Classifying (almost)-Belyi maps with Five Exceptional Points

Mark van Hoeij*
Department of Mathematics
Florida State University
Tallahassee, FL 32306, USA

Vijay Jung Kunwar*
Department of Mathematics
Albany State University
Albany, GA 31705, USA

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Abstract

We classify all rational functions $f : \mathbb{P}^1 \to \mathbb{P}^1$ whose branching pattern above $0, 1, \infty$ satisfy a certain regularity condition with precisely $d = 5$ exceptions. This work is motivated by solving second order linear differential equations, with $d = 5$ true singularities, in terms of hypergeometric functions. A similar problem was solved for $d = 4$ in [2].

1 Introduction

Our main goal in this paper is to tabulate all rational functions $f \in \mathbb{C}(x)$ whose branching patterns satisfy a regularity condition defined in Section 1.2, and to prove completeness of the table. This condition comes from solving differential equations with at most $d = 5$ true singularities in terms of hypergeometric functions. The cases $d = 3$ resp. $d = 4$ were previously studied in [6] resp. [3, 2]. The functions f in our tables are either Belyi maps or almost-Belyi maps:

Definition 1.1. *A holomorphic map f from a compact Riemann surface C to the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \bigcup \{\infty\}$ is a* Belyi map *if its branched set is in $\{0, 1, \infty\}$, i.e. f is unramified outside of $\{0, 1, \infty\}$.*

The pre-image $\subset C$ of the closed interval $[0, 1] \subset \mathbb{P}^1$ under a Belyi map f gives a bi-colored oriented graph, called dessin d'enfant. Up to equivalence, there is a 1-1 correspondence between dessins d'enfants, 3-constellations, and Belyi maps, see Section 3 for details. In our application $C = \mathbb{P}^1$ since our f's are rational functions.

Almost-Belyi maps [31] are rational maps with only 1 or 2 simple branch points outside of $\{0, 1, \infty\}$. We denote these as Belyi$^{(1)}$ resp. Belyi$^{(2)}$ maps if they have 1 resp. 2 simple branch points outside $\{0, 1, \infty\}$. Those branch points in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ are free to move, so while Belyi-maps are classified by discrete objects (dessins d'enfants, or 3-constellations), Belyi$^{(1)}$ resp. Belyi$^{(2)}$ maps naturally occur in 1 resp. 2 dimensional families.

We expect our tables to be helpful in other contexts as well; Belyi maps have wide range of application in the fields of algebra, geometry, and combinatorics. They are used to prove Davenport-Stothers-Zannier bound [21]. Shabat polynomials are the special case of Belyi maps with only one pole at infinity. A *dynamical* Belyi map $f : \mathbb{P}^1 \to \mathbb{P}^1$, which sends $\{0, 1, \infty\}$ to $\{0, 1, \infty\}$, is used to construct the situation of a Julia set with "complete chaos" [30]. Another application is the classification of elliptic fibrations of K3-surfaces [4]. Some Belyi$^{(1)}$ maps correspond to solutions of isomonodromic systems of Fuchsian equations with 4 (+1 apparent) singularities, and hence give algebraic solutions of the Painlevé VI equation by Jimbo-Miwa correspondence [20]. Such Belyi$^{(1)}$ maps are studied as deformations of dessins d'infants in [12]. The motivation for this

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paper is solving differential equations in terms of hypergeometric functions [3, 10, 2, 11, 5] but we expect that our tables will be useful for other applications as well. Indeed, some entries of the tables have already appeared in prior applications, see Section 1.2 for more.

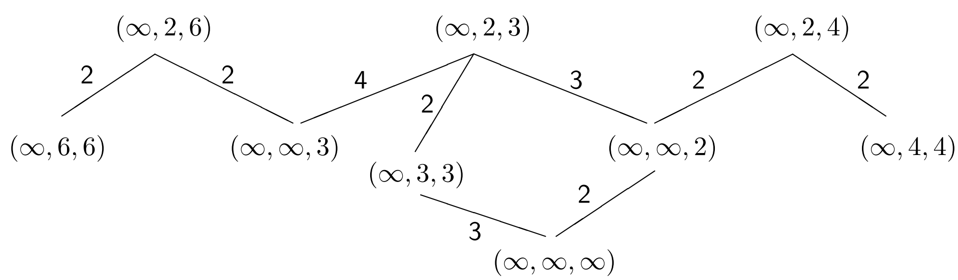
1.1 Motivation, solving differential equations

Conjecture 1. *Let L be a second order linear differential equation $a_2y'' + a_1y' + a_0y = 0$ with coefficients $a_i \in \mathbb{C}[x]$. If L is* regular singular[1] *(i.e. Fuchsian) and has, among its solutions, a non-zero power series solutions with integer coefficients, ($y \in \mathbb{Z}[[x]] - \{0\}$), then one of these cases holds:*

- *y is an* algebraic *function, or*
- *y can be written as $y = r_0S(f) + r_1(S(f))'$, where $S(f) = {}_2F_1(a, b; c \,|\, f)$. Here f, r_0, r_1 are algebraic functions, $a, b \in \mathbb{Q}$ and c is a positive integer[2].*

If y is algebraic, it can be found with Kovacic' algorithm [19], so we are mainly interested in ${}_2F_1$-type solutions. We only treat rational f's in this paper, and plan to treat algebraic f's later by exploiting their relation to the modular curve $X_0(N)$. For $d = 3$ singularities f is a Möbius transformation, and for $d = 4$ singularities (Heun's case) f's are classified in [2, 3]. So we treat $d = 5$ in this paper.

Recent algorithms for finding closed form solutions (solutions expressible in terms of well studied special functions) are given in [3, 16, 10, 11, 9, 15]. Although random differential equations are unlikely to have closed form solutions, the conjecture says that the second order differential equations that are of most interest to combinatorics[3] should have closed form solutions. We tested this on numerous differential equations obtained from the oeis.org (the Online Encyclopedia of Integer Sequences). All turned out to be ${}_2F_1$-solvable with parameters that can be related to a triplet (k, ℓ, m) (see the notation in Sections 1.2 and 8) in Diagram (1) in Takeuchi's classification [18, Section 4] of arithmetic triangle groups:



The ${}_2F_1$-functions for this diagram are said to be associated with elliptic curves, modular forms, and elliptic integrals [17]. They appear in many contexts. To cover them, it suffices[4] to cover: $(k, \ell, m) = (\infty, 2, m)$ with $m \in \{3, 4, 6\}$. Some parts of this paper focus on $(k, \ell, m) = (\infty, 2, 3)$, but our website [1] covers $m = 4$ and $m = 6$ as well.

[1] We focus on the regular singular case because for irregular singular equations of order 2, a complete algorithm to find all {Airy, Bessel, Kummer, Whittaker}-type solutions was given in [16, 9]. The regular-singular assumption can be replaced by the assumption that y in Conjecture 1 has a non-zero radius of convergence.

[2] This condition implies at least one logarithmic singularity. Because of the conjecture we focus on differential equations with at least one logarithmic singularity, however, the same table can also be used for more general "parametric" cases [2].

[3] Equations with a convergent integer power series solution (*globally nilpotent* differential equations).

[4] Solutions related to entries of the diagram can be expressed in terms of those three entries. However, the other entries can still be relevant if we want solutions of minimal size, see Section 5.3.3 (decompositions) in [11] for more. Entry (∞, ∞, ∞) corresponds to writing solutions in terms of the elliptic integrals K and E.

1.2 Project Outline

Definition 1.2. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function. Let k, ℓ, m be positive integers or ∞, see remark 1.3. The (k, ℓ, m)-exceptional points of f are:*

- *roots of f of order not divisible by k,*
- *roots of $1 - f$ of order not divisible by ℓ,*
- *roots of $\frac{1}{f}$ of order not divisible by m.*

We denote the set of (k, ℓ, m)-exceptional points of f as $E_{k\ell m}(f)$, or simply as $E(f)$ when k, ℓ, m are fixed.

Remark 1.3. *If $k = \infty$ then all roots of f are (k, ℓ, m)-exceptional points (and likewise for roots of $1 - f$ resp. $\frac{1}{f}$ if ℓ resp. m is ∞).*

Remark 1.4. *Our definition 1.2 differs slightly from that in [3] which defined exceptional points as*

$\{$*roots of* f *of order* $\neq k\} \cup \{$*roots of* $1 - f$ *of order* $\neq \ell$ $\} \cup \{$*roots of* $\frac{1}{f}$ *of order* $\neq m\}$

which contains our $E(f)$.

The goals are to:

(a) construct a database [1] that, up to Möbius-equivalence, contains all rational functions with five $(\infty, 2, 3)$-exceptional points, and likewise for $(\infty, 2, 4)$ and $(\infty, 2, 6)$.

(b) prove completeness (Sections 3 – 6)

(c) give a fast method for the following problem: Given a field $k \subseteq \mathbb{C}$ and $\{q_1, \ldots, q_5\} \subset \mathbb{P}^1$, find every $F \in k(x)$ such that $E(F) = \{q_1, \ldots, q_5\}$ (Section 7)

(d) solve linear differential equations with 5 true singularities (Section 8).

Our work continues the work (for $d = 4$ exceptional points) in [2, 3]; the main novelties are:

1. We prove completeness by giving an efficient new algorithm for finding dessins (algorithm 4.4). Such an algorithm was not needed for [2, 3]; the tables in [2] are small enough for manual enumeration, while [3] gave a method specific to $d = 4$ that did not rely on dessins.
2. We compute almost-Belyi maps ([2, 3] only consider Belyi maps), and braid orbits of almost-dessins to prove completeness.
3. Our $\text{Belyi}^{(1)}$ families turn out to be remarkably nice: they allow rational parametrizations that cover all $\text{Belyi}^{(1)}$ maps by direct substitution, without any gaps or duplicates (definitions 2.2 and 5.2).

