 1^{i_1} 2^{i_2} \cdots (n-1)^{i_{n-1}} q(\xi_1)\cdots q(\xi_n)
\end{aligned}$$

$$
\overset{\text{(c)}}{\leq} \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell n^\ell q(\xi_1) \cdots q(\xi_n)
$$

$$
\overset{\text{(d)}}{\leq} \sum_{\ell=0}^{n-1} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell (2\mathrm{e})^n \ell! q(\xi_1) \cdots q(\xi_n).
$$

In (a), the factorials coming from the Bernoulli numbers cancelled out with the the i_j from the binomial coefficients and in (b) we used

$$
\frac{(k - i_1 - \cdots - i_{k-1})!}{(k - i_1 - \cdots - i_k)!} \leq k^{i_k}.
$$

We can estimate the product of all those expressions by $(n-1)^{i_1+\cdots+i_{n-1}}$ and this by n^ℓ, since the sum over all i_j equals ℓ. In the last step (d) we used $n^\ell \leq \mathrm{e}^n \frac{n!}{(n-\ell)!} = \mathrm{e}^n \binom{n}{\ell} \ell! \leq \mathrm{e}^n 2^n \ell!$. But now we can simply estimate $|z|^\ell \leq (|z|+1)^n$ and $(n-\ell)!^R \ell! \leq n!^R$ for $R \geq 1$. We just need to count the number of summands and get

$$
\begin{aligned}
p_R(\xi_1 \star_z \cdots \star_z \xi_n) &\leq n!^R (2\mathrm{e})^n (|z|+1)^n q(\xi_1) \cdots q(\xi_n) \sum_{\ell=0}^{n-1} \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} 1 \\
&\overset{\text{(a)}}{\leq} n!^R (2\mathrm{e})^n (|z|+1)^n q(\xi_1) \cdots q(\xi_n) \sum_{\ell=0}^{n-1} \binom{n-1+\ell-1}{\ell-1} \\
&\overset{\text{(b)}}{\leq} n!^R (2\mathrm{e})^n (|z|+1)^n q(\xi_1) \cdots q(\xi_n) 2^{2n} \\
&\leq c^n n!^R q^n(\xi_1 \otimes \cdots \otimes \xi_n),
\end{aligned}
$$

with $c = 8\mathrm{e}(|z|+1)$. In (a) the estimate for the big sum is the following: for every $j = 1, \ldots, n-1$ we have surely $i_j \in \{0, 1, \ldots, n-1\}$ and the sum of all the i_j is ℓ. If we forget about all other restrictions, we will just get more terms. But then the number of summands is same as there are ways to distribute ℓ items on $n-1$ places, which is given by $\binom{n-1+\ell-1}{\ell-1}$. Then in (b) we use

$$
\binom{n-1+\ell-1}{\ell-1} \leq \binom{2n}{\ell-1}
$$

with the binomial coefficient being zero for $\ell = 0$. Then it is just the standard estimate for binomial coefficients via the sum over all ℓ. □

Proposition 5.3.16 (Functoriality) *Let $R \geq 1$, $\mathfrak{g}, \mathfrak{h}$ be AE-Lie algebras and $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ a continuous homomorphism between them. Then it lifts to a continuous unital homomorphism of locally convex algebras $\Phi_z\colon \mathscr{U}_R(\mathfrak{g}_z) \longrightarrow \mathscr{U}_R(\mathfrak{h}_z)$.*

PROOF: First, if $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ is continuous, then for every continuous seminorm q on $\mathfrak{h}$, we have a continuous seminorm r on $\mathfrak{g}$ such that for all $\xi \in \mathfrak{g}$

$$
q(\phi(\xi)) \leq r(\xi).
$$

Second, we define Ψ_z on factorizing tensors via

$$\Psi_z \colon \mathrm{T}^\bullet_R(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet_R(\mathfrak{h}), \quad \Psi_z = \widetilde{\Phi}_z \circ \mathscr{S}$$

and extend it linearly to $\mathrm{T}^\bullet_R(\mathfrak{g})$. Clearly, Φ_z and $\widetilde{\Phi}_z$ will be continuous if Ψ_z is continuous. From this, we get for a seminorm p on $\mathfrak{h}$, an asymptotic estimate q and $\xi_1, \ldots, \xi_n$

$$\begin{aligned} p_R(\Psi_z(\xi_1 \otimes \cdots \otimes \xi_n)) &= p_R(\phi(\xi_1) \star_z \cdots \star_z \phi(\xi_1)) \\ &\leq c^n n!^R q(\phi(\xi_1)) \cdots q(\phi(\xi_n)) \\ &\leq c^n n!^R r(\xi_1) \cdots r(\xi_n) \\ &= (cr)_R(\xi_1 \otimes \ldots \otimes \xi_n). \end{aligned}$$

Again, we use the infimum argument and we have the estimate on all tensor in $\mathrm{T}^\bullet_R(\mathfrak{g})$. It extends to the completion and the statement is proven. □

5.4 Alternative Topologies and an Optimal Result

In this section we set the formal parameter $z = 1$ and make some observations. So far, we found a topology on $\mathrm{S}^\bullet_R(\mathfrak{g})$ which gives a continuous star product and which has a reasonably large completion, but it is always fair to ask if we can do better than that: we have seen that our completed algebra does not contain exponential series, which would be a very good feature to have. So is it possible to put another locally convex topology on $\mathrm{S}^\bullet_R(\mathfrak{g})$ which gives a completion with exponentials? The answer is no, at least under mild additional assumptions.

Proposition 5.4.1 (Optimality of the T_R-topology) *Let $\mathfrak{g}$ be an AE Lie algebra in which one has elements ξ, η for which the Baker-Campbell-Hausdorff series does not converge. Then there is no locally convex topology on $\mathrm{S}^\bullet(\mathfrak{g})$ such that all of the following things are fulfilled:*

i.) The Gutt star product $\star_G$ is continuous.

ii.) For every $\xi \in \mathfrak{g}$ the series $\exp(\xi)$ converges absolutely in the completion of $\mathrm{S}^\bullet(\mathfrak{g})$.

iii.) For all $n \in \mathbb{N}$ the projection and inclusion maps with respect to the graded structure

$$\mathrm{S}^\bullet(\mathfrak{g}) \xrightarrow{\pi_n} \mathrm{S}^n(\mathfrak{g}) \xrightarrow{\iota_n} \mathrm{S}^\bullet(\mathfrak{g})$$

are continuous.

First of all, we should make clear what “the Baker-Campbell-Hausdorff series does not converge” actually means. This may be clear for a finite-dimensional Lie algebra, but in the locally convex setting, it is not that obvious. For simplicity, we assume our local convex space to be complete in the following, since we can always achieve this by completing it. First, we note here that a net or a sequence in a locally convex space is convergent [or Cauchy], if and only if it is convergent [or Cauchy] with respect to all $p \in \mathscr{P}$. Quite similar to a normed space, we can make the following definition.

Definition 5.4.2 *Let V be a locally convex vector space, $\mathscr{P}$ the set of continuous seminorms, $p \in \mathscr{P}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence in V. We set*

$$\varrho_p(\alpha) = \left(\limsup_{n \longrightarrow \infty} \sqrt[n]{p(\alpha_n)} \right)^{-1}$$

where $\varrho_p(\alpha) = \infty$ if $\limsup_{n\longrightarrow\infty} \sqrt[n]{p(\alpha_n)} = 0$ as usual.

From this, we immediately get the two following lemmas.

Lemma 5.4.3 (Root test in locally convex spaces) *Let V be a complete locally convex vector space, $p \in \mathscr{P}$ and $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence. Then, if $\varrho_p(\alpha) > 1$, the series*

$$\mathcal{S}_n(\alpha) = \sum_{j=0}^{n} \alpha_j$$

converges absolutely with respect to p. Conversely, if this series converges with respect to p, then we have $\varrho_p(\alpha) \geq 1$.

PROOF: The proof is completely analogous to the one in finite dimensions. □

Lemma 5.4.4 *Let V be a complete locally convex vector space, $p \in \mathscr{P}$, $\alpha = (\alpha_n)_{n\in\mathbb{N}} \subseteq V$ a sequence and $M > 0$. Then, if the power series*

$$\lim_{n\longrightarrow\infty} \sum_{j=0}^{n} \alpha_j z^j$$

converges with respect to p for all $z \in \mathbb{C}$ with $|z| \leq M$, it converges absolutely with respect to p for all $z \in \mathbb{C}$ with $|z| < M$.

PROOF: Like in the finite-dimensional setting, we use the root test: convergence for $|z| \leq M$ means $\varrho_p(\alpha_z) \geq 1$, where we have set $\alpha_z = (\alpha_n z^n)_{n\in\mathbb{N}}$. Hence, for every $z' < z$ we get $\varrho_p(\alpha_{z'}) > 1$ and absolute convergence by Lemma 5.4.3. □

But this means that a convergent power series α_z in a locally convex space has $\varrho_p(\alpha_z) = \infty$ for all $p \in \mathscr{P}$: if there was a $p \in \mathscr{P}$ with a finite radius of convergence, we could take a multiple of p in order to get a seminorm with arbitrarily small radius of convergence. Hence α_z could not converge for any $z \neq 0$. This helps us to make clear, how "BCH does not converge" can be interpreted, since we can read it as a power series in two variables:

$$\mathrm{BCH}(t\xi, s\eta) = \sum_{a,b=0}^{\infty} t^a s^b \mathrm{BCH}_{a,b}(\xi, \eta).$$

So if it converges for ξ and η, then it converges absolutely for all $t\xi$ and $s\eta$ with $t, s \in [0, 1)$. If it even converges for all $t\xi$ and $s\eta$ with $t, s \in \mathbb{K}$, then it converges absolutely for all t, s. If this is the case, we can reorder the sums as we like to, and the series

$$\sum_{n=0}^{N} \mathrm{BCH}_n(\xi, \eta)$$

converges if and only if the series

$$\mathrm{BCH}(t\xi, s\eta) = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \mathrm{BCH}_{a,b}(\xi, \eta)$$

converges. Now we can prove a Lemma, from which Proposition 5.4.1 follows immediately.

Lemma 5.4.5 *Let $\mathfrak{g}$ be an AE Lie algebra and $\mathrm{S}^\bullet(\mathfrak{g})$ is endowed with a locally convex topology, such that the conditions $(i) - (iii)$ from Proposition 5.4.1 are fulfilled. Then the Baker-Campbell-Hausdorff series converges absolutely for all $\xi, \eta \in \mathfrak{g}$.*

PROOF: First, we complete the algebra to $\widehat{\mathrm{S}}^\bullet(\mathfrak{g})$. We need the projection π_1 to the Lie algebra. Take $\xi, \eta \in \mathfrak{g}$. Now, since the the Gutt star product is continuous and the exponential series is absolutely convergent for all $\xi, \eta \in \mathfrak{g}$, we get for $t, s \in \mathbb{K}$

$$\begin{aligned}
\pi_1(\exp(t\xi) \star \exp(s\eta)) &= \pi_1\left(\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{t^n\xi^n}{n!}\right) \star \lim_{M\to\infty}\left(\sum_{m=0}^{M}\frac{s^m\eta^m}{m!}\right)\right)\\
&\overset{\text{(a)}}{=} \pi_1\left(\lim_{N\to\infty}\lim_{M\to\infty}\left(\sum_{n=0}^{N}\frac{t^n\xi^n}{n!}\right) \star_G \left(\sum_{m=0}^{M}\frac{s^m\eta^m}{m!}\right)\right)\\
&\overset{\text{(b)}}{=} \lim_{N\to\infty}\lim_{M\to\infty}\pi_1\left(\left(\sum_{n=0}^{N}\frac{t^n\xi^n}{n!}\right) \star_G \left(\sum_{m=0}^{M}\frac{s^m\eta^m}{m!}\right)\right)\\
&\overset{\text{(c)}}{=} \lim_{N\to\infty}\lim_{M\to\infty}\sum_{n,m=0}^{N,M}\mathrm{BCH}_{t\xi,s\eta}(n,m)
\end{aligned}$$

where we used the continuity of the star product in (a), the continuity of the projection in (b) and evaluated the projection in (c). Since $\exp(t\xi)$ and $\exp(s\eta)$ are elements in the completion, their star product exists and hence the double series at the end of this equation converges for any two elements $\xi, \eta \in \mathfrak{g}$. But now we can use the result that in this setting the BCH series converges absolutely. We can rearrange the terms and get the convergence of

$$\sum_{n=1}^{N}\mathrm{BCH}_n(t\xi, s\eta).$$

Therefore, the BCH series must converge globally. □

Obviously, this proves Proposition 5.4.1.

Now we want to use a result due to Wojtyński [100], who showed that for Banach-Lie algebras, global convergence of the BCH series is equivalent to the fact that for any ξ we have

$$\lim_{n\longrightarrow\infty}\|(\mathrm{ad}_\xi)^n\|^{\frac{1}{n}} = 0.$$

A Banach-Lie algebra with this property is sometimes called quasi-nilpotent, radical or nil, see for example [67] for various generalizations of nilpotency in the case of Banach algebras. For finite-dimensional Lie algebras, quasi-nilpotency implies nilpotency. Hence for a finite-dimensional Lie algebra $\mathfrak{g}$, BCH is globally convergent if and only if $\mathfrak{g}$ is nilpotent.

From this, we see that at least for "non-quasi-nilpotent" Banach-Lie algebras, our result is in some sense optimal, at least if we want the grading structure to compatible with the topology.

Remark 5.4.6 (Another topology in $\mathscr{U}(\mathfrak{g})$) In [73] Schottenloher and Pflaum mention an alternative topology on the universal enveloping algebra for finite-dimensional Lie algebras: they take the coarsest locally convex topology, such that all finite-dimensional representations of $\mathfrak{g}$ extend to continuous algebra homomorphisms. This topology is in fact even locally m-convex and has therefore an entire holomorphic calculus. In particular, this completion contains exponential functions for all Lie algebra elements. Therefore, as we have seen in Proposition 5.4.1, it can not respect the grading structure, as our topology does. The T_R-topology must hence be different from that. As we have seen in Corollary 5.3.12, it is finer for $R \geq 1$, and since the topologies are different, it is even strictly finer. One could argue that the R-topology is "just" locally convex, but its advantage (for our purpose) is that the grading is necessary for the holomorphic dependence on the formal parameter which is a feature that we want.

Chapter 6

Nilpotent Lie Algebras

At the end of the last chapter, we have seen that the Baker-Campbell-Hausdorff series and its convergence play an important role for a topology on the universal enveloping algebra. Thus it is natural to ask whether things will change, if we restrict our observations to Lie algebras with globally convergent BCH series. To make things not too complicated from the beginning, we focus on locally convex and truly nilpotent Lie algebras. Recall that a Lie algebra $\mathfrak{g}$ is nilpotent, if there exists a $N \in \mathbb{N}$, such that for all $n > N$ and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ we have

$$\mathrm{ad}_{\xi_1} \circ \cdots \circ \mathrm{ad}_{\xi_n} = 0. \tag{6.0.1}$$

In the infinite-dimensional case, this is a priori *not* the same as

$$(\mathrm{ad}_\xi)^n = 0 \tag{6.0.2}$$

for all $\xi \in \mathfrak{g}$ and and $n > N$, but something stronger. In the case of finite-dimensional Lie algebras, the notions (6.0.1) and (6.0.2) coincide due to the Engel theorem, which makes use of the existence of a finite descending series of nilpotent ideals in the Lie algebra. Such a terminating series does not need to exist in infinite dimensions. At least as soon as we are in the setting of Banach-Lie algebras, these two notions coincide again, but note that one can give different forms of *quasi-nilpotency*, which weaken the statements from (6.0.1) and (6.0.2), respectively. Those generalized notions do not coincide any more, so we have to be careful.

Before we look at this case more closely, we first want to make a list of things, which we expect to change or not when we go to this more particular setting.

i.) In Example 5.3.5 we have seen that we can not expect to get a continuous algebra structure for $R < 1$, even for very simple nilpotent, but non-abelian Lie algebras. Therefore, we should not expect to get much larger completions now.

ii.) In [96], Waldmann showed that for example the Weyl-Moyal star product converges in the T_R-topology for $R \geq \frac{1}{2}$. This so-called Weyl algebra is, however, nothing but a quotient of the Heisenberg algebra. It would be interesting to understand this a bit better, since we know that we need $R \geq 1$ for the Heisenberg algebra. The quotient procedure must therefore have some strong influence on this construction. Can we reproduce the value $R \geq \frac{1}{2}$ somehow by dividing out an ideal?

iii.) The argument we used in Proposition 5.4.1, namely the non-global convergence of BCH, is not given any more. Now there is no longer a reason to expect that exponentials are not part of the completion. In this sense, it would be at least nice to have something more than "just" $R = 1$. Can we do that?

iv.) As already mentioned, there are generalizations or weaker forms of nilpotency in infinite-dimensions, especially for Banach-Lie algebras, which are equivalent to the usual notion of nilpotency in finite dimensions. If we get a stronger result for nilpotent Lie algebras, will it be possible to extend it to some of these generalizations?

The very fascinating and highly interesting answer to the three questions from $(ii) - (iv)$ is: yes, we can. The first section of this chapter will be devoted to the question from point (iii): we get a bigger completion by using a projective limit. We will also see how to get again the good functorial properties we had before. The next section treats another phenomenon, which was not there in the generic case: we can observe bimodule-structures within $\mathrm{S}^\bullet_R(\mathfrak{g})$. In third section, we will reproduce one of the results of Waldmann's, at least for the finite-dimensional case. The fourth part will take care of some generalizations of nilpotentcy for Banach-Lie algebras and will extend the result of the projective limit to a particular subcase there.

6.1 The Projective Limit

6.1.1 Continuity of the Product

As already mentioned, it is possible to extend the continuity result. Therefore, we take a locally convex, nilpotent Lie algebra $\mathfrak{g}$ and look at

$$\mathrm{S}^\bullet_{1^-}(\mathfrak{g}) = \operatorname*{proj\,lim}_{\epsilon \longrightarrow 0} \mathrm{S}^\bullet_{1-\epsilon}(\mathfrak{g}).$$

A tensor will be in the completed vector space $\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$, if and only if it lies for every $\epsilon > 0$ in the completion of $\mathrm{S}^\bullet_{1-\epsilon}(\mathfrak{g})$. Otherwise stated: let $\mathscr{P}$ be the set of all continuous seminorms of the Lie algebra $\mathfrak{g}$, then

$$f \in \widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g}) \quad \Longleftrightarrow \quad p_{1-\epsilon}(f) < \infty \quad \forall_{p\in\mathscr{P}} \forall_{\epsilon>0}.$$

So, if we want to show, that the Gutt star product is continuous on $\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$, we need to show that for every $p \in \mathscr{P}$ and $R < 1$, there exists a $q \in \mathscr{P}$ and a R' with $R \leq R' < 1$, such that we have for all $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$

$$p_R(x \star_z y) \leq q_{R'}(x) q_{R'}(y).$$

Before we prove the next theorem, we want to remind that locally convex, nilpotent Lie algebras are always AE Lie algebras. So the results we have found so far are valid in this case, too.

Theorem 6.1.1 *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra with continuous Lie bracket and $N \in \mathbb{N}$ such that $N + 1$ Lie brackets nested into each other vanish.*

i.) *If $0 \leq R < 1$, the C_n-operators are continuous and fulfil the estimate*

$$p_R(C_n(x, y)) \leq \frac{1}{2 \cdot 8^n} (32\mathrm{e}q)_{R+\epsilon}(x)(32\mathrm{e}q)_{R+\epsilon}(y), \tag{6.1.1}$$

for all $x, y \in \mathrm{S}^\bullet_R(\mathfrak{g})$, where p is a continuous seminorm, q an asymptotic estimate and $\epsilon = \frac{N-1}{N}(1 - R)$.

ii.) *The Gutt star product $\star_z$ is continuous for the locally convex projective limit $\mathrm{S}^\bullet_{1^-}(\mathfrak{g})$ and we have*

$$p_R(x \star_z y) \leq (cq)_{R+\epsilon}(x)(cq)_{R+\epsilon}(y) \tag{6.1.2}$$

with $c = 32\mathrm{e}(|z|+1)$ and the ϵ from the first part. The Gutt star product extends continuously to $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$, where it converges absolutely and coincides with the formal series.

PROOF: We use again $\star_z$ on the whole tensor algebra and compute the estimate for $\xi_1 \otimes \cdots \otimes \xi_k$ and $\eta_1 \otimes \cdots \otimes \eta_\ell$. The important point is that now, we get restrictions for the values of n. Recall that $k + \ell - n$ is the symmetric degree of $C_n(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell)$ and that we can have at most N letters in one symmetric factor. This means

$$(k + \ell - n)N \geq k + \ell \quad \iff \quad n \leq (k + \ell)\frac{N-1}{N}.$$

Hence we can estimate $n!^{1-R}$ in Equation (5.3.6): set $\delta = \frac{N-1}{N}$ and also denote a factorial where we have non-integers, meaning the gamma function. We get

$$\begin{aligned} n!^{1-R} &\leq (\delta(k+\ell)!)^{1-R} \\ &\leq (\delta(k+\ell))^{(1-R)\delta(k+\ell)} \\ &\leq (k+\ell)^{(1-R)\delta(k+\ell)} \\ &= \Big((k+\ell)^{(k+\ell)}\Big)^{(1-R)\delta} \\ &\leq \Big(\mathrm{e}^{k+\ell} 2^{k+\ell} k!\ell!\Big)^{(1-R)\delta} \\ &= \Big((2\mathrm{e})^{\delta(1-R)}\Big)^{k+\ell} k!^{\epsilon} \ell!^{\epsilon}, \end{aligned}$$

using $\epsilon = \delta(1-R)$. Hence

$$\begin{aligned} p_R\big(C_n\big(\xi_1 \otimes \cdots \otimes \xi_k, &\eta_1 \otimes \cdots \otimes \eta_\ell\big)\big) \\ &\leq \frac{\big((2\mathrm{e})^{\delta(1-R)}\big)^{k+\ell} k!^{\epsilon}\ell!^{\epsilon}}{2 \cdot 8^n} (16q)_R(\xi_1 \otimes \cdots \otimes \xi_k)(16q)_R(\eta_1 \otimes \cdots \otimes \eta_\ell) \\ &\leq \frac{1}{2 \cdot 8^n} (cq)_{R+\epsilon}(\xi_1 \otimes \cdots \otimes \xi_k)(cq)_{R+\epsilon}(\eta_1 \otimes \cdots \otimes \eta_\ell) \end{aligned}$$

with $c = 16(2\mathrm{e})^{\delta(1-R)} \leq 32\mathrm{e}$. We then get the estimate on all tensors by the infimum argument and extend it to the completion. Note, that for every $R < 1$ we also have $R + \epsilon < 1$ with the $\epsilon = \delta(1-R)$ from above. Iterating this continuity estimate, we get closer and closer to 1 and it is not possible to repeat this process an arbitrary number of times and stop at some value strictly less than 1. For the second part, we can conclude analogously to the second part of Theorem 5.3.2. □

We have proven one of the four statements. This projective limit is interesting, because it has a bigger completion than just $R = 1$. For example, we get the following result.

Corollary 6.1.2 *Let $\mathfrak{g}$ be a nilpotent, locally convex Lie algebra.*

i.) Let $\exp(\xi)$ be the exponential series for $\xi \in \mathfrak{g}$, then we have $\exp(t\xi) \in \widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$ for all $t \in \mathbb{K}$.

ii.) For $\xi, \eta \in \mathfrak{g}$ and $z \neq 0$ we have $\exp(\xi) \star_z \exp(\eta) = \exp\big(\frac{1}{z}\mathrm{BCH}(z\xi, z\eta)\big)$.

iii.) For $s, t \in \mathbb{K}$ and $\xi \in \mathfrak{g}$ we have $\exp(t\xi) \star_z \exp(s\xi) = \exp((t+s)\xi)$.

PROOF: For the first part, recall that the completion of the projective limit 1^- contains all those series $(a_n)_{n \in \mathbb{N}_0}$ such that

$$\sum_{n=0}^{\infty} a_n n!^{1-\epsilon} c^n < \infty$$

for all $c > 0$. This is the case for the exponential series of $t\xi$ for $t \in \mathbb{K}$ and $\xi \in \mathfrak{g}$. The second part follows from the fact that all the projections π_n onto the homogeneous subspaces S^n_π are continuous. The third part is then a direct consequence of the second. □

So the exponential series is what we want it to be, somehow. It generates a one parameter group. Recall that we can just exponentiate *vectors*. If we wanted to exponentiate a quadratic tensor, this would yield something like a Gaussian, which is again *not* part of the completion.

In the general case, we could find easier proof by assuming submultiplicativity of the seminorms. This is again the case for nilpotent Lie algebras. We get something like an alternative version of Lemma 5.3.4.

Lemma 6.1.3 *Let $\mathfrak{g}$ be a locally m-convex, nilpotent Lie algebra such that more than N Lie brackets nested into each other vanish. Let p be a continuous seminorm, $z \in \mathbb{K}$ and $R \geq 0$. Then, for every tensor $x \in \mathrm{S}^\bullet_R(\mathfrak{g})$ of degree at most $k \in \mathbb{N}$ and $\eta \in \mathfrak{g}$, we have the estimate*

$$p_R(x \star_z \eta) \leq c(k+1)^R k^{N(1-R)} p_R(x) p(\eta) \tag{6.1.3}$$

with the constant $c = \sum_{n=0}^N \frac{|B_n^*|}{n!}|z|^n$.