The first page on our website [1] gives the database for goal (a), while another page (follow the link on the line "Completeness") gives examples and all algorithms needed for goals (b), (c) and (d). For (c), one has to select every f in the database whose $E(f)$ match $\{q_1, \ldots, q_5\}$ up to a Möbius transformation $x \mapsto \frac{ax+b}{cx+d}$. We do this by computing *five-point-invariants* (functions of $\{q_1, \ldots, q_5\}$ whose values are invariant under Möbius transformations of the input[5]).

(k, ℓ, m)	Belyi maps				$\text{Belyi}^{(1)}$ families		$\text{Belyi}^{(2)}$ families
	$\notin \text{Belyi}^{(1)}$	$\in \text{Belyi}^{(1)}$		Total	$\notin \text{Belyi}^{(2)}$	$\in \text{Belyi}^{(2)}$	
		indirectly	directly				
$(\infty, 2, 3)$	*411*	9	266	686	*65*	3	*2*
$(\infty, 2, 4)$	*121*	3	23	147	*20*	0	*0*
$(\infty, 2, 6)$	*54*	2	5	61	*12*	0	*0*

Table 1: Summary of the online table [1].

[5] A four-point invariant is given by the j-invariant of $y^2 = (x - q_1)(x - q_2)(x - q_3)(x - q_4)$.

All maps in Table 1 are listed on our website, and all have $|E(f)| = 5$. The three columns highlighted in italics (Column "$\notin \mathrm{Belyi}^{(1)}$", Column "$\notin \mathrm{Belyi}^{(2)}$", and Column "$\mathrm{Belyi}^{(2)}$") contain precisely those entries of Table 1 that would still have 5 exceptional points if we used the definition in Remark 1.4. So although our application uses definition 1.2, both definitions are useful for the construction of the database. The files on our website give algorithms and tables for both definitions, using the phrase "count=5" to refer to our definition 1.2, and the phrase "Count=5" for Remark 1.4.

All entries of the table are needed for goal (b), proving completeness. However, the only entries that are needed for goal (c) are the entries in italics plus Column "indirectly", as will be explained in Section 5.6.

Belyi maps and almost-Belyi maps have other applications as well. Indeed, some entries of our table have appeared elsewhere. The almost-Belyi maps of degree ≤ 4 are constructed in [14]. In addition, 14 out of the 68 Belyi$^{(1)}$ maps corresponding to $(k, \ell, m) = (\infty, 2, 3)$ in Table 1 are constructed in [12, 13, 14].

Our main goal for this paper to prove completeness of our online table (summarized in Table 1) since that will be useful for our application. The Belyi and Belyi$^{(1)}$ columns have so many entries that algorithms are needed for this proof. Algorithms that are key to the proof will be described in this paper. Particularly important is Algorithm 4.4 which is key to proving that our Belyi tables are complete. It computes all "dessins" (conjugacy classes of 3-constellations) relevant for our project. Although there is another implemented algorithm for the same task [8] based on group theory, it was not efficient enough for higher degrees, so we developed a novel algorithm instead.

2 Rational functions with a prescribed branching pattern

This section will cover goal (a) from the introduction. Section 2.1 will enumerate the relevant branching patterns. Finding function(s) for a branching pattern, as in [5, 7], is shown here by an example:

Example 2.1. *Suppose we want to find a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ with branching pattern $(1, 1, 3, 5)$, $(2, 2, 2, 2, 2)$, $(1, 3, 3, 3)$ above $0, 1, \infty$. We abbreviate this as $(1^2, 3, 5), (2^5), (1, 3^3)$. The degree is $n = 1 + 1 + 3 + 5 = 2 + 2 + 2 + 2 + 2 = 1 + 3 + 3 + 3 = 10$. The sum of $e_p - 1$ (where e_p denotes the branching index) for all p above $\{0, 1, \infty\}$ is $(1 - 1) + \cdots + (3 - 1) = 17$. However, the Hurwitz formula for genus zero:*

$$\sum_{p \in \mathbb{P}^1} (e_p - 1) = 2n - 2 \tag{1}$$

gives 18, so there must be 1 more ramification point p, with $e_p = 2$, above some point $t \notin \{0, 1, \infty\}$. So, f is a Belyi$^{(1)}$ *map as it has a simple branch point outside $\{0, 1, \infty\}$.*

We aim to find all such f up to Möbius-equivalence (Definition 3.1). We use the three degrees of freedom in Möbius-transformations to move the order-1 pole of f to $x = 1$, and the roots of orders 3 and 5 to $x = 0$ and $x = \infty$. That brings f in this form:

$$f := \frac{(Ax^2 + Bx + C)x^3}{(x-1)(x^3 + a_2x^2 + a_1x + a_0)^3}, \quad 1 - f = \frac{c\left(x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0\right)^2}{(x-1)\left(x^3 + a_2x^2 + a_1x + a_0\right)^3}.$$

Equating f with $1 - (1 - f)$ produces equations for the unknowns. Our implementation eliminates unknowns as long as it finds an equation that is linear in an unknown. Three unknowns b_2, b_3, b_4 in two large equations remain. Factoring the resultant produces one equation in two unknowns, i.e. an algebraic curve which turned out to have genus 0 (remarkably, the same happened for all $68 + 20 + 12$ cases in Tables 2 and 3). That means the solutions to this equation can be written as rational functions in a new variable s, which we can find with Maple's `parametrization`. *After simplification we obtain $b_3 = \frac{1}{3}s^4 - \frac{8}{3}s^3 + 18s^2 - 96s + 18$ and $b_4 = s^2 - 14$. Substitution followed by a gcd produces the value of b_2. Substituting into f, followed by two simple transformations ($x \mapsto 1 - x$, and $s \mapsto s + 3$) to reduce its size, produces:*

$$f = \frac{64s^8(x-1)^3(9x^2 - 6s^2x - 28sx + s^4 - 12s^3 + 36s^2)}{x(9x^3 - 6s^2x^2 - 36sx^2 + s^4x - 4s^3x + 60s^2x + 8s^4 - 32s^3)^3} \in \mathbb{Q}(s)(x).$$

This branches above $0, 1, \infty$ plus one more point, denoted t. To find it we first compute the ramification point p above t. This p must be the only root of $f' = 0$ not in $f^{-1}(\{0, 1, \infty\})$. We find $p = \frac{1}{15}s^2 - \frac{2}{3}s - \frac{8}{5}$ and

$$t = f(p) = \frac{3125\,(s-9)^4\,s^8\,(3s-2)}{4\,(s-4)^4\,(s^3-9s^2+324s-216)^3\,(s-1)^2}. \tag{2}$$

If $g : \mathbb{P}^1 \to \mathbb{P}^1$ has the same branching pattern above $0, 1, \infty$, one could ask if it is Möbius-equivalent to f for some value of s. More generally, how to prove completeness for the entire database? Before we can answer that in Section 5 we first need a definition:

Definition 2.2. *For a* Belyi$^{(1)}$ *map $f = f(s, x) \in \mathbb{C}(s)(x)$, let $\phi_f \in \mathbb{C}(s)$ be the function that expresses (as for example in equation (2)) the branch point $t \notin \{0, 1, \infty\}$ in terms of s. A point $s_0 \in \mathbb{P}^1$ is* degenerate *if $f(s_0, x)$ does not evaluate to some $g \in \mathbb{C}(x)$ with the same x-degree as f. It is called* generic *if $\phi_f(s_0) \notin \{0, 1, \infty\}$, and* special *if it is not degenerate nor generic. We define f's* family *as $\{f(s_0, x) \,|\, s_0$ not degenerate$\}$, and call it* gap-free *if no generic s_0 degenerates.*

2.1 Enumerating branching patterns

Let $B = (e_{1,1}, \ldots, e_{1,n_1}), (e_{2,1}, \ldots, e_{2,n_2}), (e_{3,1}, \ldots, e_{3,n_3})$ be a *branching pattern of degree* n, which means that the $e_{i,j}$ are positive integers with $n = \sum_{j=1}^{n_i} e_{i,j}$ for each $i = 1, 2, 3$.

Since we only tabulate rational functions, we only consider *planar* (genus zero) branching patterns. Then $S \leq 2n - 2$, where $S := \sum_{i=1}^{3} \sum_{j=1}^{n_i} (e_{i,j} - 1)$ is the part of the Hurwitz formula (1) coming from points p above $\{0, 1, \infty\}$. Let $\delta = 2n - 2 - S$. If $\delta = 0$ then we call B a *Belyi branching pattern*, if $\delta > 0$ then we call B a Belyi$^{(\delta)}$ branching pattern (Example 2.1 was planar and Belyi$^{(1)}$).

Definition 2.3. *Let B and $e_{i,j}$ as above and k, ℓ, m be positive integers or ∞. Let $(A_1, A_2, A_3) = (k, \ell, m)$. We define* E$(B) := \{(i, j) \mid A_i = \infty$ *or* $A_i \nmid e_{i,j}\}$.

Remark 2.4. *Definition 2.3 corresponds to Definitions 1.2 in that if B is the branching pattern of a rational function f, then* $|\,\mathrm{E}(B)\,| = |\,\mathrm{E}(f)\,|$.

If B is planar Belyi$^{(\delta)}$ of degree n, then $\#e_{i,j} = \sum e_{i,j} - \sum(e_{i,j} - 1) = 3n - S = n + 2 + \delta$. The number of $e_{i,j}$ divisible by A_i is at most $n/k + n/\ell + n/m$. So if $(k, \ell, m) = (\infty, 2, 3)$ and $d = |\,\mathrm{E}(B)\,|$, then $d \geq n + 2 + \delta - (n/\infty + n/2 + n/3)$ and hence

$$n \leq 6(d - 2 - \delta). \tag{3}$$

Our website [1] has a routine (similar to [3, Section 3]) to enumerate the necessary branching patterns.

3 Riemann existence theorem and (almost)-Belyi maps

Definition 3.1. *Two rational functions $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ are called* Möbius-equivalent *if $f = g \circ m$ for some $m \in \mathrm{Aut}(\mathbb{P}^1)$ (= the group of Möbius transformations $\{\frac{ax+b}{cx+d} \,|\, ad - bc \neq 0\}$).*

Definition 3.2. *[30] A list $[g_1, \ldots, g_k]$ of permutations in S_n is called a k-*constellation *if the group $\langle g_1, \ldots, g_k \rangle$ acts transitively on $\{1 \ldots n\}$ and $g_1 \cdots g_k = 1$. Here n is the* degree*, and $\langle g_1, \ldots, g_k \rangle$ is the* monodromy group *of $[g_1, \ldots, g_k]$.*

Definition 3.3. *$[g_1, \ldots, g_k]$ and $[h_1, \ldots, h_k]$ are* conjugated *if $\exists_{\tau \in S_n} \forall_i\ h_i = \tau^{-1} g_i \tau$. We denote the conjugacy class of $[g_1, \ldots, g_k]$ as $[g_1, \ldots, g_k]_\sim$.*

Theorem 3.4. Riemann Existence Theorem *(formulation from [22], for more see [23, 24, 25]). Let $p_1, \ldots, p_k$ be distinct points of $\mathbb{P}^1$. For any transitive representation $\rho : \pi_1(\mathbb{P}^1 \setminus \{p_1, \ldots, p_k\}) \to S_n$ there is a connected Riemann surface X and a proper holomorphic map $f : X \to \mathbb{P}^1$ of degree n which realizes ρ as its monodromy homomorphism. Moreover X and f are unique up to equivalence.*

Remark 3.5. *If $p_1, \ldots, p_k \in k \cup \{\infty\}$ for a subfield $k \subseteq \mathbb{C}$ then f can be defined over some algebraic extension of k (Cor. 7.10 in [23]). In Example 2.1, $p_1, \ldots, p_4 \in k \bigcup \{\infty\}$ where $k := \mathbb{Q}(t)$, while f is defined over $\mathbb{Q}(s)$, an algebraic extension of k (equation (2) shows $t \in \mathbb{Q}(s)$).*

If the branched set $\{p_1, \ldots, p_k\}$ is $\{0, 1, \infty\}$ then the pair X, f is called a *Belyi map*. The representation ρ is given by a k-constellation $[g_1, \ldots, g_k]$. We use this for Belyi and Belyi$^{(1)}$ maps in Sections 4 and 5, but not for Belyi$^{(2)}$ maps where we only have 2 cases (Section 6).

$$p_1, \ldots, p_k = \begin{cases} 0, 1, \infty & (k = 3, \text{ Belyi case}), \\ 0, 1, t, \infty & (k = 4 \text{ where } g_t \text{ is a 2-cycle, Belyi}^{(1)} \text{ case}). \end{cases}$$

We only use *planar* k-constellations, i.e. $X = \mathbb{P}^1$. Then $[g_1, \ldots, g_k]_\sim$ determines f up to Aut($\mathbb{P}^1$):

$$[g_1, \ldots, g_k]_\sim \iff f \text{ up to Möbius-equivalence.} \tag{4}$$

In the Belyi case $[g_1, \ldots, g_k]_\sim$ corresponds to a *dessin d'enfant* as well.

3.1 Dessins d'enfants

Definition 3.6. *A* dessin d'enfant *[29, 30] is a connected and oriented graph with black and white vertices, where any edge joins a black and a white vertex. The* degree *is the number of edges.*

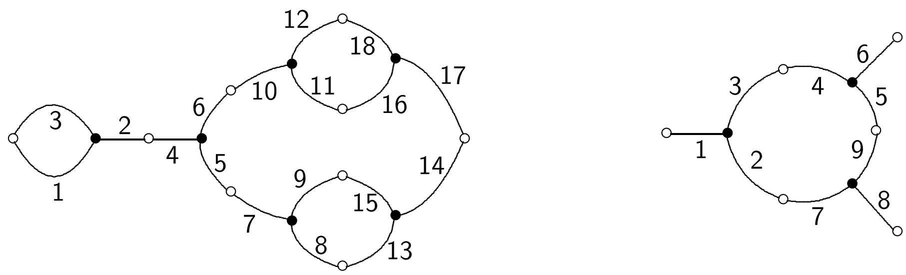


Figure 1: two planar dessins d'enfants (the labels are not part of the dessins d'enfants)

There is a one-to-one correspondence [4] between:

1. 3-constellations up to conjugation,
2. Belyi maps up to equivalence,
3. dessins d'enfants up to equivalence.

$1 \mapsto 2$: The Riemann existence theorem.

$2 \mapsto 3$: The dessin d'enfant of a Belyi map f is the graph $f^{-1}([0, 1])$ where: $f^{-1}(\{0\}) = \{$black vertices$\}$, $f^{-1}(\{1\}) = \{$white vertices$\}$, and $f^{-1}(\,(0, 1)\,) = \{$edges$\}$. Faces correspond 1-1 to $f^{-1}(\{\infty\})$.
Example: $f_1 = 4(x^6 - 4x^5 + 5x^2 + 4x + 4)^3/(27(x - 4)(5x^2 + 4x + 4)^2x^5)$ and $f_2 = 4(x^3 - 1)^3/(27x^3)$. Plotting $f_1^{-1}([0, 1])$ and $f_2^{-1}([0, 1])$ we find Figure 1 up to homeomorphism, without the labels [1].

$2 \mapsto 1$: The monodromy command in Maple computes a k-constellation for any algebraic function. Applying this (see Section 3.1 in [1]) to f_1 produces:
$g_0 = (1\ 5\ 3)(2\ 4\ 6)(7\ 9\ 11)(8\ 12\ 10)(13\ 14\ 15)(16\ 18\ 17)$
$g_1 = (1\ 4)(2\ 8)(3\ 7)(5\ 9)(6\ 12)(10\ 13)(11\ 14)(15\ 16)(17\ 18)$
$g_\infty = (1\ 2\ 10\ 15\ 17\ 16\ 14\ 9)(3\ 11\ 13\ 12\ 4)(5\ 7)(6\ 8)$.

$3 \mapsto 1$: After adding a label to each edge (the labels are not part of the dessin d'enfant itself) we can read the 3-constellation $[g_0, g_1, g_\infty]$ from the "labelled dessin" as follows. Reading labels counter-clockwise around each black vertex produces each cycle of g_0 (some of which may be 1-cycles, the valence of a vertex is the length of the corresponding cycle). Likewise, each white vertex corresponds to a cycle of g_1. There are two ways to find g_∞, one could compute it as $(g_0 g_1)^{-1}$, but one can also read g_∞ directly from the "labelled dessin"; each cycle of g_∞ is found by following the labels inside each face. From the first "labelled dessin" we read:

$h_0 = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18)$

$h_1 = (1\ 3)(2\ 4)(5\ 7)(6\ 10)(8\ 13)(9\ 15)(11\ 16)(12\ 18)(14\ 17)$

$h_\infty = (3)(1\ 2\ 6\ 12\ 17\ 13\ 7\ 4)(5\ 9\ 14\ 16\ 10)(8\ 15)(11\ 18)$.

Algorithm 4.3 in Section 4.3 can verify that the 3-constellations $[g_0, g_1, g_\infty]$ and $[h_0, h_1, h_\infty]$ are conjugated. Several algorithms in Section 4 use permutations in *expanded form*, which means the 1-cycles are written as well. For example, the 3-constellation of the second "labelled dessin" is:

$[(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9),\ (1)(6)(8)(2\ 7)(3\ 4)(5\ 9),\ (1\ 7\ 8\ 5\ 6\ 3)(2\ 4\ 9)]$.

Remark 3.7. *If D is the dessin of a Belyi map f with branching pattern B, then $|\,E(f)\,|$, $|\,E(B)\,|$, $|\,E(D)\,|$ denote the number of exceptional: points of f resp. branchings in B resp. cycle-lengths in D. Since these numbers are equal, we will also use the shorter notation $|E|$ if f, B, or D is clear from the context.*

4 Belyi maps

The goal in this section is Algorithm 4.4 which can compute all dessins with $|\,\mathsf{E}\,| = 5$, which is the key step to proving that all Belyi maps f with $|\,\mathsf{E}\,| = 5$ appear in our table.