Proof: We do the estimate on factorizing tensors and apply the infimum argument later. So let $\xi_1, \dots, \xi_k, \eta \in \mathfrak{g}$, $k \in \mathbb{N}$, p a continuous seminorm on $\mathfrak{g}$ and $z \in \mathbb{K}$. Then, we have for $R \geq 0$

$$\begin{aligned}
&p_R\big(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta\big) \\
&= \sum_{n=0}^k (k+1-n)!^R \binom{k}{n} |B_n^*||z|^n p^{k+1-n}\left(\frac{1}{k!}\sum_{\sigma \in S_k} \xi_{\sigma(1)} \cdots \xi_{\sigma_{k-n}} \Big(\mathrm{ad}_{\xi_{\sigma(k-n+1)}} \circ \cdots \circ \mathrm{ad}_{\xi_{\sigma(k)}}\Big)(\eta)\right) \\
&\leq (k+1)^R \sum_{n=0}^N \frac{k!(k-n)!^R}{(k-n)!n!} |B_n^*||z|^n p(\xi_1)\cdots p(\xi_k) p(\eta) \\
&= (k+1)^R p_R(\xi_1 \otimes \cdots \otimes \xi_k) p(\eta) \sum_{n=0}^N \left(\frac{k!}{(k-n)!}\right)^{1-R} \frac{|B_n^*||z|^n}{n!} \\
&\leq (k+1)^R k^{N(1-R)} p_R(\xi_1 \otimes \cdots \otimes \xi_k) p(\eta) \sum_{n=0}^N \frac{|B_n^*||z|^n}{n!}. \qquad \square
\end{aligned}$$

Now, we can iterate Lemma 6.1.3 in the same way we did in Chapter 5:

Proof (Alternative Proof of Theorem 6.1.1): The calculation is done only on factorizing tensors. We need to transform the $k^{N(1-R)}$ into a very small factorial somehow. This is possible, since for given $N \in \mathbb{N}$ and $0 \leq R < 1$, the sequence

$$\left(\frac{k^N}{\sqrt{k!}}\right)^{1-R}$$

converges to 0 for $k \longrightarrow \infty$ and is therefore bounded by some $\kappa_N > 0$. Hence we get

$$k^{N(1-R)} \leq \kappa_N \sqrt{k!}^{1-R},$$

and together with Lemma 6.1.3 we find

$$p_R(x \star_z \eta) \leq c\kappa_N (k+1)^R k!^{\frac{1-R}{2}} p_R(x) p(\eta)$$

for any tensor x of degree at most k. Now, we can iterate this result for $\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_\ell \in \mathfrak{g}$, $R \geq 0$:

$$p_R\big(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta_1 \cdots \eta_\ell\big)$$

$$
\begin{aligned}
&= p_R\left(\frac{1}{\ell!}\sum_{\tau\in S_\ell}\xi_1\otimes\cdots\otimes\xi_k\star_z\eta_{\tau(1)}\star_z\cdots\star_z\eta_{\tau(\ell)}\right)\\
&\le \frac{1}{\ell!}\sum_{\tau\in S_\ell} c\kappa_N(k+\ell)^R(k+\ell-1)!^{\frac{1-R}{2}}\\
&\quad\cdot p_R\left(\frac{1}{\ell!}\sum_{\tau\in S_\ell}\xi_1\otimes\cdots\otimes\xi_k\star_z\eta_{\tau(1)}\star_z\cdots\star_z\eta_{\tau(\ell-1)}\right)p\big(\eta_{\tau(\ell)}\big)\\
&\le \quad\vdots\\
&\le (c\kappa_N)^\ell\left(\frac{(k+\ell)!}{k!}\right)^R(k+\ell-1)!^{\frac{1-R}{2}}\dots k!^{\frac{1-R}{2^N}}p_R(\xi_1\otimes\cdots\otimes\xi_k)p\big(\eta_{\tau(1)}\big)\cdots p\big(\eta_{\tau(\ell)}\big)\\
&\le (c\kappa_N)^\ell\binom{k+\ell}{k}^R\ell!^R(k+\ell)!^{\frac{(2^N-1)(1-R)}{2^N}}p_R(\xi_1\otimes\cdots\otimes\xi_k)p\big(\eta_{\tau(1)}\big)\cdots p\big(\eta_{\tau(\ell)}\big)\\
&\le (c\kappa_N)^\ell 2^{(k+\ell)R}\ell!^R k!^{\frac{(2^N-1)(1-R)}{2^N}}\ell!^{\frac{(2^N-1)(1-R)}{2^N}}2^{(k+\ell)\frac{(2^N-1)(1-R)}{2^N}}\\
&\quad\cdot p_R(\xi_1\otimes\cdots\otimes\xi_k)p\big(\eta_{\tau(1)}\big)\cdots p\big(\eta_{\tau(\ell)}\big)\\
&\le (2p)_{R+\epsilon}(\xi_1\otimes\cdots\otimes\xi_k)(2c\kappa_N p)_{R+\epsilon}(\eta_1\otimes\cdots\otimes\eta_\ell),
\end{aligned}
$$

where we have set $\epsilon = \frac{(2^N-1)(1-R)}{2^N}$. From this, we have $R+\epsilon<1$ and get the wanted result for the projective limit. □

Again, just like in the case of AE-Lie algebras for $R\ge 1$, we can show that the star product depends analytically on the formal parameter.

Proposition 6.1.4 (Dependence on z) *Let $\mathfrak{g}$ be a nilpotent locally convex Lie algebra, $0\le R<1$ and $z\in\mathbb{K}$, then for all $x,y\in\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g})$ the map*

$$
\mathbb{K}\ni z\longmapsto x\star_z y\in\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g}) \tag{6.1.4}
$$

is analytic with (absolutely convergent) Taylor expansion at $z=0$ given by Equation (4.1.11). For $\mathbb{K}=\mathbb{C}$, the collection of algebras $\left\{\left(\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g}),\star_z\right)\right\}_{z\in\mathbb{C}}$ is an entire holomorphic deformation of the completed symmetric tensor algebra $\left(\widehat{\mathrm{S}}^\bullet_{1^-}(\mathfrak{g}),\vee\right)$.

Proof: The proof is completely analogue to the case of AE Lie algebras when $R=1$. □

6.1.2 Representations and Functoriality

In the general AE case, we had some useful results concerning representations of Lie algebras and the functorialty of our construction. These results can be extended to the projective limit $\mathrm{S}^\bullet_{1^-}(\mathfrak{g})$.

Proposition 6.1.5 (Universal Property) *Let $\mathfrak{g}$ be a locally convex nilpotent Lie algebra, $\mathscr{A}$ an associative AE algebra and $\phi\colon\mathfrak{g}\longrightarrow\mathscr{A}$ is a continuous homomorphism of Lie algebras. Then, the lifted homomorphisms from $\mathrm{S}^\bullet_{1^-}(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g}_z)$ to $\mathscr{A}$ are continuous.*

Proof: The proof is exactly the same as in the general AE case, since there, we actually only needed $R\ge 0$. □

Again, this construction is not universal in the categorial sense, since $\mathrm{S}^\bullet_{1^-}(\mathfrak{g})$ fails to be AE. But also here, we get the case of continuous representations into a Banach space (and in particular into a finite-dimensional space) as a corollary.

Corollary 6.1.6 (Continuous Representations) *Let $\mathscr{U}_R(\mathfrak{g}_z)$ the universal enveloping algebra of locally convex nilpotent Lie algebra $\mathfrak{g}$ with bracket scaled by $z \in \mathbb{C}$, then for every continuous representation ϕ of $\mathfrak{g}$ into the bounded linear operators $\mathfrak{B}(V)$ on a Banach space V, the induced homomorphism of associative algebras $\Phi\colon \mathscr{U}_R(\mathfrak{g}_z) \longrightarrow \mathfrak{B}(V)$ is continuous.*

We can also extend the functoriality statement to the projective limit, but we need to get another version of Lemma 5.3.15 for nilpotent Lie algebras, since this is the corner stone of the functoriality proof.

Lemma 6.1.7 *Let $\mathfrak{g}$ be locally convex nilpotent Lie algebra and $N \in \mathbb{N}$ such that $N+1$ Lie brackets vanish, $0 \leq R < 1$ and $z \in \mathbb{C}$. Then for p a continuous seminorm, q an asymptotic estimate, $n \in \mathbb{N}$ and all $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ the estimate*

$$p_R(\xi_1 \star_z \cdots \star_z \xi_n) \leq c^n n!^{R+\epsilon} q^n(\xi_1 \otimes \cdots \otimes \xi_n) \tag{6.1.5}$$

holds with $c = 16\mathrm{e}^2(|z|+1)$ and $\epsilon = \frac{N-1}{N}(1-R)$ and the estimate is locally uniform in z.

Proof: We take $R < 1$ and go directly into the proof of Lemma 5.3.15 at (5.3.14). We know that, since we may have at most N brackets, also the values for ℓ are restricted to

$$\ell \leq \frac{N-1}{N} n = \delta n.$$

Using that in the proof of Lemma 5.3.15 leads to

$$\begin{aligned}
&p_R(\xi_1 \star_z \cdots \star_z \xi_n) \\
&\leq \sum_{\ell=0}^{\delta n} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell \frac{1!(2-i_1)!\cdots(n-1-i_1-\cdots-i_{n-2})!}{(1-i_1)!\cdots(n-1-i_1-\cdots-i_{n-1})!} q(\xi_1)\cdots q(\xi_n) \\
&\leq \sum_{\ell=0}^{\delta n} (n-\ell)!^R \sum_{\substack{1 \leq j \leq n-1 \\ i_j \in \{0,\ldots,j\} \\ \sum_{j=1}^{n-1} i_j = \ell}} |z|^\ell (2\mathrm{e})^n \ell! q(\xi_1)\cdots q(\xi_n) \\
&\leq (2\mathrm{e})^n (|z|+1)^n q(\xi_1)\cdots q(\xi_n) \sum_{\ell=0}^{\delta n} (n-\ell)!^R \ell! \binom{n+\ell-2}{\ell-1}.
\end{aligned}$$

We have

$$\ell! = \ell!^R \ell!^{1-R} \leq \ell!^R \Big((\delta n)^{\delta n}\Big)^{1-R} \leq \ell!^R n^{\delta n(1-R)} \leq \ell!^R n!^{\delta(1-R)} \mathrm{e}^{\delta n(1-R)}.$$

Together with $\ell!^R (n-\ell)!^R \leq n!^R$ this gives

$$\begin{aligned}
p_R(\xi_1 \star_z \cdots \star_z \xi_n) &\leq (2\mathrm{e})^n (|z|+1)^n n!^R n!^{\delta(1-R)} q(\xi_1)\cdots q(\xi_n) \sum_{\ell=0}^{\delta n} \binom{n+\ell-2}{\ell-1} e^{\delta n(1-R)} \\
&\leq (2\mathrm{e})^n (|z|+1)^n \Big(\mathrm{e}^{(1-R)\delta}\Big)^n 4^n n!^{R+\epsilon} q(\xi_1)\cdots q(\xi_n),
\end{aligned}$$

with $\epsilon = \delta(1-R)$. It is clear that for all $R < 1$ we have $R + \epsilon < 1$. Set

$$c = 8\mathrm{e}(|z|+1)\mathrm{e}^{(1-R)\delta} \leq 16\mathrm{e}^2(|z|+1)$$

and note that the estimate is locally uniform in z. □

Proposition 6.1.8 *Let $R \geq 1$, $\mathfrak{g}, \mathfrak{h}$ be two locally convex nilpotent Lie algebras and $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{h}$ a continuous homomorphism between them. Then it lifts to a continuous unital homomorphism of locally convex algebras $\Phi_z\colon \mathscr{U}_R(\mathfrak{g}_z) \longrightarrow \mathscr{U}_R(\mathfrak{h}_z)$.*

PROOF: The proof is analogous to the one of Proposition 5.3.16. □

6.2 Module Structures

The projective limit 1^- is not the only additional structure we will get, if our Lie algebra $\mathfrak{g}$ is nilpotent. For every $R \in \mathbb{R}$, the symmetric tensor algebra $\mathrm{S}^\bullet_R(\mathfrak{g})$ is a locally convex vector space. For $R \geq 0$, the (symmetric) tensor product is continuous, which is very important for many estimates, and for $R \geq 1^-$, we have an algebra structure. In between however, we have more than "only" vector spaces: the spaces $\mathrm{S}^\bullet_R(\mathfrak{g})$ form locally convex bimodules over the $\mathrm{S}^\bullet_{R'}(\mathfrak{g})$ for certain values of R'. The next proposition makes this more exact.