4.1 Computing 3-constellations

Definition 4.1. *Let $g \in S_n$. Then g' denotes an element of S_{n-1} defined as follows: for $i \in \{1 \ldots n-1\}$ define $g'(i)$ as $g(i)$ if $g(i) \neq n$, and $g(g(i))$ if $g(i) = n$. When g is written in disjoint cycle notation, one obtains $g' \in S_{n-1}$ by simply erasing n.*

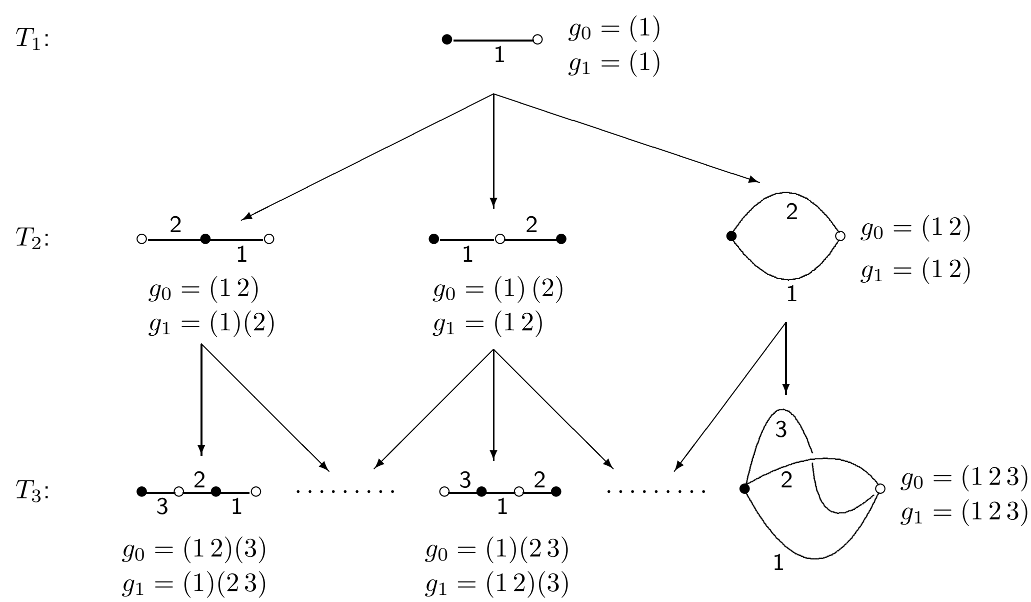


Figure 2: Computing repeat-transitive 3-constellations (definition 4.2)

Definition 4.2. *A pair (g, h) where $g, h \in S_n$ is called repeat-transitive when $n = 1$, or, when $< g, h >$ is a transitive subgroup of S_n and (g', h') is repeat-transitive in S_{n-1}.*

Figure 2 shows (for $N = 3$) how one can compute all repeat-transitive 3-constellations of degrees $1, 2, \ldots, N$ recursively. Start with the 3-constellation of degree 1. Then, given the set T_{n-1} of repeat-transitive 3-constellations of degree $n-1$, insert one more edge in all possible ways to obtain all repeat-transitive 3-constellations of degree n. Algorithm 4.1 below shows how to implement this. It represents 3-constellations as $[g_0, g_1]$ (since g_∞ can be recovered from $g_0 g_1 g_\infty = 1$) with $g_0, g_1 \in S_{n-1}$ written in *expanded form*, i.e., including 1-cycles. Inserting an edge means doing the following to both g_0 and g_1:

(i) insert the new number n into an existing cycle, or

(ii) add a new 1-cycle (n).

One can not choose (ii) for both g_0 and g_1, because the resulting pair would not be transitive (the graph would not be connected). This leaves $n^2 - 1$ ways in Step 2 of Algorithm 4.1 to add an edge to $[g_0, g_1]$. Indeed, $|T_2| = 3 = (2^2 - 1) \cdot |T_1|$ in Figure 2. In Step 2, the call $\texttt{Insert}(g_0, i, n)$ (with $g_0 \in S_{n-1}$ and $i \in \{1, \ldots, n\}$) inserts edge $\#n$ at the i^{th} position, as shown here:

Example 4.3. *Let $g_0 = (1\,2)(4\,5)(6\,8) \in S_8$. The program call $\texttt{Insert}(g_0, 6, 9)$ computes:*
Step 1: *Write g_0 in expanded form (including 1-cycles) so that all edges 1–8 appear. Placeholders (asterisks) indicate all 9 possible positions in g_0 where 9 can be inserted:*

$$g_0 = (3\,*)(7\,*)(1 * 2\,*)(4 * 5\,*)(6 * 8\,*)(*)$$

Step 2: *Insert 9 in the 6^{th} placeholder:*

$$\texttt{Insert}(g_0, 6, 9) := (3)(7)(1\,2)(4\,5\,9)(6\,8) \quad \text{(written in expanded form).}$$

Algorithm 4.1: Compute all repeat-transitive 3-constellations of degree $\leq N$.
Step 1: $T_1 := \{[(1), (1)]\}$
Step 2: For n from 2 to N do:
$T_n := \{[\texttt{Insert}(g_0, i, n), \texttt{Insert}(g_1, j, n)] \mid i, j \in \{1 \ldots n\},\ (i, j) \neq (n, n),\ [g_0, g_1] \in T_{n-1}\}$.

From $|T_1| = 1$ and $|T_n| = (n^2 - 1) \cdot |T_{n-1}|$ one finds

$$|T_n| = (n-1)!(n+1)!/2 = 1,\ 3,\ 24,\ 360,\ 8640,\ 302400,\ 14515200,\ 914457600,\ \ldots$$

Remark 4.4. *There are twenty-six 3-constellations of degree 3. Two of them are not repeat-transitive:*

So the set T_3 computed by Algorithm 4.1 has twenty-four 3-constellations.

The construction in Figure 2 is complete up to re-labeling (e.g., compare the two 3-constellations from Remark 4.4 with T_3 in Figure 2). So Algorithm 4.1 does find all 3-constellations up to conjugation. Dessins with $|\,\mathsf{E}\,| = 5$ have degrees ≤ 18, see inequality (3). To find them, we need to implement several improvements because T_{18} is much too large for the computer.

4.2 Discarding unnecessary 3-constellations

Let $\#g_0$ denote the number of cycles in g_0, including 1-cycles. For a 3-constellation $[g_0, g_1, g_\infty]$ of degree n, the genus g of the corresponding dessin d'enfant is given by Euler's formula:

$$2 - 2g = \#\text{vertices} - \#\text{edges} + \#\text{faces} = \#g_0 + \#g_1 - n + \#g_\infty.$$

Our aim is rational functions, which correspond to planar (i.e. $g = 0$) 3-constellations. Adding edges to a non-planar dessin d'enfant can not make it planar, so we may discard non-planar 3-constellations in Algorithm 4.1 as soon as they occur. This reduces the growth of T_n but more improvements are needed since it still grows much too fast.

The goal is $|\mathsf{E}| = 5$, however, we can not simply discard 3-constellations with $|\mathsf{E}| > 5$ as soon as they occur, because adding an edge can lower the value of $|\mathsf{E}|$. We solve this problem with *weighted* counts.

Definition 4.5. *Notations as in Section 2.1, Definition 2.3.*
If $A_i|e_{i,j}$ *then* $s_{i,j} := 0$*, if* $e_{i,j} \equiv -1 \bmod A_i$ *then* $s_{i,j} := \frac{1}{2}$*, otherwise* $s_{i,j} := 1$ *(if* $A_i = \infty$ *then* $s_{i,j} = 1$*).*
The weighted-count of B *is the sum of the* $s_{i,j}$*.*

If we replace $\frac{1}{2}$ in Definition 4.5 by 1, then we get $|\mathsf{E}|$ from Definition 2.3. Thus,

$$|\mathsf{E}| \geq \text{weighted-count} \tag{5}$$

Proposition 4.6. *Adding an edge does not decrease the weighted-count if the dessin d'enfant stays planar and* $A_3 = \infty$*.*

Proof: Let S_i be the sum of the $s_{i,j}$, and let $\tilde{S}_i$ be the sum after adding one edge. Then $\tilde{S}_1 = S_1 + 1$ if the number of black vertices increased, otherwise, $\tilde{S}_1 \geq S_1 - \frac{1}{2}$. Likewise, $\tilde{S}_2 = S_2 + 1$ if the number of white vertices increased, otherwise, $\tilde{S}_2 \geq S_2 - \frac{1}{2}$. The number of faces is S_3 if $A_3 = \infty$, and does not decrease when adding an edge ($\tilde{S}_3 \geq S_3$) if the result remained planar. Now $\tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 \geq S_1 + S_2 + S_3$ because if no black or white vertices were added, then adding an edge increases the number of faces. □

We now switch from $(\infty, 2, 3)$ to $(3, 2, \infty)$, an easily reversible transformation (for Belyi maps it means $f \mapsto 1/f$). Then we may discard 3-constellations with weighted-count > 5 in Algorithm 4.1 as soon as they occur; adding edges can not lead to $|\mathsf{E}| \leq 5$ by Proposition 4.6 and inequality (5). This drastically reduces the growth of T_n, but a problem still remains, which we handle next.

4.3 Finding a unique representative of a conjugacy class

The 3-constellations $[(1\,2)(3), (1)(2\,3)]$ and $[(1)(2\,3), (1\,2)(3)]$ in Figure 2 are conjugated. We should remove all but one constellation in each conjugacy class, not only because this gives an another drastic reduction in the growth of T_n, but also because we need 3-constellations *up to conjugacy* for the correspondence from Section 3. For $\tau \in S_n$, denote $g^\tau := \tau^{-1} g \tau$.

Algorithm 4.2: Sort With Base point (SWB)
Input: Transitive $g_0 \dots g_s$ in S_n and a base point $b \in \{1, \dots, n\}$.
Output: $[g_0^\tau \dots g_s^\tau]$ for some $\tau \in S_n$ with the **property**: $[g_0 \dots g_s]$ is conjugated to $[h_0 \dots h_s]$ if and only if $\{\mathtt{SWB}(g_0 \dots g_s, b) \,|\, 1 \leq b \leq n\} = \{\mathtt{SWB}(h_0 \dots h_s, b) \,|\, 1 \leq b \leq n\}$.
Step 1: $\pi_1 := b$.
Step 2: For k from 1 to $n-1$ let $\pi_{k+1} := g_i(\pi_l)$ where (i, l) is the first pair in $\{0 \dots s\} \times \{1 \dots k\}$ with $g_i(\pi_l) \notin \{\pi_1, \dots, \pi_k\}$.
Step 3: Let $\tau \in S_n$ with $\tau(i) = \pi_i$ and return $[g_0^\tau \dots g_s^\tau]$.