Proposition 6.2.1 (Bimodules in $\mathrm{S}^\bullet_R(\mathfrak{g})$) *Let $\mathfrak{g}$ be a nilpotent, locally convex Lie algebra, $N \in \mathbb{N}$ such that $N+1$ Lie brackets vanish, $z \in \mathbb{K}$ and $0 \leq R < 1$. Then, for all $x, y \in \mathrm{S}^\bullet(\mathfrak{g})$ and every continuous seminorm p, we have a continuous seminorm q, such that the estimates*

$$p_R(x \star_z y) \leq (16q)_R(x)(16cq)_{R+N(1-R)}(y) \tag{6.2.1}$$

and

$$p_R(x \star_z y) \leq (16cq)_{R+N(1-R)}(x)(16q)_R(y) \tag{6.2.2}$$

hold with $c = (N\mathrm{e})^{N(1-R)}$. Hence, the vector space $\widehat{\mathrm{S}}^\bullet_R(\mathfrak{g})$ forms a bimodule over the algebra $\widehat{\mathrm{S}}^\bullet_{R+N(1-R)}(\mathfrak{g})$. In particular, if $\mathfrak{g}$ is 2-step nilpotent, the vector space $\widehat{\mathrm{S}}^\bullet_0(\mathfrak{g})$ is a $\widehat{\mathrm{S}}^\bullet_1(\mathfrak{g})$-bimodule.

PROOF: Note that for every degree n we loose, we get a bracket of ξ's and η's. Since we can not have too highly nested brackets, we get the following bounds:

$$n \leq Nk \quad \text{and} \quad n \leq N\ell.$$

We prove the statement only on factorizing tensors again. We want to show Estimate (6.2.1) and take $x = \xi_1 \otimes \cdots \otimes \xi_k$ and $y = \eta_1 \otimes \cdots \otimes \eta_\ell$. So

$$\begin{aligned}(\ell N)!^{1-R} &\leq (\ell N)^{(\ell N(1-R))} \\ &\leq (N\mathrm{e})^{\ell N(1-R)}\ell!^{N(1-R)}.\end{aligned}$$

This allows us again to go back to the proof of Theorem 6.1.1 and we find

$$\begin{aligned}&p_R\big(C_n(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell)\big) \\ &\qquad \leq \frac{(N\mathrm{e})^{\ell N(1-R)}\ell!^{N(1-R)}}{2 \cdot 8^n}(16q)_R(\xi_1 \otimes \cdots \otimes \xi_k)(16q)_R(\eta_1 \otimes \cdots \otimes \eta_\ell)\end{aligned}$$

$$\leq \frac{1}{2 \cdot 8^n}(16q)_R(\xi_1 \otimes \cdots \otimes \xi_k)(cq)_{R-N(1-R)}(\eta_1 \otimes \cdots \otimes \eta_\ell)$$

with $c = 16(N\mathrm{e})^{N(1-R)}$. The rest of the proof is analogue to the proofs of the Theorems 6.1.1 or 5.3.2. □

Once again, assuming submultiplicativity of the seminorms, it is possible to give an easier proof for Proposition 6.2.1 which relies on Lemma 6.1.3.

Proof (Alternative proof for Proposition 6.2.1): We do the calculation on factorizing tensors: let $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$, $R \geq 0$, $k, \ell \in \mathbb{N}$. Using Lemma 6.1.3, we get

$$\begin{aligned}
&p_R\big(\xi_1 \otimes \cdots \otimes \xi_k \star_z \eta_1 \otimes \cdots \otimes \eta_\ell\big) \\
&\quad = p_R\left(\frac{1}{\ell!}\sum_{\tau \in S_\ell} \xi_1 \otimes \cdots \otimes \xi_k \star \eta_{\tau(1)} \star_z \cdots \star_z \eta_{\tau(\ell)}\right) \\
&\quad \leq c(k+\ell)^R(k+\ell-1)^{N(1-R)}\frac{1}{\ell!}\sum_{\tau \in S_\ell} p_R\big(\xi_1 \otimes \cdots \otimes \xi_k \star \eta_{\tau(1)} \star_z \cdots \star_z \eta_{\tau(\ell-1)}\big)p\big(\eta_{\tau(\ell)}\big) \\
&\quad \leq \ \vdots \\
&\quad \leq c^\ell\left(\frac{(k+\ell)!}{k!}\right)^R\left(\frac{(k+\ell-1)!}{(k-1)!}\right)^{N(1-R)} p_R(\xi_1 \otimes \cdots \otimes \xi_k)p\big(\eta_{\tau(1)}\big)\cdots p\big(\eta_{\tau(\ell)}\big) \\
&\quad \leq c^\ell 2^{k+l} 2^{N(k+\ell)} \ell!^{N(1-R)} p_R(\xi_1 \otimes \cdots \otimes \xi_k)p\big(\eta_{\tau(1)}\big)\cdots p\big(\eta_{\tau(\ell)}\big) \\
&\quad = \big(2^{N+1}p\big)_R(\xi_1 \otimes \cdots \otimes \xi_k)\big(2^{N+1}cp\big)_{R+N(1-R)}(\eta_1 \otimes \cdots \otimes \eta_\ell).
\end{aligned}$$

The proof of the second estimate is analogous. □

Remark 6.2.2 (Possible extensions) This result immediately raises new questions, like the one about possible generalizations to "weaker forms" of nilpotency, for example. They may be issues of some future work, but can not be addressed here, since we rather want to present something like the "big picture", instead of getting lost in its details too much. There are, without any doubt, questions that are more significant than extending those estimates to very special cases and finding sharp bounds there, although this is interesting and important, too.

Though it seems clear from the construction that these bimodules can not be there for general Lie algebras, we can give a concrete counter-example, which shows that there are Lie algebras, which don not allow them.

Example 6.2.3 Choose $R < 1$ and take $\mathfrak{g} = \mathbb{R}^3$ with the basis e_1, e_2, e_3 and the vector product as Lie bracket:

$$[e_1, e_2] = e_3 \qquad [e_2, e_3] = e_1 \qquad [e_3, e_1] = e_2$$

Again, we take a ℓ^1-norm n such that $n(e_1) = n(e_2) = n(e_3) = 1$. It has the nice property that for $k, \ell, m \in \mathbb{N}$ we get on the projective tensor product

$$n^{k+\ell+m}\left(e_1^k e_2^\ell e_3^m\right) = 1.$$

Now choose an $\epsilon > 0$ such that $R + \epsilon < 1$ and we define the sequence $(a_k)_{k \in \mathbb{N}}$

$$a_k = \frac{1}{k!^R}e_1^k,$$

for which we get $\lim_{k \longrightarrow \infty} n_R(a_k) = 0$. Now, we want to show that $a_k \star_z e_2$ grows faster than exponentially:

$$
\begin{aligned}
n_R(a_k \star_z e_2) &= n_R\left(\sum_{j=0}^{k} \binom{k}{j} B_j^* \frac{1}{k!^{R+\epsilon}} e_1^{n-j} (\mathrm{ad}_{e_1})^j (e_2) \right) \\
&= \sum_{j=0}^{k} \binom{k}{j} |B_j^*| \frac{1}{k!^{R+\epsilon}} (k-j+1)!^R \underbrace{n^{k-j}(e_1(e_2 \wedge e_3))}_{=1} \\
&= \sum_{j=0}^{k} (k-j+1)^R \binom{k}{j} \frac{|B_j^*|}{j!} \frac{(k-j)!^R j^R}{k!^{R+\epsilon}} j!^{1-R} \\
&= \sum_{j=0}^{k} (k-j+1)^R \binom{k}{j}^{1-R} \frac{|B_j^*|}{j!} \frac{j!^{1-R}}{k!^{\epsilon}} \\
&\geq \sum_{j=0}^{k} \frac{|B_j^*|}{j!} \frac{j!^{1-R}}{k!^{\epsilon}} \\
&\geq \frac{|B_k^*|}{k!^{R+\epsilon}}.
\end{aligned}
$$

We know that for $R + \epsilon < 1$ and any $c > 0$

$$
\limsup_{n \longrightarrow \infty} \frac{|B_n^*|}{c^n n!^{R+\epsilon}} = \infty.
$$

Hence the Limes superior of $n_R(a_k \star_z e_2)$ grows faster than any exponential function and can not be absorbed into the seminorm of e_2. So the multiplication in the module can not be continuous.

6.3 The Heisenberg and the Weyl Algebra

Now we want to see how we get the link to the Weyl algebra from [96], since we have something like a discrepancy for the parameter R concerning the continuity of the product in the Weyl and the Heisenberg algebra. In the following, we will show that this gap actually makes a lot of sense. For simplicity, we consider the easiest case of the Weyl/Heisenberg algebra with two generators Q and P, but the calculation for the Heisenberg/Weyl algebra in $2n+1$ $[2n]$ dimensions is done exactly in the same way. Recall that the Weyl algebra is a quotient of the enveloping algebra of the Heisenberg algebra $\mathfrak{h}$ which one gets from dividing out its center. So let $h \in \mathbb{K}$ and we have a projection

$$
\pi \colon \mathrm{S}_R^\bullet(\mathfrak{h}) \longrightarrow \mathcal{W}_R(\mathfrak{h}) = \frac{\mathrm{S}_R^\bullet(\mathfrak{h})}{\langle E - h\mathbb{1} \rangle} \tag{6.3.1}
$$

Of course we want to know if this projection is continuous.

Proposition 6.3.1 *The projection π is continuous for $R \geq 0$.*

Proof: We extend π to the whole tensor algebra by symmetrizing beforehand. Let then p be a continuous seminorm on $\mathfrak{h}$ and $k, \ell, m \in \mathbb{N}_0$. We have

$$
\begin{aligned}
p_R(\pi(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m})) &= p_R(Q^k P^\ell h^m) \\
&= |h|^m (k+\ell)!^R p^{k+\ell}(Q^k P^\ell)
\end{aligned}
$$

$$\begin{aligned}
&\leq (|h|+1)^{k+\ell+m}(k+\ell+m)!^R p(Q)^k p(P)^\ell p(E)^m \\
&= ((|h|+1)p)_R(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}).
\end{aligned}$$

Then we do the usual infimum argument and have the result on arbitrary tensors again. □

To establish the link to the continuity results of the Weyl algebra, we need more: $\pi \circ \star_z$ should to be continuous for $R \geq \frac{1}{2}$.

Proposition 6.3.2 *Let $R \geq \frac{1}{2}$ and π the projection from* (6.3.1). *Then the map $\pi \circ \star_z$ is continuous.*

PROOF: Since we are in finite dimensions, we can choose a submultiplicative norm p with $p(Q) = p(P) = p(E)$ without restrictions. Moreover, let $k, k', \ell, \ell', m, m' \in \mathbb{N}_0$. Then we have to get an estimate for $p_R\Big(\pi\Big(Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'}\Big)\Big)$. If we calculate the star product explicitly, we will see that we only get Lie brackets where we have P's and Q's. Let $r = k + \ell + m$ and $s = k' + \ell' + m'$, then we can actually simplify the calculations by

$$\begin{aligned}
p_R\Big(\pi\Big(Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'}\Big)\Big) &= (p_R \circ \pi)\left(\sum_{n=0}^{r+s-1} z^n C_n(Q^k P^\ell E^m, Q^{k'} P^{\ell'} E^{m'})\right) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n (p_R \circ \pi)\Big(C_n(Q^k P^\ell E^m, Q^{k'} P^{\ell'} E^{m'})\Big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n (p_R \circ \pi)(C_n(Q^r, P^s)) \\
&= \sum_{n=0}^{r+s-1} |z|^n \frac{r!s!}{(r-n)!(s-n)!n!} (p_R \circ \pi)\big(Q^{r-n} P^{s-n} E^n\big) \\
&= \sum_{n=0}^{r+s-1} |z|^n |h|^n \frac{r!s!}{(r-n)!(s-n)!n!} p_R\big(Q^{r-n} P^{s-n}\big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n |h|^n \frac{r!s!}{(r-n)!(s-n)!n!} \frac{(r+s-2n)!^R}{r!^R s!^R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\leq \sum_{n=0}^{r+s-1} |z|^n |h|^n \binom{r}{n}\binom{s}{n} \frac{(r+s-2n)!^R n!}{r!^R s!^R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\overset{(a)}{\leq} \sum_{n=0}^{r+s-1} |z|^n |h|^n \binom{r}{n}\binom{s}{n} \binom{r+s}{s}^R \binom{r+s}{2n}^{-R} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\leq \sum_{n=0}^{r+s-1} (|z|+1)^n (|h|+1)^n 4^{r+s} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&\overset{(b)}{\leq} \underbrace{(8(|z|+1)(|h|+1))^{r+s}}_{=\tilde{c}^{r+s}} p_R\big(Q^{\otimes r}\big) p_R\big(P^{\otimes s}\big) \\
&= (\tilde{c}p)_R\big(Q^{\otimes r}\big)(\tilde{c}p)_R\big(P^{\otimes s}\big) \\
&\overset{(c)}{=} (\tilde{c}p)_R\Big(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}\Big)(\tilde{c}p)_R\Big(Q^{\otimes k'} \otimes P^{\otimes \ell'} \otimes E^{\otimes m'}\Big),
\end{aligned}$$

where we have set $c = 8(|z|+1)(|h|+1)$. We rearranged the factorials in (a) and used $R \geq \frac{1}{2}$. The estimates (b) are the standard binomial coefficient estimates. In (c) we used $p(Q) = p(P) = p(E)$.

Now we just use

$$\left(Q^{\otimes k} \otimes P^{\otimes \ell} \otimes E^{\otimes m}\right) \star_z \left(Q^{\otimes k'} \otimes P^{\otimes \ell'} \otimes E^{\otimes m'}\right) = Q^k P^\ell E^m \star_z Q^{k'} P^{\ell'} E^{m'}$$

and the infimum argument to expand this estimate to all tensors. This concludes the proof. □

The previous proposition can be seen as something like the "finite-dimensional version" of Lemma 3.10 in [96], just that we took a large detour for proving it. One could, most probably, redo some more results of this paper using finite-dimensional versions the Heisenberg algebra and the projection onto the Weyl algebra, but this would yield, also most probably, nothing new. It is good to know that this connections exists, but it is not something which is very helpful to pursue, since an evident generalization to infinite dimensions does not seem be obvious.

6.4 Banach-Lie Algebras

Now we want to focus a bit on weaker notions than true nilpotency. Since there are many of them, we want to restrict to the easier case of Banach-Lie algebras, where a somewhat developed theory already exists.

6.4.1 Generalizations of Nilpotency

In [67], Müller gives a list of weaker forms of nilpotency for associative Banach algebras. We can mostly copy the ideas and use them for Banach-Lie algebras, too

Definition 6.4.1 *Let $\mathfrak{g}$ be a Banach-Lie algebra in which the Lie bracket fulfils the estimate*

$$\|[\xi,\eta]\| \leq \|\xi\|\|\eta\|.$$

Denote by $\mathbb{B}_1(0)$ all elements $\xi \in \mathfrak{g}$ with $\|\xi\| = 1$. We say that

i.) $\mathfrak{g}$ is topologically nil (or radical, or quasi-nilpotent), if every $\xi \in \mathfrak{g}$ is quasi-nilpotent, i.e.

$$\lim_{n\longrightarrow\infty} \|\mathrm{ad}_\xi^n\|^{\frac{1}{n}} = 0.$$

ii.) $\mathfrak{g}$ is uniformly topologically nil, if

$$\lim_{n\longrightarrow\infty} \mathcal{N}_1(n) = 0$$

for

$$\mathcal{N}_1(n) = \sup\Big\{ \|\mathrm{ad}_\xi^n\|^{\frac{1}{n}} \Big| \xi \in \mathbb{B}_1(0)\Big\}. \tag{6.4.1}$$

iii.) $\mathfrak{g}$ is topologically nilpotent, if for every sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{B}_1(0)$ we have

$$\lim_{n\longrightarrow\infty} \|\mathrm{ad}_{\xi_1} \circ \ldots \circ \mathrm{ad}_{\xi_n}\|^{\frac{1}{n}} = 0.$$

iv.) $\mathfrak{g}$ is uniformly topologically nilpotent, if

$$\lim_{n\longrightarrow\infty} \mathcal{N}(n) = 0$$

for

$$\mathcal{N}(n) = \sup\Big\{ \|\mathrm{ad}_{\xi_1} \circ \ldots \circ \mathrm{ad}_{\xi_n}\|^{\frac{1}{n}} \Big| \xi_1, \ldots, \xi_n \in \mathbb{B}_1(0)\Big\}. \tag{6.4.2}$$

It is clear that *(ii)* $\Rightarrow$ *(i)* and *(iv)* $\Rightarrow$ *(iii)*. In the associative case, we have *(iii)* $\Leftrightarrow$ *(iv)* and hence *(iii)* $\Rightarrow$ *(ii)*. Of course, it is a good question, if this remains true for Banach-Lie algebras. We have already encountered notion *(i)*: in [100] Wojtyński gave a proof that it is equivalent to the global convergence of the BCH series. In the following, we will make use of notion *(iv)*: we will show, that it is possible to generalize the result of Theorem 6.1.1 to this case.

6.4.2 An Adapted Topology for the Tensor Algebra

The idea consists in changing the T_R-topology a bit: instead of taking $n!^R$ as weights with $0 \leq R < 1$, we take sequence $(\alpha_n)_{n\in\mathbb{N}}$ with a certain asymptotic behaviour for $n \longrightarrow \infty$ and use $\frac{n!}{\alpha_n}$ as weights. This will generalize the idea of $n!^R$ and will be the starting point for estimates.

First, we observe that every uniformly topologically nilpotent Banach-Lie algebra $\mathfrak{g}$ comes with a characteristic, monotonously decreasing sequence

$$\omega_n = \sup_{m\geq n} \mathcal{N}(m). \tag{6.4.3}$$

If there exists a $N \in \mathbb{N}$, such that $\omega_n = 0$ for all $n \geq N$, then $\mathfrak{g}$ is actually nilpotent and we can use the results of the first section in this chapter. We may hence restrict to those Banach-Lie algebras, where we have $\omega_n > 0$ for all $n \in \mathbb{N}$. This allows the next definition.

Definition 6.4.2 (Rapidly increasing sequences) *Let $\mathfrak{g}$ be a uniformly topologically nilpotent Banach-Lie algebra and $(\omega_n)_{n\in\mathbb{N}}$ the sequence defined in* (6.4.3)*. Then we define the characteristic sequence $(\chi_n^{\mathfrak{g}})_{n\in\mathbb{N}}$ of $\mathfrak{g}$ by*

$$\chi_n^{\mathfrak{g}} = \max\left\{\frac{1}{\omega_n}, 2\right\}. \tag{6.4.4}$$

Furthermore, we will say that a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(1,\infty)$ is $\mathfrak{g}$-rapidly increasing, if it fulfils the following properties:

i.) It grows fast than exponentially, i.e.

$$\lim_{n\longrightarrow\infty} \frac{\log(\alpha_n)}{n} = \infty.$$

ii.) There exists a constant $c > 0$, such that

$$\alpha_n \leq c^n \chi_n^{\mathfrak{g}}.$$

We denote by $\mathfrak{I}_{\mathfrak{g}}$ the set of all $\mathfrak{g}$-rapidly increasing sequences.

Clearly, $(\chi_n^{\mathfrak{g}})_{n\in\mathbb{N}}$ is a $\mathfrak{g}$-rapidly increasing sequence itself. Note that with this definition we get for every sequence $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{B}_1(0)$

$$\|[\ldots[[\xi_1, \xi_2], \xi_3], \ldots \xi_n]\| \leq \frac{2^n}{\chi_n^{\mathfrak{g}}}. \tag{6.4.5}$$

Remark 6.4.3 The number 2 in (6.4.4) may look a bit confusing at the first sight, since it is somehow arbitrary, but for technical reasons, we will need $\chi_n^{\mathfrak{g}} > 1$ later. So actually every real number $c > 1$ could have been used there. In this sense, the previous definition is rather a technical tool than a "general concept".

Each $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{I}_{\mathfrak{g}}$ gives a continuous seminorm.

Definition 6.4.4 (Adapted seminorms) *Let $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{I}_{\mathfrak{g}}$. Then*

$$p_\alpha = \sum_{n=0}^{\infty} \frac{n!}{\alpha_n} \|\cdot\|^{\otimes_\pi n}$$

defines a seminorm on the tensor algebra with the projective tensor product $\mathrm{T}^\bullet_\pi(\mathfrak{g})$. We denote the set of all continuous seminorms with respect to those coming from such sequences by $\mathscr{P}$.

It will be important to see that every rapidly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ yields a continuous function f_α by

$$f_\alpha \colon \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+, \quad f_\alpha(0) = 2,\ f_\alpha(n) = \log(\alpha_n),\ \forall_{n\in\mathbb{N}} \tag{6.4.6}$$

and linear interpolation between the values at the integers. The idea behind is that this will allow us to use a technical lemma, which we now introduce. This lemma shows that there are always "many" rapidly increasing functions in a certain sense. It is taken from a work [65] by Mitiagin, Rolewicz and Żelazko, where it is stated in Lemma 2.1 and Lemma 2.2.

Lemma 6.4.5 *Let $f \colon \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+$ be a continuous functions, such that*

$$\lim_{n\longrightarrow\infty} \frac{f(x)}{x} = \infty. \tag{6.4.7}$$

Then there exists a convex, continuous function $g \colon \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+$, fulfilling (6.4.7) and

$$g(t_1 + \cdots + t_n) \leq 8(g(t_1) + \cdots + g(t_n)) + f(n), \quad \forall_{n\in\mathbb{N}} \text{ and all } t_i \in \mathbb{R}^+_0. \tag{6.4.8}$$

PROOF: We refer the reader to the paper [65], since we just want to use this lemma and do not want to go too much into details here. □

Note that if we have $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{I}_{\mathfrak{g}}$, then we can apply Lemma 6.4.5 to the function f_α, which is defined according to Equation 6.4.6.

6.4.3 A New Continuity Result

Now, we have finally prepared our toolbox well enough to prove a new result.

Proposition 6.4.6 *Let $\mathfrak{g}$ be a uniformly topologically nilpotent Banach-Lie algebra, $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{I}_{\mathfrak{g}}$, p_α the corresponding seminorm according to (6.4.6) and $z \in \mathbb{K}$. Then, there exists a series $(\beta_n)_{n\in\mathbb{N}} \in \mathfrak{I}_{\mathfrak{g}}$, such that for all $x, y \in \mathrm{T}^\bullet(\mathfrak{g})$ we have the estimate*

$$p_\alpha(x \star_z y) \leq (cp)_\beta(x)(cp)_\beta(y) \tag{6.4.9}$$

with a $c > 0$, which only depends on α, z and the Lie algebra $\mathfrak{g}$.

PROOF: We compute the estimate on factorizing tensors and extend it with the infimum argument later. Let hence $k, \ell \in \mathbb{N}$ and $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_\ell \in \mathfrak{g}$. We need to estimate the C_n-operators for $n = 0, 1, \ldots, k + \ell - 1$. Therefore we note $r = k + \ell - n$ and use the short-hand notation for the sums, which appear in the Gutt star product, again. We take $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{I}_{\mathfrak{g}}$ and get for p_α

$$p_\alpha\big(C_n(\xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_\ell)\big)$$

$$
\begin{aligned}
&= p_\alpha\Bigg(\frac{1}{r!}\sum_{\sigma,\tau}\sum_{a_i,b_j}\widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)};\eta_{\tau(j)}\big)\cdots\widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)};\eta_{\tau(j)}\big)\Bigg)\\
&\leq \frac{1}{r!}\frac{r!}{\alpha_r}\sum_{\sigma,\tau}\sum_{a_i,b_j}\|\widetilde{\mathrm{BCH}}_{a_1,b_1}\big(\xi_{\sigma(i)};\eta_{\tau(j)}\big)\|\cdots\|\widetilde{\mathrm{BCH}}_{a_r,b_r}\big(\xi_{\sigma(i)};\eta_{\tau(j)}\big)\|\\
&\leq \frac{k!\ell!}{\alpha_r}\sum_{a_i,b_j}\frac{2}{\chi^{\mathfrak{g}}_{a_1+b_1}}\cdots\frac{2}{\chi^{\mathfrak{g}}_{a_r+b_r}}\|\xi_1\|\cdots\|\xi_k\|\|\eta_1\|\cdots\|\eta_\ell\|,
\end{aligned}
$$

where we have used the estimate from Lemma 5.3.1 (*iii*) and Estimate (6.4.5) in the last step. Rearranging this, we have

$$
p_\alpha\Big(C_n\Big(\xi^{\otimes k},\eta^{\otimes\ell}\Big)\Big) \leq k!\ell!2^r\|\xi_1\|\cdots\|\xi_k\|\|\eta_1\|\cdots\|\eta_\ell\|\sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}}\frac{1}{\alpha_r\cdot\chi^{\mathfrak{g}}_{a_1+b_1}\cdots\chi^{\mathfrak{g}}_{a_r+b_r}},
$$

and we would like to find a $(\beta_n)_{n\in\mathbb{N}}\in\mathfrak{I}_{\mathfrak{g}}$ such that

$$
\sup\left\{\frac{\beta_k\cdot\beta_\ell}{\alpha_r\cdot\chi^{\mathfrak{g}}_{a_1+b_1}\cdots\chi^{\mathfrak{g}}_{a_r+b_r}}\,\middle|\, k,\ell\in\mathbb{N},\ a_i+b_i\geq 1,\ \sum_i a_i=k,\ \sum_j b_j=\ell\right\}\leq\kappa^{k+\ell} \tag{6.4.10}
$$

for some $\kappa>0$, just depending on (α). Then we would have

$$
\begin{aligned}
&p_\alpha\big(C_n\big(\xi_1\otimes\cdots\otimes\xi_k,\eta_1\otimes\cdots\otimes\eta_\ell\big)\big)\\
&\qquad\leq\frac{k!\ell!}{\beta_k\beta_\ell}2^r\|\xi_1\|\cdots\|\xi_k\|\|\eta_1\|\cdots\|\eta_\ell\|\sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}}\kappa^{k+\ell}\\
&\qquad=2^{-n}(2\kappa)^{k+\ell}p_\beta(\xi_1\otimes\cdots\otimes\xi_k)p_\beta(\eta_1\otimes\cdots\otimes\eta_\ell)\sum_{\substack{a_1,b_1,\ldots,a_r,b_r\geq 0\\ a_i+b_i\geq 1\\ a_1+\ldots+a_r=k\\ b_1+\ldots+b_r=\ell}}1\\
&\qquad\leq\frac{1}{2\cdot 8^n}(16\kappa)^{k+\ell}p_\beta(\xi_1\otimes\cdots\otimes\xi_k)p_\beta(\eta_1\otimes\cdots\otimes\eta_\ell)\\
&\qquad=\frac{1}{2\cdot 8^n}(16\kappa p)_\beta(\xi_1\otimes\cdots\otimes\xi_k)(16\kappa p)_\beta(\eta_1\otimes\cdots\otimes\eta_\ell).
\end{aligned}
$$