Verifying that $\mathtt{SWB}(g_0 \dots g_s, b) = \mathtt{SWB}(g_0^\tau \dots g_s^\tau, \tau^{-1}(b))$ for $\tau \in S_n$ proves the claimed property.

Algorithm 4.3: UniqueRepresentative
Input: Transitive $g_0 \dots g_s$ in S_n.
Output: A unique representative in the S_n-conjugacy class of $[g_0 \dots g_s]$.
Step 1: $S := \{\mathtt{SWB}(g_0 \dots g_s, b) \,|\, 1 \leq b \leq n\}$.
Step 2: Return the first (we use a lexicographic ordering) element of S.

4.4 Computing dessins to prove that the Belyi table is complete

From here on, the phrase "dessin" is short for "conjugacy class of 3-constellations" represented by the output of Algorithm 4.3. The bound $6(d-2)$ comes from Equation (3).

Algorithm 4.4: Compute all planar dessins with $|E| = d$
Step 1: $T_1 := \{[(1),(1)]\}$
Step 2: For n from 2 to $6(d-2)$ do:
$T_n := \{[\texttt{Insert}(g_0,i,n), \texttt{Insert}(g_1,j,n)] \mid i,j \in \{1\ldots n\},\ (i,j) \neq (n,n),\ [g_0,g_1] \in T_{n-1}\}$
$T_n := \{[g_0,g_1] \in T_n \mid [g_0,g_1,(g_0g_1)^{-1}]$ is planar and has weighted-count $\leq d\}$
$T_n := \{\text{UniqueRepresentative}(g_0,g_1) \mid [g_0,g_1] \in T_n\}$.
Step 3: Return $\{[g_0,g_1] \mid n \leq 6(d-2) \text{ and } [g_0,g_1] \in T_n$ with $|\,\mathsf{E}\,| = d\}$.

Algorithm 4.4 produces the following dessins for $d \in \{4,5\}$:

d	Number of planar dessins with $(3,2,\infty)$-count d.
4	0, 1, 3, 5, 3, 10, 4, 6, 4, 4, 0, 6
5	0, 0, 2, 10, 18, 40, 50, 71, 76, 103, 36, 108, 40, 42, 32, 32, 0, 26

Another way to generate maps, using parenthesis systems, was given in [28]. Although we are mainly interested in $d = 5$, we ran Algorithm 4.4 for $d \leq 7$, for both definitions (see Section 1.2). The entries of degree $n = 6(d-2)$ form sequence $2, 6, 26, 191, 1904, \ldots$ (`www.oeis.org/A112948`). Beukers and Montanus [4] computed the dessins and Belyi maps for $d = 6$, $n = 24$ with a combination of machine and hand computation, but at the time they missed one of the 191 dessins. To avoid the likelihood of a gap in a large table, it is important to verify it with machine-only computation.

Proving Completeness for the Belyi table: To prove that our website [1] lists all rational Belyi maps with $|\,\mathsf{E}\,| = 5$, it is not enough to check that its number of functions of degree n matches row $d = 5$ in the above table. That would leave open the possibility of Möbius-equivalent (Definition 3.1) duplicates while missing other ones. So we implemented a more rigorous check [1]. It computes the dessin for each f in our table by applying $2 \mapsto 1$ from Section 3.1, followed by Algorithm 4.3. Completeness is then proven by comparing these dessins with the independently-computed set of dessins from Algorithm 4.4.

Invariants offer a much faster way to prove completeness of our Belyi table, without the time-consuming computation $2 \mapsto 1$ from Section 3.1. Each time a pair $f_1 \neq f_2$ in our Belyi table had the same branching pattern, it turned out that their five point invariants are not equal, i.e; $I_5(f_1) \neq I_5(f_2)$. More details about these invariants are given in Section 7. This proves that the table has no Möbius-equivalent duplicates. To prove completeness it now suffices to compare (for each branching pattern) the number of Belyi maps in the table with the number of dessins from Algorithm 4.4. □

Our $(\infty,2,3)$-Belyi table [1] has 255+9+99 = 363 entries representing 411+9+266 = 686 functions $F_1^B, \ldots, F_{686}^B \in \mathbb{C}(x)$ (If $f \in \mathbb{Q}(\alpha)(x)$ with $[\mathbb{Q}(\alpha):\mathbb{Q}] = d$, then it represents d elements of $\mathbb{C}(x)$, one for each of the d complex roots of the minimal polynomial of α.) Their dessins are precisely the 686 dessins produced by Algorithm 4.4 (there are no obstruction issues as in [3, Section 6]).

5 Belyi$^{(1)}$ maps

We consider planar 4-constellations $[g_0, g_1, g_t, g_\infty]$ where g_t is a 2-cycle. The phrase "almost-dessin" in this section refers to: conjugacy class of such a 4-constellation, represented by the output of Algorithm 4.3. Almost-dessins corresponds to Belyi$^{(1)}$ maps up to Möbius-equivalence, see (4) in Section 3.

5.1 Finding almost-dessins

Suppose for example we want to find all (up to conjugation) planar 4-constellations $[g_0, g_1, g_t, g_\infty]$ where the cycle structures of g_1 and g_∞ are (2^6) and (3^4) (g_t is always a 2-cycle). Up to conjugacy we may assume that $g_\infty = (1\,2\,3)(4\,5\,6)(7\,8\,9)(10\ 11\ 12)$. The number of elements of S_{12} of type (2^6) is 10395, and the number of 2-cycles is 66. One could, for all 10395×66 combinations of (g_1, g_t), compute $g_0 = (g_1 g_t g_\infty)^{-1}$, check if $[g_0, g_1, g_t, g_\infty]$ is transitive and planar, and if so, apply Algorithm 4.3. This works fine, but it can easily be sped up.

Since g_t is a 2-cycle and $<g_1, g_t, g_\infty>$ should be transitive, it follows that $<g_1, g_\infty>$ may have at most two orbits in $\{1 \ldots 12\}$. So g_1 must connect some of the g_∞-orbits $\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}$. Up to conjugation, we may assume g_1 connects the first two orbits with the 2-cycle $(1\,4)$ (we may also assume that g_1 contains either $(2\,7)$ or $(7\ 10)$ since g_1 must connect more than one pair of g_∞-orbits). This way all almost-dessins (all 4-constellations up to conjugation) with such branching patterns can be found with little CPU time.

5.2 Braid orbits

The braid group, generated by the braids $\sigma_1, \ldots, \sigma_{k-1}$, acts on k-constellations in the following way:

$$\sigma_i : [g_1 \ldots g_k] \mapsto [g_1 \ldots g_{i-1},\ g_{i+1},\ g_{i+1}^{-1}\, g_i\, g_{i+1},\ g_{i+2} \ldots g_k].$$

The points p_i, p_{i+1} are swapped by σ_i with a half-rotation. We will use orbits under the *pure braid group* (Def. 9.11 in [23]) which consists of products of σ_i's that return $p_1, \ldots, p_n$ to their original locations. The diagram in Figure 3, taken from Section 1 in [26], illustrates σ_1^2. An algorithm is given in [27] for computing braid orbits of k-constellations $[g_1, \ldots, g_k]$. Combining this with Algorithm 4.3 we obtain an algorithm that computes braid orbits of almost-dessins.

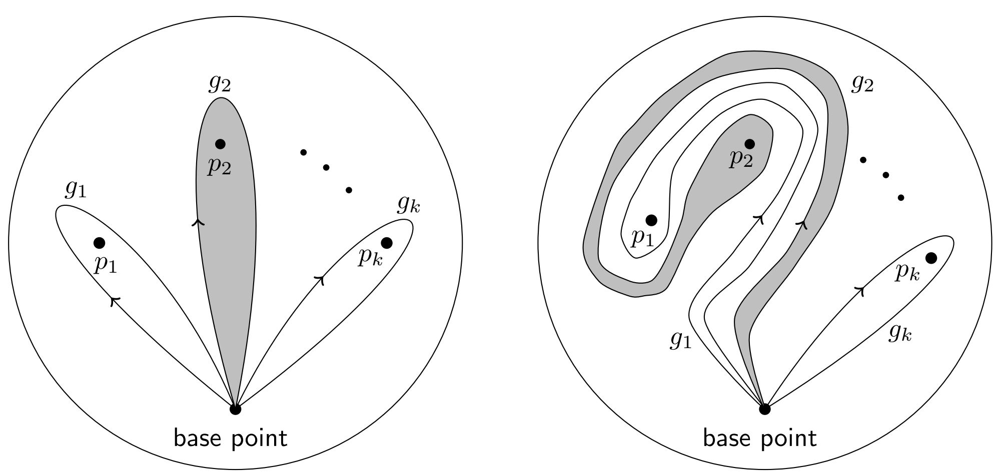


Figure 3: Action of $\sigma_1^2 : [g_1 \ldots g_k] \mapsto [g_1^\tau, g_2^\tau,\ g_3 \ldots g_k]$ (here $\tau = g_1 g_2$).