From this we could conclude analogously to the procedure in the proof of Theorem 5.3.2 and the statement would follow. However, we need to show the existence of a $(\beta_n)_{n\in\mathbb{N}}\in\mathfrak{I}_{\mathfrak{g}}$ and a $\kappa>0$, such that (6.4.10) holds.

Lemma 6.4.7 *For f_α defined as in* (6.4.6)*, we take the function g we get from Lemma 6.4.5. Then the sequence $((\beta_n)_{n\in\mathbb{N}}$ defined by*

$$
\beta_n=\exp\bigg(\frac{g(n)}{8}\bigg) \tag{6.4.11}
$$

and $\kappa=c\mathrm{e}^{8g(1)}$ fulfil (6.4.10)*, where $c>0$ is a constant such that $\alpha_n\leq c^n\chi^{\mathfrak{g}}_n$.*

Proof: First, note that there is a fixed $c\geq 1$ such that

$$
\alpha_n\leq c^n\chi^{\mathfrak{g}}_n\quad\iff\quad\frac{1}{\chi(\mathfrak{g})_n}\leq\frac{c'^n}{\alpha_n}
$$

by the definition of a rapidly increasing sequence. Denote $a_i + b_i = n_i$. Then we have

$$\frac{\beta_k \cdot \beta_\ell}{\alpha_r \cdot \chi^{\mathfrak{g}}_{n_1} \cdots \chi^{\mathfrak{g}}_{n_r}} \leq \frac{c^{k+\ell}\beta_k \cdot \beta_\ell}{\alpha_r \cdot \alpha_{n_1} \cdots \alpha_{n_r}}$$

and hence

$$\begin{aligned}
&\log\left(\frac{\beta_k \cdot \beta_\ell}{\alpha_r \cdot \chi^{\mathfrak{g}}_{n_1} \cdots \chi^{\mathfrak{g}}_{n_r}}\right)\\
&\leq \log\left(\frac{c^{k+\ell}\beta_k \cdot \beta_\ell}{\alpha_r \cdot \alpha_{n_1} \ldots \alpha_{n_r}}\right)\\
&= (k+\ell)\log(c) + \frac{g(k)}{8} + \frac{g(\ell)}{8} - f_\alpha(r) - f_\alpha(n_1) - \cdots - f_\alpha(n_r)\\
&\overset{\text{(a)}}{\leq} (k+\ell)\log(c) + \frac{g(k+\ell)}{8} - f_\alpha(r) - f_\alpha(n_1) - \cdots - f_\alpha(n_r)\\
&\overset{\text{(b)}}{\leq} (k+\ell)\log(c) + g(n_1) + \cdots + g(n_r) + \frac{f_\alpha(r)}{8}\\
&\qquad - f_\alpha(r) - f_\alpha(n_1) - \cdots - f_\alpha(n_r)\\
&\overset{\text{(c)}}{\leq} (k+\ell)\log(c) + (g(n_1) - f_\alpha(n_1)) + \cdots + (g(n_r) - f_\alpha(n_r))\\
&\overset{\text{(d)}}{\leq} (k+\ell)\log(c') + 8g(1)n_1 + \cdots + 8g(1)n_r\\
&= (k+\ell)(\log(c') + 8g(1)).
\end{aligned}$$

In (a), we have used the convexity of g and Lemma 6.4.5 in (b). Then, (c) is just $\frac{f_\alpha(n)}{8} \leq f_\alpha(n)$. In (d) we used Lemma 6.4.5 again by estimating

$$g(n) = g(1 + \ldots + 1) \leq 8ng(1) + f_\alpha(n) \quad \Longleftrightarrow \quad g(n) - f_\alpha(n) \leq 8ng(1).$$

Now we need to exponentiate the inequality we just found and get

$$\frac{\beta_k \cdot \beta_\ell}{\alpha_r \cdot \chi^{\mathfrak{g}}_{n_1} \cdots \chi^{\mathfrak{g}}_{n_r}} \leq \mathrm{e}^{k+\ell)(\log(c')+8g(1))} = \left(c\mathrm{e}^{8g(1)}\right)^{k+\ell}.$$

Since the $n_1, \ldots, n_r$ have been arbitrary, $(\beta_n)_{n\in\mathbb{N}}$ and κ fulfil (6.4.10) and the Lemma is proven.$\triangledown$

Now, we only need to take the sum of the C_n. We know this is possible from Chapter 5, and we finally get a continuity estimate which is locally uniform in z. $\square$

From the result of the previous proposition, we see that the next definition is useful:

Definition 6.4.8 (The S_{1--}-Topology) *Let $\mathfrak{g}$ be a uniformly topologically nilpotent Banach-Lie algebra. Then we denote by $\mathrm{S}^\bullet_{1--}(\mathfrak{g})$ the symmetric tensor algebra endowed with the topology, which is defined by the set of seminorms $\mathscr{P}$ from Definition 6.4.4 and which is based $\mathfrak{g}$-rapidly increasing sequences.*

Now we can rephrase the statement of Proposition 6.4.6 in the following way: $\mathrm{S}^\bullet_{1--}(\mathfrak{g})$ endowed with the Gutt star product is a locally convex algebra. We also see analogously to Corollary 6.1.2 that exponentials functions are in the completion $\widehat{\mathrm{S}}^\bullet_{1--}(\mathfrak{g})$.

At the end of Chapter 5, we showed that a finite-dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ admitted a locally convex topology, such that the following three things are fulfilled.

i.) The product in $\mathscr{U}(\mathfrak{g})$ is continuous.

ii.) For every $\xi \in \mathfrak{g}$ the series $\exp(\xi)$ converges absolutely in the completion of $\mathscr{U}(\mathfrak{g})$.

iii.) Pulling back the topology to the symmetric tensor algebra, the projection and inclusion maps with respect to the graded structure

$$\mathrm{S}^\bullet(\mathfrak{g}) \xrightarrow{\pi_n} \mathrm{S}^n(\mathfrak{g}) \xrightarrow{\iota_n} \mathrm{S}^\bullet(\mathfrak{g})$$

are continuous for all $n \in \mathbb{N}$.

For Banach-Lie algebras, we came quite close to a similar statement: we know from Proposition 5.4.1 and the result of Wojtyǹski, that a Banach-Lie algebra must at least be *topologically nil* to satisfy the three upper points, so being topologically nil is *necessary*. We also know, that a uniformly topologically nilpotent Banach-Lie algebra $\mathfrak{g}$ allows us to construct such a locally convex topology on $\mathscr{U}(\mathfrak{g})$ explicitly, hence uniform topological nilpotency is *sufficient*. Maybe it is possible to find a notion of generalized nilpotency, which is equivalent to those three points.

Chapter 7

The Hopf Algebra Structure

As already pointed out in Chapter 3, universal enveloping algebras of a Lie algebras are more than just associative, unital algebras: they constitute one of the most important types of *Hopf algebras*, which are very common structures in mathematics. Hopf algebras are a particular *bialgebras*, which in turn are on one hand associative, unital algebras, and on the other hand coassociative, counital coalgebras. Those two substructures of bialgebras must of course fulfil certain compatibility conditions. Note at this point, that coalgebras play a crucial role in the theory of formal deformation quantization, which is due to Kontsevich, but which has been a lot further developed since. Good references on this so-called formality theory are given by Esposito [41] and Manetti [62].

In a Hopf algebra however, we have an additional map, called the *antipode*, which again must fulfil some compatibility relations. So, in brief, a Hopf algebra over a field $\mathbb{K}$ is a tuple $(H, \cdot, \eta, \Delta, \varepsilon, S)$ of a vector space H together with the following maps

$$
\begin{aligned}
\cdot\colon &\quad H \otimes H \longrightarrow H, && \text{multiplication} \\
\eta\colon &\quad \mathbb{K} \longrightarrow H, && \text{unit} \\
\Delta\colon &\quad H \longrightarrow H \otimes H, && \text{coproduct} \\
\varepsilon\colon &\quad H \longrightarrow \mathbb{K}, && \text{counit} \\
S\colon &\quad H \longrightarrow H, && \text{antipode,}
\end{aligned}
$$

such that we have a certain set of commuting diagrams. A very nice introduction to the theory of Hopf algebras with a lot of examples can e.g. be found in [83].

In the previous chapters, we have mostly studied the properties of the multiplication in $\mathscr{U}(\mathfrak{g}_z)$, endowed with a particular topology. In this last chapter of this thesis, we will treat the remaining Hopf algebra structure maps. In the first section, we will see that the comultiplication and the antipode are not touched by our deformation procedure. Hence, it will be enough to show the continuity of the undeformed versions of those maps, which we will do in second section.

7.1 An Undeformed Hopf Structure

To get show the continuity of the remaining structure maps of $\mathscr{U}_R(\mathfrak{g}_z)$, we have to put estimates on seminorms again. For this purpose, we need explicit formulas for the antipode

$$
S_z\colon \mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathscr{U}(\mathfrak{g}_z) \tag{7.1.1}
$$

and the coproduct

$$
\Delta_z\colon \mathscr{U}(\mathfrak{g}_z) \longrightarrow \mathscr{U}(\mathfrak{g}_z) \otimes \mathscr{U}(\mathfrak{g}_z) \tag{7.1.2}
$$

in the $\mathscr{U}_R(\mathfrak{g}_z)$ and in $\mathrm{S}^\bullet(\mathfrak{g})$. We pull them back to the symmetric algebra and extend them to the whole tensor algebra by symmetrizing beforehand. We define

$$\widetilde{S}_z\colon \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}), \quad \widetilde{S}_z = \mathfrak{q}_z^{-1} \circ S_z \circ \mathfrak{q}_z \circ \mathscr{S} \tag{7.1.3}$$

and

$$\widetilde{\Delta}_z\colon \mathrm{T}^\bullet(\mathfrak{g}) \longrightarrow \mathrm{S}^\bullet(\mathfrak{g}) \otimes \mathrm{S}^\bullet(\mathfrak{g}), \quad \widetilde{\Delta}_z = (\mathfrak{q}_z^{-1} \otimes \mathfrak{q}_z^{-1}) \circ \Delta_z \circ \mathfrak{q}_z \circ \mathscr{S}, \tag{7.1.4}$$

to avoid that the maps on $\mathrm{S}^\bullet(\mathfrak{g})$ and on $\mathscr{U}(\mathfrak{g}_z)$ are denoted by the same symbols. The next lemma gives us the two explicit formulas we need.

Lemma 7.1.1 *For $\xi_1, \ldots, \xi_n \in \mathfrak{g}$ we have the identities*

$$\widetilde{S}_z(\xi_1 \otimes \cdots \otimes \xi_n) = (-1)^n \xi_1 \cdots \xi_n \tag{7.1.5}$$

and

$$\widetilde{\Delta}_z(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{I \subseteq \{1,\ldots,n\}} \xi_I \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n \tag{7.1.6}$$

where ξ_I denotes the symmetric tensor product of all ξ_i with $i \in I$ and $\widehat{\xi_I}$ means that the ξ_i with $i \in I$ are left out.

Proof: First, we derive Formula 7.1.5: the antipode gives $S_z(\xi) = -\xi$ for $\xi \in \mathfrak{g}$ and extends to $\mathscr{U}(\mathfrak{g}_z)$ by algebra antihomomorphism, hence

$$S_z(\xi_1 \odot \cdots \odot \xi_n) = (-1)^n \xi_n \odot \cdots \odot \xi_1$$

in $\mathscr{U}(\mathfrak{g}_z)$ for $\xi_1, \ldots, \xi_n \in \mathfrak{g}$. This means

$$\widetilde{S}_z(\xi_1 \star_z \cdots \star_z \xi_n) = (-1)^n \xi_n \star_z \cdots \star_z \xi_1$$

in $\mathrm{S}^\bullet(\mathfrak{g})$. But now, using the linearity of $\widetilde{S}_z$ we get

$$\begin{aligned}
\widetilde{S}_z(\xi_1 \cdots \xi_n) &= \widetilde{S}_z\left(\frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\right) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \widetilde{S}_z\big(\xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\big) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^n \xi_{\sigma(n)} \star_z \cdots \star_z \xi_{\sigma(1)} \\
&= (-1)^n \xi_1 \cdots \xi_n.
\end{aligned}$$

For the coproduct, we have well-known formula with shuffle permutations:

$$\Delta_z(\xi_1 \odot \cdots \odot \xi_n) = \sum_{k=0}^{n} \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ I = \{i_1, \ldots, i_k\}}} \xi_{i_1} \odot \cdots \odot \xi_{i_k} \otimes \xi_1 \odot \cdots \widehat{\xi_I} \cdots \odot \xi_n.$$

We can derive it from the fact that Δ_z has the following form on Lie algebra elements:

$$\Delta_z(\xi) = \xi \otimes \mathbb{1} + \mathbb{1} \otimes \xi.$$

The coproduct extends to $\mathscr{U}(\mathfrak{g}_z)$ by algebra homomorphism. This yields

$$\begin{aligned}\Delta_z(\xi_1 \odot \cdots \odot \xi_n) &= \Delta_z(\xi_1) \odot \cdots \odot \Delta_z(\xi_n) \\ &= (\xi_1 \otimes \mathbb{1} + \mathbb{1} \otimes \xi_1) \odot \cdots \odot (\xi_n \otimes \mathbb{1} + \mathbb{1} \otimes \xi_n) \\ &= \sum_{k=0}^{n} \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ I=\{i_1,\ldots,i_k\}}} \xi_{i_1} \odot \cdots \odot \xi_{i_k} \otimes \xi_1 \odot \cdots \widehat{\xi_I} \cdots \odot \xi_n.\end{aligned}$$

Now we can pull this back to $\mathrm{S}^\bullet(\mathfrak{g}_z)$. We get a $\star_z$ for every $\odot$. For symmetric tensors we have by linearity

$$\begin{aligned}&\widetilde{\Delta}_z(\xi_1 \cdots \xi_n) \\ &= \widetilde{\Delta}_z\left(\frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\right) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \widetilde{\Delta}_z\big(\xi_{\sigma(1)} \star_z \cdots \star_z \xi_{\sigma(n)}\big) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{k=0}^{n} \sum_{\substack{1 \leq i_1 < \ldots < i_k \leq n \\ I=\{i_1,\ldots,i_k\}}} \xi_{i_{\sigma(1)}} \star_z \cdots \star_z \xi_{i_{\sigma(k)}} \otimes \xi_{\sigma(1)} \star_z \cdots \widehat{\xi_{\sigma(I)}} \cdots \star_z \xi_{\sigma(n)} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \sum_{\sigma \in S_k} \sum_{\tau \in S_{n-k}} \sum_{\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}} \frac{n!}{k!(n-k)!} \cdot \xi_{i_{\sigma(1)}} \star_z \cdots \star_z \xi_{i_{\sigma(k)}} \otimes \xi_{\tau(1)} \star_z \cdots \widehat{\xi_I} \cdots \star_z \xi_{\tau(n)} \\ &= \sum_{k=0}^{n} \sum_{\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,n\}} \xi_{i_1} \cdots \xi_{i_k} \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n. \qquad \square\end{aligned}$$

This lemma yields immediately the following result.

Corollary 7.1.2 *If we deform the symmetric tensor algebra of an AE-Lie algebra* $\mathrm{S}^\bullet(\mathfrak{g})$ *with the Gutt star product, the coproduct and the antipode will remain undeformed.*

Remark 7.1.3 (Hopf structures on the tensor algebra)

i.) On the first sight, this result looks astonishing, but actually it is not. The Hopf algebra structure we found here is just the Hopf algebra structure which comes from the tensor algebra of $\mathfrak{g}$. To be more precise: it is the Hopf algebra structure using the tensor product and the shuffle coproduct. There is also a second Hopf algebra structure: the one using the deconcatenation coproduct and the shuffle product. We find these structures on every tensor algebra of a vector space, it does not need to be a tensor algebra over a Lie algebra. Since the coalgebra structure on $\mathscr{U}(\mathfrak{g}_z)$ is inherited from $\mathrm{T}^\bullet(\mathfrak{g})$ and does not make use of the Lie bracket, there is no reason why a rescaling of the Lie bracket should change it.

ii.) What we have seen is hence just *one* possibility to deform the symmetric algebra. Another way to do so would be to deform the costructure and leave the product untouched. Such a deformation of a Hopf algebra would lead to the notion of *quantum groups*.

7.2 Continuity of the Hopf Structure

We need a topology on the tensor product in (7.1.6), since we want to prove the continuity of this map. As we have always used the projective tensor product for our construction so far, it seems just logic to do so again. The continuity of the two maps is then very easy to show.

Proposition 7.2.1 *Let $\mathfrak{g}$ be an AE-Lie algebra and $R \geq 0$. For every continuous seminorm p and all $x \in \widehat{\mathrm{T}}^{\bullet}_R(\mathfrak{g})$ the following estimates hold:*

$$p_R\Big(\widetilde{S}_z(x)\Big) \leq p_R(x) \tag{7.2.1}$$

and

$$(p_R \otimes p_R)\Big(\widetilde{\Delta}_z(x)\Big) \leq (2p)_R(x). \tag{7.2.2}$$

Proof: We just need to show both estimates on factorizing tensors and extend them with the infimum argument. Inequality (7.2.1) is clear, since we only get a sign. To get Equation (7.2.2), we compute:

$$\begin{aligned}
(p_R \otimes p_R)\Big(\widetilde{\Delta}_z(\xi_1 \otimes \cdots \otimes \xi_n)\Big) &= (p_R \otimes p_R)\left(\sum_{I \subseteq \{1,\dots,n\}} \xi_I \otimes \xi_1 \cdots \widehat{\xi_I} \cdots \xi_n\right) \\
&\leq \sum_{I \subseteq \{1,\dots,n\}} |I|!^R (n-|I|)!^R p^{|I|}(\xi_I) p^{n-|I|}\Big(\xi_1 \cdots \widehat{\xi_I} \cdots \xi_n\Big) \\
&\leq \sum_{I \subseteq \{1,\dots,n\}} |I|!^R (n-|I|)!^R p(\xi_1) \cdots p(\xi_n) \\
&\leq \sum_{I \subseteq \{1,\dots,n\}} n!^R p(\xi_1) \cdots p(\xi_n) \\
&= 2^n n!^R p(\xi_1) \cdots p(\xi_n) \\
&= (2p)_R(\xi_1 \otimes \cdots \otimes \xi_n).
\end{aligned}$$

This shows the statement. □

Remark 7.2.2 We see that we get no dependence on the parameter R. Also this is not surprising: the Hopf structure of the symmetric or the universal enveloping algebra over a vector space is cocommutative. The symmetric tensor product is commutative and its continuity estimate does not depend on the parameter R either. Note that for an abelian Lie algebra, also the product structure remains undeformed and all Hopf algebra maps are continuous for $R \geq 0$. In this sense, the independence of R fits into the picture.

The only maps which are left to consider are the unit and the counit. Since their continuity is clear by the definition of the T_R-topology, we get the a final result.

Theorem 7.2.3 *Let $\mathfrak{g}$ be an AE-Lie algebra and $z \in \mathbb{K}$. Then, if $R \geq 1$, $\widehat{\mathrm{S}}^{\bullet}_R(\mathfrak{g})$ will be a locally convex Hopf algebra. The same will hold for $\widehat{\mathrm{S}}^{\bullet}_{1^-}(\mathfrak{g})$, if $\mathfrak{g}$ is a nilpotent locally convex Lie algebra with continuous Lie bracket and for $\widehat{\mathrm{S}}^{\bullet}_{1^{--}}(\mathfrak{g})$, if $\mathfrak{g}$ is a uniformly topologically nilpotent Banach-Lie algebra.*

Bibliography

[1] Aigner, M.: *A course in enumeration*, vol. 238 in *Graduate texts in mathematics*. Springer-Verlag, 2000. 38

[2] Arakawa, T., Ibukiyama, T., Kaneko, M.: *Bernoulli Numbers and Zeta functions. Springer Monographs in Mathematics*. Springer-Verlag, 2014. 32

[3] Arnol'd, V. I.: *Mathematical Methods of Classical Mechanics*, vol. 60 in *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1989. 11

[4] Baker, H.: *Alternants and Continuous Groups*. Proc. Lond. Math. Soc. **3** (1905), 24–47. 29

[5] Bär, C., Ginoux, N., Pfäffle, F.: *Wave equations on Lorentzian manifolds and quantization. ESI Lectures in Mathematics and Physics*. European Mathematical Society (EMS), Zürich, 2007. 17

[6] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Quantum Mechanics as a Deformation of Classical Mechanics*. Lett. Math. Phys. **1** (1977), 521–530. 19

[7] Bayen, F., Flato, M., Frønsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization*. Ann. Phys. **111** (1978), 61–151. 19

[8] Beiser, S.: *A Convergent Algebra for the Wick Star Product on the Poincaré Disc*. PhD thesis, Fakultät für Mathematik und Physik, Physikalisches Institut, Albert-Ludwigs-Universität, Freiburg, 2011. Available at `https://www.freidok.uni-freiburg.de/data/8369`. 21

[9] Beiser, S., Waldmann, S.: *Fréchet algebraic deformation quantization of the Poincaré disk*. Crelle's J. reine angew. Math. **688** (2014), 147–207. 21

[10] Berezin, F. A.: *Some remarks on the associative envelope of a Lie algebra*. Funkcion. Anal. Priloz. **1**.5 (1967), 1–14. 27, 45

[11] Berezin, F. A.: *General Concept of Quantization*. Commun. Math. Phys. **40** (1975), 153–174. 18

[12] Berezin, F. A.: *Quantization*. Math. USSR Izvestija **8**.5 (1975), 1109–1165. 18

[13] Berezin, F. A.: *Quantization in Complex Symmetric Spaces*. Math. USSR Izvestija **9**.2 (1975), 341–379. 18

[14] Bertelson, M., Cahen, M., Gutt, S.: *Equivalence of Star Products*. Class. Quant. Grav. **14** (1997), A93–A107. 20

[15] Bieliavsky, P.: *Strict Quantization of Solvable Symmetric Spaces*. J. of Symplectic Geometry **1**.2 (2002), 269–320. 21

[16] Bieliavsky, P., Gayral, V.: *Deformation Quantization for Actions of Kählerian Lie Groups*, vol. 236.1115 in *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, RI, 2015. 21

[17] Blackadar, B.: *Operator algebras*, vol. 122 in *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, Heidelberg, 2006. 56

[18] Bogfjellmo, G., Dahmen, R., Schmedig, A.: *Character groups of Hopf algebras as infinite-dimensional Lie groups*. Preprint **arXiv: 1410.6468 [math.DG]** (2015). 60

[19] Bongaarts, P.: *Quantum Theory*. Springer-Verlag, Cham, Heidelberg, New York, Dordrecht, London, 2015. A Mathematical Approach. 14

[20] Bordemann, M., Neumaier, N., Waldmann, S.: *Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications*. J. Geom. Phys. **29** (1999), 199–234. 35

[21] Bordemann, M., Waldmann, S.: *A Fedosov Star Product of Wick Type for Kähler Manifolds.* Lett. Math. Phys. **41** (1997), 243–253. 20

[22] Boseck, H., Czichowski, G., Rudolph, K.-P.: *Analysis on topological groups - General Lie Theory*, vol. 37 in *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics].* BSB B. G. Teubner Verlagsgesellschaft, 1981. 59, 60

[23] Bourbaki, N.: *Lie Groups and Lie Algebras.* Springer-Verlag, Berlin, Heidelberg, New York, 1989. 27

[24] Campbell, J. E.: *On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups.* Proc. Lond. Math. Soc. **28** (1896), 381–390. 29

[25] Campbell, J. E.: *On a Law of Combination of Operators.* Proc. Lond. Math. Soc. **29** (1897), 14–32. 29

[26] Carlitz, L.: *Bernoulli Numbers.* Fibonacci Quarterly **6** (1968), 71–85. 38