Our implementation [1] automatically generates Table 2 below. First, it computes all planar Belyi$^{(1)}$ branching patterns with $|\,E\,| = 5$ with the algorithm mentioned in Section 2.1. Next, it computes all almost-dessins for these branching patterns as in Section 5.1. These are grouped into braid orbits labeled $N_1, \ldots, N_{68}$. Here N_{66}, N_{67}, N_{68} are inside Belyi$^{(2)}$ families. The table shows the length (number of almost-dessins) of each orbit and its branching pattern. Most branching patterns have one orbit, but some have zero or two.

n	branching pattern	name	$\lvert\mathcal{O}\rvert$	decomp.	n	branching pattern	name	$\lvert\mathcal{O}\rvert$	decomp.
2	(2), (1^2), (1^2)	N_1	1		8	$(1^2,2,4)$, (2^4), $(2,3^2)$	N_{38}	9	
	(1^2), (1^2), (2)	N_2	1			$(1^2,3^2)$, (2^4), $(2,3^2)$	N_{39}	4	$4\circ 2$
3	$(1,2)$, (1^3), (3)	N_3	1			$(1,2^2,3)$, (2^4), $(2,3^2)$	N_{40}	9	
	(3), $(1,2)$, (1^3)	N_4	1			(2^4), (2^4), $(2,3^2)$	—	—	
	$(1,2)$, $(1,2)$, $(1,2)$	N_5	4		9	$(1^3,6)$, $(1,2^4)$, (3^3)	N_{41}	3	$3\circ 3$
4	$(1,3)$, $(1^2,2)$, $(1,3)$	N_6	6			idem	N_{42}	9	
	(2^2), $(1^2,2)$, $(1,3)$	N_7	3			$(1^2,2,5)$, $(1,2^4)$, (3^3)	N_{43}	18	
	(4), (2^2), (1^4)	N_8	1	$2\circ 2$		$(1^2,3,4)$, $(1,2^4)$, (3^3)	N_{44}	15	
	$(1,3)$, (2^2), $(1^2,2)$	N_9	3			$(1,2^2,4)$, $(1,2^4)$, (3^3)	N_{45}	12	$3\circ 3$
	(2^2), (2^2), $(1^2,2)$	N_{10}	2	$2\circ 2$		$(1,2,3^2)$, $(1,2^4)$, (3^3)	N_{46}	12	
	$(1^2,2)$, (2^2), (2^2)	N_{11}	2	$2\circ 2$		$(2^3,3)$, $(1,2^4)$, (3^3)	N_{47}	3	$3\circ 3$
	(1^4), (2^2), (4)	N_{12}	1	$2\circ 2$	10	$(1^3,7)$, (2^5), $(1,3^3)$	N_{48}	15	
5	$(1,4)$, $(1,2^2)$, $(1^2,3)$	N_{13}	10			$(1^2,2,6)$, (2^5), $(1,3^3)$	N_{49}	15	
	$(2,3)$, $(1,2^2)$, $(1^2,3)$	N_{14}	7			$(1^2,3,5)$, (2^5), $(1,3^3)$	N_{50}	15	
	$(1^2,3)$, $(1,2^2)$, $(2,3)$	N_{15}	7			$(1^2,4^2)$, (2^5), $(1,3^3)$	N_{51}	12	
	$(1,2^2)$, $(1,2^2)$, $(2,3)$	N_{16}	6			$(1,2^2,5)$, (2^5), $(1,3^3)$	N_{52}	15	
6	$(1^2,4)$, $(1^2,2^2)$, (3^2)	N_{17}	3	$3\circ 2$		$(1,2,3,4)$, (2^5), $(1,3^3)$	N_{53}	18	
	idem	N_{18}	6			$(1,3^3)$, (2^5), $(1,3^3)$	—	—	
	$(1,2,3)$, $(1^2,2^2)$, (3^2)	N_{19}	12			$(2^3,4)$, (2^5), $(1,3^3)$	—	—	
	(2^3), $(1^2,2^2)$, (3^2)	N_{20}	3	$3\circ 2$		$(2^2,3^2)$, (2^5), $(1,3^3)$	N_{54}	6	
	$(1,5)$, (2^3), $(1^3,3)$	N_{21}	5		12	$(1^4,8)$, (2^6), (3^4)	N_{55}	3	$3\circ 2\circ 2$
	$(2,4)$, (2^3), $(1^3,3)$	N_{22}	2	$2\circ 3$		idem	N_{56}	4	
	(3^2), (2^3), $(1^3,3)$	N_{23}	2			$(1^3,2,7)$, (2^6), (3^4)	N_{57}	7	
	$(1^2,4)$, (2^3), $(1,2,3)$	N_{24}	9			$(1^3,3,6)$, (2^6), (3^4)	N_{58}	4	$4\circ 3$
	$(1,2,3)$, (2^3), $(1,2,3)$	N_{25}	10			$(1^3,4,5)$, (2^6), (3^4)	N_{59}	10	
	(2^3), (2^3), $(1,2,3)$	N_{26}	2	$2\circ 3$		$(1^2,2^2,6)$, (2^6), (3^4)	N_{60}	9	$3\circ 4$
7	$(1^2,5)$, $(1,2^3)$, $(1,3^2)$	N_{27}	21			$(1^2,2,3,5)$, (2^6), (3^4)	N_{61}	15	
	$(1,2,4)$, $(1,2^3)$, $(1,3^2)$	N_{28}	24			$(1^2,2,4^2)$, (2^6), (3^4)	N_{62}	6	$3\circ 2\circ 2$
	$(1,3^2)$, $(1,2^3)$, $(1,3^2)$	N_{29}	12			$(1^2,3^2,4)$, (2^6), (3^4)	—	—	
	$(2^2,3)$, $(1,2^3)$, $(1,3^2)$	N_{30}	9			$(1,2^3,5)$, (2^6), (3^4)	—	—	
8	$(1^2,6)$, (2^4), $(1^2,3^2)$	N_{31}	4	$4\circ 2$		$(1,2^2,3,4)$, (2^6), (3^4)	N_{63}	9	$3\circ 4$
	idem	N_{32}	12			$(1,2,3^3)$, (2^6), (3^4)	N_{64}	4	$4\circ 3$
	$(1,2,5)$, (2^4), $(1^2,3^2)$	N_{33}	10			$(2^4,4)$, (2^6), (3^4)	N_{65}	3	$S_3\circ 2$
	$(1,3,4)$, (2^4), $(1^2,3^2)$	N_{34}	15			$(2^3,3^2)$, (2^6), (3^4)	—	—	
	$(2^2,4)$, (2^4), $(1^2,3^2)$	N_{35}	6	$2\circ 4$	4	(1^4), (4), $(1,3)$	N_{66}	1	
	$(2,3^2)$, (2^4), $(1^2,3^2)$	N_{36}	4	$4\circ 2$	6	$(1^4,2)$, $(2,4)$, (3^2)	N_{67}	4	
	$(1^3,5)$, (2^4), $(2,3^2)$	N_{37}	6			$(1^4,2)$, (2^3), (6)	N_{68}	2	$2\circ 3$

Table 2: Braid orbits of the almost-dessins with $|\,E\,| = 5$

Column $|\mathcal{O}|$ in the above table gives the number of almost-dessins in each braid orbit. The complete table of almost-dessins themselves is given in [1]. The notation $3\circ 4$ means that any Belyi$^{(1)}$ map f for this orbit equals $g\circ h$ for some g,h of degrees 3,4. The notation $S_3\circ 2$ means $f = g\circ h$ where g has three $3\circ 2$-decompositions and one $2\circ 3$-decomposition. We do not need explicit $f\in\mathbb{C}(x)$ in order to find any of the information listed in Table 2, including the decomposition structure of f (the almost-dessins $[g_0,g_1,g_t,g_\infty]$ suffice). Decompositions of f correspond to subfields $\mathbb{C}(f)\subseteq E\subseteq\mathbb{C}(x)$, which in turn correspond to subgroups of $G := \langle g_0,g_1,g_t,g_\infty\rangle$ that contain $\mathrm{Stab}(1)=\{g\in G \mid g(1)=1\}$.

The sections below can use Table 2 to prove that our database covers all Belyi$^{(1)}$ maps, as everything in Table 2 was computed independently of these functions.

5.3 Continuation of Example 2.1

Let B be the third branching pattern under $n = 10$ in Table 2. Example 2.1 gave a Belyi$^{(1)}$ map $f(s,x)$ for B. The table shows that B has 15 distinct almost-dessins, in one braid orbit named N_{50}. Let $\phi_f(s) \in \mathbb{Q}(s)$ be the rational function of degree 15 in Equation (2), as in Definition 2.2. Choose any $t_0 \in \mathbb{P} - \{0, 1, \infty\}$, and let $S := \phi_f^{-1}(\{t_0\}) \subset \mathbb{P}^1$. If $\alpha \in S$, then $f(\alpha, x)$ is a Belyi$^{(1)}$ map for B that ramifies only above $\{0, 1, t_0, \infty\}$, *assuming f's family is gap-free* as in Definition 2.2. One could compute ($2 \mapsto 1$ in Section 3.1) the almost-dessin D_α of $f(\alpha, x)$ for each $\alpha \in S$, then take $D_* := \{D_\alpha | \alpha \in S\}$, and check that $N_{50} = D_*$. However, it is not hard to see that this check is not necessary for this B.

Let γ be a loop in $\mathbb{P} - \{0, 1, \infty\}$ with base point t_0. Applying analytic continuation to $\phi_f^{-1}(\{t\})$, with t following γ, gives a map from S to S. This gives an action of the fundamental group $\pi_1(\mathbb{P} - \{0, 1, \infty\}, t_0)$ on S. Since D_* is an image of S, the fundamental group acts on D_* as well. Figure 3 illustrates how this corresponds to an action of the pure braid group.

Table 2 implies $D_* \subseteq N_{50}$ because according to Table 2, all 15 almost-dessins for B are in N_{50}. Then $D_* = N_{50}$ because the pure braid group acts on D_* and N_{50} is an orbit.

Proposition 5.1. *Let f and B be as above. If $g \in \mathbb{C}(x)$ has branching pattern B then it is Möbius-equivalent to a member of f's family.*

Proof: Let t_0 be the branch point of g not in $\{0, 1, \infty\}$. $D_* = N_{50}$ (we checked that f's family is gap-free [1]). The almost-dessin of g has branching pattern B, is thus in N_{50} and hence equals D_α for some $\alpha \in S$. Then $f(\alpha, x)$ is Möbius-equivalent to g, see correspondence (4) in Section 3. □

5.4 A branching pattern with two orbits

Let

$$f_1 = \frac{3\,s(x-1)x^2 + 4}{4\,(s(x-1)x^2+1)^3} \quad \text{and} \quad f_2 = -\frac{s^2\,((4s-3)x^3 + 6\,(s-1)x^2 + 3\,(3s^2-2s-1)x - 4s))}{4\,(x^3+2x^2+(2s+1)\,x+s)^3}.$$

Both are gap-free, have branching $(1^3, 6), (1, 2^4), (3^3)$ above $0, 1, \infty$ and one more branch point $t = \phi_{f_1}(s)$ and $t = \phi_{f_2}(s)$ respectively. The degree of ϕ_{f_1} is 3. This, combined with argument from Section 5.3, suffices to prove that f_1 covers N_{41} in Table 2. However, the fact that ϕ_{f_2} has degree 9 is not enough to demonstrate that f_2 covers N_{42} because, in the notations from Section 5.3, the cardinality of $\{D_\alpha | \alpha \in S\}$ could be less than the cardinality of S.

Definition 5.2. *A Belyi$^{(1)}$ map $f \in \mathbb{C}(s)(x)$ is called* duplicate-free *if $\{f(\alpha, x) \,|\, \alpha \in \phi_f^{-1}(\{t_0\})\}$ has $\deg_s(\phi_f)$ distinct almost-dessins for any $t_0 \notin \{0, 1, \infty\}$.*

After verifying that f_2 is duplicate-free we may conclude that it covers N_{42}, since it is the only orbit for this branching pattern of length 9.

Remark 5.3. *If $f \in \mathbb{C}(s)(x)$ is a duplicate-free Belyi$^{(1)}$ map then $\phi_f \in \mathbb{C}(s)$ is a Belyi map, and its dessin can be computed directly from a 4-constellation $[g_0, g_1, g_t, g_\infty]$ of f.*

Proof: Definition 5.2 immediately implies $|\phi_f^{-1}(\{t_0\})| \geq \deg_s(\phi_f)$ for any $t_0 \notin \{0, 1, \infty\}$, in other words, ϕ_f is a Belyi map. Take braid actions that correspond to looping t around $0, 1, \infty$. Let h_0, h_1, h_∞ be the corresponding permutations of the almost-dessins, then $[h_0, h_1, h_\infty]_\sim$ is the dessin of ϕ_f. □

It was fortunate these dessins were always planar in our database, otherwise our Belyi$^{(1)}$ maps could not have been in $\mathbb{Q}(s)(x)$, complicating the algorithms.

5.5 Proving completeness of our table of Belyi$^{(1)}$ maps

To our surprise, Section 2 often produced Belyi$^{(1)}$ maps f that were not duplicate-free, where the degree of ϕ_f was twice the number of distinct almost-dessins. For such cases, we computed automorphisms $\tau \in \mathrm{Aut}(\mathbb{Q}(s))$ of order 2 for which $\tau(\phi_f) = \phi_f$, in order to find τ for which $\tau(f)$ is Möbius-equivalent to f. Let $\tilde{s}$ be a generator of the subfield of $\mathbb{Q}(s)$ fixed by τ.

We write ϕ_f as element of $\mathbb{Q}(\tilde{s})$ and use it to search for a $\tilde{f}(s,x)$ for which $\tilde{f}(\tilde{s},x)$ is Möbius-equivalent to f. Then $\phi_{\tilde{f}}$ has half the degree of ϕ_f. This way, we managed to make every member of our Belyi$^{(1)}$ table duplicate-free. After suitable Möbius transformations, we managed to make them gap-free as well. The arguments of the previous two subsections now suffice to prove that our Belyi$^{(1)}$ table [1] is complete. But we implemented a more direct verification as well:

Let $F_1^{(1)} \ldots F_{68}^{(1)}$ be the explicit Belyi$^{(1)}$ maps at [1]. For each i we check that $F_i^{(1)}$ is gap-free, compute almost-dessin for $F_i^{(1)}$ and check that it is in N_i. This suffices to prove that, up to Möbius-equivalence, the families of $F_1^{(1)} \ldots F_{68}^{(1)}$ contain all rational Belyi$^{(1)}$ maps with $|\,E\,| = 5$. We also compute the degree of $\phi_{F_i^{(1)}}$ and check that it equals $|\mathcal{O}|$ which denotes the number of elements of the braid orbit.

As for the Belyi case, we also implemented a faster approach, based on five point invariants, to prove the completeness of the Belyi$^{(1)}$ tables. Here we used not one, but two algebraically independent five point invariants I_5 and $\tilde{I}_5$. Section 7 gives more details about these invariants. For each f in the Belyi$^{(1)}$ table, $I_5(f)$ and $\tilde{I}_5(f)$ are both in $\mathbb{Q}(s)$, and thus satisfy an algebraic relation. If two Belyi$^{(1)}$ maps give distinct algebraic relations, then they can not be part of the same family. This turned out to be the case for any pair in our table with the same branching pattern.

5.6 Belyi maps inside Belyi$^{(1)}$ families

The family of $F_i^{(1)}$ contains, up to Möbius-equivalence, all Belyi$^{(1)}$ maps with almost-dessin in N_i. But it often contains Belyi maps as well; if s_0 is special (Definition 2.2) then $F_i^{(1)}(s_0, x)$ is a Belyi map. Depending on whether $\phi_{F_i^{(1)}}(s_0)$ is 0, 1, or ∞, the dessin of this Belyi map is $[g_0 g_t^{\tau},\, g_1, g_\infty]_\sim$ (where $\tau = g_1^{-1}$), $[g_0,\, g_1 g_t,\, g_\infty]_\sim$ or $[g_0, g_1,\, g_t g_\infty]_\sim$. All dessins for which $|E|$ is larger in the definition from Remark 1.4 than in our definition 1.2 can be obtained this way. So one would expect that the families of $F_1^{(1)} \ldots F_{68}^{(1)}$ contain a Belyi maps for each of those dessins. However, there are a few exceptions; some dessins that can be obtained this way from N_i do not *directly* appear in $F_i^{(1)}$'s family because they correspond to degenerate values of s. They do appear *indirectly*, i.e. in another, less favorable, parametrization of $F_i^{(1)}$:

Example 5.4. *Let* $f := -27(sx^4 - 2sx^3 + sx^2 + 1)^2/(sx^4 - 2sx^3 + sx^2 - 3)^3$. *Let* $\tilde{s} := \sqrt[4]{s}$ *and* $\tilde{f} := f(s, x/\tilde{s})$. *The point* $\tilde{s} = 0$ *is degenerate for* f *but special for* $\tilde{f}$, *where it evaluates to a Belyi map* $g = -27(x^4+1)^2/(x^4-3)^3$. *Although* f*'s family is a proper subset of* $\tilde{f}$*'s family, we prefer* f *because it is duplicate-free.*

6 Belyi$^{(2)}$ maps

Definition 6.1. *Let* S *be a subset of* $\mathbb{P}^1$ *with* n *elements. The* n-point-polynomial $P_S \in \mathbb{C}[x]$ *is the product of* $x - p$ *taken over all* $p \in S - \{\infty\}$. *It has degree* $n-1$ *if* $\infty \in S$ *and degree* n *otherwise. Let* k *be a subfield of* $\mathbb{C}$. *We say* S *is* defined over k *if* $P_S \in k[x]$.

If $f \in k(x)$ then its set of (k, ℓ, m)-exceptional points is defined over k. Let

$$F_4^{(2)}(a,b,c,d,x) = 1 - \frac{(x^2+ax+b)^2}{c(x+d)^3}, \quad F_6^{(2)}(a,b,c,d,x) := 1 - \frac{(x^3+3ax^2+bx+c)^2}{(x^2+2ax+d)^3}.$$

Their branching patterns are $B_4 = (1^4), (2^2), (1, 3)$ and $B_6 = (1^4, 2), (2^3), (3^2)$. Both are two-dimensional families up to Möbius-equivalence (two of the 4 parameters a, b, c, d can be eliminated with a linear transformation on x).

Lemma 6.2. *Let k be a subfield of $\mathbb{C}$ and $f \in k(x)$ a Belyi$^{(2)}$ map with $|\,E\,| = 5$. Then f has branching pattern B_4 or B_6, and there exist unique $m \in \{1/(x-p)\,|\,p \in k\} \bigcup \{x\}$ and $a, b, c, d \in k$ such that f equals $F_4^{(2)}(a, b, c, d, m)$ if f has B_4, and $F_6^{(2)}(a, b, c, d, m)$ if f has B_6.*

Proof: Our implementation mentioned in Section 2.1 shows (it is also easy to show by hand) that B_4 and B_6 are the only planar Belyi$^{(2)}$ branching patterns with $|\,E\,| = 5$. If f has B_6, then $(1^4, 2)$ indicates that it has a unique root p of order 2, and four roots of order 1. The part (2^3) of B_6 indicates that numerator of $1 - f$ must be a square, while (3^2) indicates that the denominator is a cube. If $p = \infty$ then $f(\infty) = 0$ with multiplicity 2, which implies that the numerator and denominator of $1 - f$ must have the same degree, same leading coefficient, and the same x^5-coefficient as well. Then f must equal $F_6^{(2)}(a, b, c, d, x)$ for some a, b, c, d, uniquely determined by f, and hence in k. If $p \neq \infty$, the Möbius-transformation m moves p to ∞, after which the same argument applies.