[27] Casas, F., Murua, A.: *An efficient algorithm for computing the Baker–Campbell–Hausdorff series and some of its applications.* J. Math. Phys. **50** (2009). 30

[28] Cattaneo, A. S., Felder, G., Tomassini, L.: *From local to global deformation quantization of Poisson manifolds.* Duke Math. J. **115**.2 (2002), 329–352. 21

[29] Chari, V., Pressley, A.: *A Guide to Quantum Groups.* Cambridge University Press, Cambridge, 1994. 26

[30] Connes, A.: *Noncommutative Geometry.* Academic Press, San Diego, New York, London, 1994. 18

[31] Cortiñas, G.: *An explicit formula for PBW quantization.* Comm. Algebra **30**.4 (2002), 1705–1713. 49

[32] Deligne, P.: *Déformations de l'Algèbre des Fonctions d'une Variété Symplectique: Comparaison entre Fedosov et DeWilde, Lecomte.* Sel. Math. New Series **1**.4 (1995), 667–697. 20

[33] DeWilde, M., Lecomte, P. B. A.: *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds.* Lett. Math. Phys. **7** (1983), 487–496. 20

[34] DeWilde, M., Lecomte, P. B. A.: *Star-Products on Cotangent Bundles.* Lett. Math. Phys. **7** (1983), 235–241. 20

[35] Dixmier, J.: *Enveloping Algebras. North-Holland Mathematical Library* no. **14**. North-Holland Publishing Co., Amsterdam, New York, Oxford, 1977. 45

[36] Dolgushev, V. A.: *Covariant and equivariant formality theorems.* Adv. Math. **191** (2005), 147–177. 21

[37] Drinfel'd, V. G.: *On constant quasiclassical solutions of the Yang-Baxter quantum equation.* Sov. Math. Dokl. **28** (1983), 667–671. 21, 28

[38] Dynkin, E. B.: *Calculation of the Coefficients in the Campbell-Hausdorff formula.* Doklady Akademii Nauk SSSR **57** (1947), 323–326. English version in "Selected papers of E. B. Dynkin with commentary", by AMS in 2000. 29

[39] Dynkin, E. B.: *Normed Lie algebras and analytic groups.* Uspekhi Mat. Nauk. **5** (1950), 135–186. English version in "AMS Translation", series 1, volume 9, 1962. 29

[40] El kinani, A., Oudadess, M.: *Entire functions and m-convex structure in commutative Baire algebras.* Bulletin of the Belgian Mathematical Society **4**.5 (1997), 685 – 687. 60

[41] Esposito, C.: *Formality theory. From Poisson structures to deformation quantization.* Springer-Verlag, Heidelberg, Berlin, 2015. 16, 97

[42] Esposito, C., Stapor, P., Waldmann, S.: *Convergence of the Gutt Star Product.* Preprint **arXiv:1509.09160** (2015), 33. 7

[43] Fedosov, B. V.: *A Simple Geometrical Construction of Deformation Quantization.* J. Diff. Geom. **40** (1994), 213–238. 20

[44] Glöckner, H., Neeb, K.-H.: *When unit groups of continuous inverse algebras are regular Lie groups.* Studia Math. **211** (2012), 95–109. 60

[45] Goldberg, K.: *The formal power series for* $log(e^x e^y)$. Duke Math. J. **23** (1956), 13–21. 31

[46] Goodman, R.: *Differential Operators of Infinite Order on a Lie Group, II.* Indiana Math. J. **21** (1971), 383–409. 66

[47] Groenewold, H. J.: *On the principles of elementary quantum mechanics.* Physica **12** (1946), 405–460. 18

[48] Gutt, S.: *An Explicit $*$-Product on the Cotangent Bundle of a Lie Group.* Lett. Math. Phys. **7** (1983), 249–258. 21, 28, 45

[49] Gutt, S., Rawnsley, J.: *Natural Star Products on Symplectic Manifolds and Quantum Moment Maps.* Lett. Math. Phys. **66** (2003), 123–139. 20

[50] Haag, R.: *Local Quantum Physics.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 1993. 17

[51] Hall, B. C.: *Lie Groups, Lie Algebras, and Representations*, vol. 222 in *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 2003. 30

[52] Hall, B. C.: *Quantum Theory for Mathematicians*, vol. 267 in *Graduate Texts in Mathematics.* Springer-Verlag, Heidelberg, Berlin, New York, 2013. 14

[53] Hausdorff, F.: *Die symbolische Exponentialformel in der Gruppentheorie.* Sächsische Akademie der Wissenschaften **58** (1906), 19–48. 29

[54] Hilgert, J., Neeb, K.-H.: *Structure and geometry of Lie groups. Springer Monographs in Mathematics.* Springer-Verlag, Heidelberg, New York, 2012. 26, 30

[55] Hörmander, L.: *The Analysis of Linear Partial Differential Operators I. Classics in Mathematics.* Springer-Verlag, Berlin, Heidelberg, New York, 2. edition, 2003. Reprint of the 2nd Edition 1990. 56

[56] Jacobson, N.: *Lie algebras.* Dover Publications Inc., New York, 1979. Republication of the 1962 original. 30

[57] Jarchow, H.: *Locally Convex Spaces.* B. G. Teubner, Stuttdart, 1981. 56, 61

[58] Kathotia, V.: *Kontsevich's universal formula for deformation quantization and the Campbell-Baker-Hausdorff Formula, I.* Preprint **math.QA/9811174v2** (December 1998). 45

[59] Kontsevich, M.: *Deformation Quantization of Poisson Manifolds, I.* Preprint **q-alg/9709040** (September 1997). 21

[60] Kontsevich, M.: *Deformation Quantization of Poisson manifolds.* Lett. Math. Phys. **66** (2003), 157–216. 21

[61] Majid, S.: *Foundations of Quantum Group Theory.* Cambridge University Press, 1995. 26

[62] Manetti, M.: *Lectures on deformations of complex manifolds.* Dipartimento di Matematica "G. Castelnuovo", Universit'a di Roma "La Sapienza", 2005. Available at `math.AG/0507286`. 97

[63] Marsden, J. E., Ratiu, T. S.: *Introduction to Mechanics and Symmetry. Texts in applied mathematics* no. **17**. Springer-Verlag, New York, Heidelberg, 1999. 11

[64] Michael, E. A.: *Locally Multiplicatively-Convex Topological Algebras. Mem. Amer. Math. Soc.* no. **11**. AMS, Providence, R. I., 1952. 56

[65] Mitiagin, B. S., Rolewicz, S., Żelazko, W.: *Entire functions in B_0-algebras.* Studia Math. **21** (1962), 291–306. 60, 93

[66] Moyal, J. E.: *Quantum Mechanics as a Statistical Theory.* Proc. Camb. Phil. Soc. **45** (1949), 99–124. 18

[67] Müller, V.: *Nil, Nilpotent and PI-Algebras.* Functional Analysis and Operator Theory **30** (1994). 80, 91

[68] Natsume, T., Nest, R., Peter, I.: *Strict Quantizations of Symplectic Manifolds.* Lett. Math. Phys. **66** (2003), 73–89. 21

[69] Nest, R., Tsygan, B.: *Algebraic Index Theorem.* Commun. Math. Phys. **172** (1995), 223–262. 20

[70] Neumaier, N.: *Sternprodukte auf Kotangentenbündeln und Ordnungs-Vorschriften.* master thesis, Fakultät für Physik, Albert-Ludwigs-Universität, Freiburg, 1998. 45

[71] Newman, M., Thompson, R. C.: *Numerical Values of Goldberg's Coefficients in the Series* $\log(e^x e^y)$. Mathematics of Computation **48**.177 (1987), 265–271. 34

[72] Omori, H., Maeda, Y., Yoshioka, A.: *Weyl Manifolds and Deformation Quantization.* Adv. Math. **85** (1991), 224–255. 20

[73] Pflaum, M. J., Schottenloher, M.: *Holomorphic deformation of Hopf algebras and applications to quantum groups.* J. Geom. Phys. **28** (1998), 31–44. 80

[74] Raševskiĭ, P. K.: *Associative hyper-envelopes of Lie algebras, their regular representations and ideals.* Trans. Mosc. Math. Soc. **15** (1966), 3–54. 66

[75] Reed, M., Simon, B.: *Methods of modern mathematical physics I. Functional analysis.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second. edition, 1980. 56

[76] Reichert, T., Waldmann, S.: *Classification of Equivariant Star Products on Symplectic Manifolds.* Preprint **arXiv:1507.00146** (2015), 33. 21

[77] Rieffel, M.: *Deformation Quantization and Operator Algebras.* In: Arveson, W. B., Douglas, R. G. (eds.): *Proceedings of Symposia in Pure Mathematics*, vol. 51 in *Proceedings of Symposia in Pure Mathematics.* American Mathematical Society, Providence, RI, 1990. Proceedings of the Summer Research Institute held at the University of New Hampshire, Durham, New Hampshire, July 3–23, 1988. 21

[78] Rieffel, M. A.: *Deformation Quantization of Heisenberg Manifolds.* Commun. Math. Phys. **122** (1989), 531–562. 21

[79] Rieffel, M. A.: *Deformation quantization for actions of* $\mathbb{R}^d$. Mem. Amer. Math. Soc. **106**.506 (1993), 93 pages. 21

[80] Rieffel, M. A.: *Questions on quantization.* Contemporary Mathematics **228** (1998). 21

[81] Rudin, W.: *Functional Analysis.* McGraw-Hill Book Company, New York, 2. edition, 1991. 56, 58, 73

[82] Schötz, M.: *Konvergente Sternprodukte auf Hilberträumen.* master thesis, Mathematisches Institut der Universität Würzburg, Würzburg, 2014. 22, 23

[83] Schweigert, C.: *Hopf algebras, quantum groups and topological field theory.* Hamburg University, Department of Mathematics, 2015. Lecture notes. 97

[84] Serre, J.-P.: *Lie algebras and Lie groups*, vol. 1500 in *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2. edition, 2006. 1964 lectures given at Harvard University, Corrected fifth printing. 30

[85] Sloane, N. J. A.: *Denominator of Bernoulli number* B_n. The On-Line Encyclopedia of Integer Sequences, 1964, 2015. https://oeis.org/A027641. 32

[86] Sloane, N. J. A.: *Numerator of Bernoulli number* B_n. The On-Line Encyclopedia of Integer Sequences, 1964, 2015. https://oeis.org/A027641. 32

[87] Souriau, J.-M.: *Structure des systèmes dynamiques. Maîtrises de mathématiques.* Dunod, Paris, 1970. 18

[88] Tamarkin, D. E.: *Formality of Chain Operad of Little Discs.* Lett. Math. Phys. **66** (2003), 65–72. 21

[89] Thompson, R. C.: *Cyclic relations and the Goldberg coefficients in the Campbell–Baker–Hausdorff formula.* Proc. Amer. Math. Soc. **86**.1 (1982), 12–14. 31

[90] Thompson, R. C.: *Convergence proof for Goldberg's exponential series.* Linear Algebra and its Applications **121** (1989), 3–7. 34

[91] van Hove, L.: *Sur certaines représentations unitaires d'un groupe infini de transformations.* Mem. de l'Acad. Roy. de Belgique (Classe des Sci.) **XXVI** (1951), 61–102. 18

[92] Varadarajan, V. S.: *Lie Groups, Lie Algebras, and Their Representations*, vol. 102 in *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1974. 26, 30

[93] Várilly, J. C.: *An introduction to noncommutative geometry. EMS Series of Lectures in Mathematics.* European Mathematical Society (EMS), Zürich, 2006. 18

[94] von Neumann, J.: *Mathematische Grundlagen der Quantenmechanik.* Springer-Verlag, Heidelberg, Berlin, New York, 2. edition, 1996. 14

[95] Waldmann, S.: *Poisson-Geometrie und Deformationsquantisierung. Eine Einführung.* Springer-Verlag, Heidelberg, Berlin, New York, 2007. 16, 20, 24

[96] Waldmann, S.: *A nuclear Weyl algebra.* J. Geom. Phys. **81** (2014), 10–46. 21, 22, 23, 65, 81, 89, 91

[97] Waldmann, S.: *Operator-Algebraic Methods in Quantum Mechanics*, 2014. Lecture notes in preparation. 56, 61

[98] Waldmann, S.: *Differential Geometry*, 2015. Lecture notes in preparation. 24, 27

[99] Weyl, H.: *The Theory of Groups and Quantum Mechanics.* Dover, New York, 1931. 16

[100] Wojtynski, W.: *Quasi-nilpotent Banach-Lie-Algebras are Baker-Campbell-Hausdorff.* Journal of Functional Analysis **153** (1998), 405–413. 80, 92

[101] Żelazko, W.: *Concerning entire functions in* B_0*-algebras.* Studia Math. **110**.3 (1994), 283–290. 60