If f has B_4, then let p be the unique pole of order 1. If $p = \infty$, then the denominator of f must be a cube and the numerator of $1 - f$ a square, hence $f = F_4^{(2)}(a, b, c, d, x)$ for unique $a, b, c, d \in k$. The case $p \neq \infty$ again reduces to this under m. □

As there are only two cases, it is not hard to solve the Belyi$^{(2)}$ part of goal (c) from the introduction:

Algorithm 6.1: FindBelyi2
Input: A field $k \subseteq \mathbb{C}$ and a 5-element subset $S = \{q_1 \ldots q_5\} \subset \mathbb{P}^1$ defined over k.
Output includes: Every Belyi$^{(2)}$ $f \in k(x)$ such that $E(f) = S$.

For each p in $S \bigcap (k \bigcup \{\infty\})$ do:

- **Step 1**. Let m be as in Lemma 6.2 and $\tilde{m}$ be its inverse (x if $p = \infty$, otherwise $1/x + p$).
- **Step 2**. Comparing the numerator of $F_4^{(2)}(a, b, c, d, x)$ with $m(S)$ gives 4 equations in a, b, c, d.
- **Step 3**. Two equations are linear in a variable, solving these leaves 2 equations in 2 unknowns.
- **Step 4**. Compute all solutions over k with a resultant.
- **Step 5**. For each solution, append $F_4^{(2)}(a, b, c, d, m)$ to the output.
- **Step 6**. Doing the same for $F_6^{(2)}(a, b, c, d, x)$ gives 4 equations, one of which is linear.
- **Step 7**. With a pre-computed [11] elimination we obtain an equation of degree 12 for a.
- **Step 8**. After computing its roots in k, two equations in two unknowns remain.
- **Step 9**. Compute solutions as in Step 4 and for each, append $F_6^{(2)}(a, b, c, d, m)$ to the output.

The program finds all Belyi$^{(2)}$ maps for S in $k(x)$ but it also finds certain Belyi or Belyi$^{(1)}$ maps: $F_{66}^{(1)}$ is a special case of $F_4^{(2)}$ while $F_{67}^{(1)}$ and $F_{68}^{(1)}$ are special cases of $F_6^{(2)}$. So we can remove these three from our Belyi$^{(1)}$ table without interfering with goal (c). To cover goal (c) for Belyi and Belyi$^{(1)}$ maps we need one more ingredient, which will be the topic of the next section.

7 Five point invariants

Given k and S, our goal is to quickly find, if it exists, $f \in k(x)$ such that $E(f) = S$. After running Algorithm FindBelyi2 we may assume that f is Belyi or Belyi$^{(1)}$. Such f must be Möbius-equivalent to a member of our Belyi or Belyi$^{(1)}$ table because they were proved to be complete.

It is not efficient to search for a Möbius-equivalence between S and the exceptional points of each of the many entries of the Belyi table. For the Belyi$^{(1)}$ table, one first needs to find the correct value of the parameter s before a Möbius-equivalence could occur.

Let k_5 be the set of 5-element subsets $S \subset \mathbb{P}^1$ that are defined over k. A *five-point-invariant* is a function $k_5 \to k$ that is invariant under Möbius-transformations. We implemented two such functions. The first, called I_5, maps S to $\sum_{q \in S} j(S - \{q\})$ where $j(T)$ refers to the j-invariant of a set T with 4 points. More precisely, if $T = \{q_1, q_2, q_3, q_4\}$ then $j(T)$ is the j-invariant of the elliptic curve $y^2 = (x - q_1)(x - q_2)(x - q_3)(x - q_4)$, where a factor $x - q_i$ is omitted if $q_i = \infty$. The second invariant $\tilde{I}_5$ is similar, except that it uses the sum of the squares of the j-invariants.

If f has 5 exceptional points $S = \{q_1, \ldots, q_5\}$, then $I_5(f)$ denotes $I_5(S)$. We attach $I_5(f)$ to each Belyi map f in our database. To each Belyi$^{(1)}$ map $f \in \mathbb{Q}(s)(x)$, we attach $I_5(f)$ and $\tilde{I}_5(f)$, which are elements of $\mathbb{Q}(s)$. For a Belyi map f, the invariant $I_5(f)$ is either a rational or an algebraic number (we insert its minimal polynomial over $\mathbb{Q}$ into the table). We do not use five-point invariants for Belyi$^{(2)}$ maps because there were only two cases.

These invariants give an efficient solution to goal (c), they rapidly eliminate nearly all entries that do not lead to a solution.

Algorithm 7.1: FindF (goal (c))
Input: A field $k \subseteq \mathbb{C}$ and a 5-element subset $S = \{q_1 \ldots q_5\} \subset \mathbb{P}^1$ defined over k.
Output: Every element of $f \in k(x)$ such that $E(f) = S$.

Step 1. $A :=$ FindBelyi2$(S) \subset k(x), \quad i_5 := I_5(S) \in k, \quad \tilde{i}_5 := \tilde{I}_5(S) \in k.$
Step 2. For each f in the Belyi table whose I_5 matches i_5, adjoin $f(m)$ to A for every (if any) Möbius-transformation m that sends S to $E(f)$.
Step 3. For each f in the Belyi$^{(1)}$ table, compute the gcd of the numerators of $I_5(f) - i_5$ and $\tilde{I}_5(f) - \tilde{i}_5$. If this gcd is not 1, then compute all its roots in k. For each non-degenerate root s_0, evaluate f at $s = s_0$ and then proceed as in Step 2.
Step 4. Return A.

For each f in our Belyi table, if $\alpha = I_5(f)$, then f turned out to be in $\mathbb{Q}(\alpha)(x)$. But if for example $\alpha = \mathrm{RootOf}(x^2 - x - 1)$ while the input of `FindF` is defined over say $k = \mathbb{Q}(\sqrt{5})$, then we must replace α by its corresponding element(s) of k before one can use f (use $\alpha \mapsto i_5$ to map f to an element of $k(x)$). Computing m requires some care too, for details see our implementation [1]. In Step 3 it is important that every member of our Belyi$^{(1)}$ table is duplicate-free, this ensures that if a Belyi$^{(1)}$ map in $k(x)$ has 5 exceptional points, then the corresponding value of s is unique and thus in $k \bigcup \{\infty\}$. The algorithm does not consider $s = \infty$ since it is degenerate for every member of our Belyi$^{(1)}$ table.

8 goal (d)

The Gauss Hypergeometric Function ${}_2F_1(a, b; c \,|\, x)$ satisfies the so-called Gauss Hypergeometric Equation

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0. \tag{6}$$

It has singularities at $0, 1, \infty$ with *exponents* $\{0, 1-c\}$, $\{0, c-a-b\}$, $\{a, b\}$ respectively. The exponent differences are $(e_0, e_1, e_\infty) = (1-c, c-a-b, b-a)$ up to sign. The numbers (k, ℓ, m) from Definition 1.2 correspond to a GHE (equation (6)) with the following exponent differences:

$$(e_0, e_1, e_\infty) = (1/k, 1/\ell, 1/m). \tag{7}$$

Finding a ${}_2F_1$-type solution of a second order differential equation L is equivalent to finding a combination of transformations (i),(ii),(iii) that sends the GHE (6) to L:

(i) Change of variables: $y(x) \mapsto y(f)$
(ii) Gauge transformation: $y \mapsto r_0 y + r_1 y'$
(iii) Exponential product: $y \mapsto \exp(\int r\,dx) \cdot y$ (in Conjecture 1, $\exp(\int r\,dx)$ will be algebraic).

Let L be as in Conjecture 1, with coefficients $a_i \in k(x)$ for some field $k \subseteq \mathbb{C}$, and with 5 true singularities $S = \{q_1 \dots q_5\}$, at least one of them logarithmic. Our tasks are (1): Use algorithm `FindF` to find $f \in k(x)$ (if it exists) such that $E(f) = S$ and (2): Find a combination of transformations (i),(ii),(iii) that sends the GHE (6) with $(e_0, e_1, e_\infty) = (0, \frac{1}{2}, \frac{1}{3})$ to L.

8.1 Example

Let L be:

$$y'' + \frac{\left(8\,x^4 - x^2 + 2\,x - 3\right)}{x\,(x+1)\,(4\,x+3)\,(x^2-2\,x+3)}y' - \frac{4\,x^2}{\left(x^2-2\,x+3\right)^2(x+1)^2\,(4\,x+3)}y = 0$$

1. Find the true (= non-removable) singularities [11]. In this example, all singularities except $x = \infty$. The 5-point polynomial is $P = x(x+1)(x+3/4)(x^2-2x+3)$.

2. `FindF` finds the following functions f such that $E(f)$ are given by P:
$F_{\text{list}} := [\frac{-x^8}{4(x^2-2\,x+3)(4\,x+3)(x+1)^2}, \frac{-4(x+1)^2\left(x^2-2\,x+3\right)x^4}{(4\,x+3)^2}, \frac{(x+1)^4\left(x^2-2\,x+3\right)^2}{4(4\,x+3)x^4}, \frac{-64\left(x^2-2\,x+3\right)(x+1)^2x^{12}}{(4\,x+3)(8\,x^4+36\,x+27)^3},$
$\frac{64(x+1)^6\left(x^2-2\,x+3\right)^3x^4}{(4\,x+3)(8\,x^4-4\,x-3)^3}, 1+3\,\frac{\left(x^2-10\,x-3\right)^2}{(5\,x-3)^3(x+1)}]$.

3. $H\,:\,y'' - \frac{(-3/2\,x+1)}{x(x-1)}y' + \frac{5}{144}\,\frac{1}{x(x-1)}y = 0$ is the GHE (6) with $(e_0, e_1, e_\infty) = (0, \frac{1}{2}, \frac{1}{3})$. Among its solutions is $y(x) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \,|\, x\right)$. We need to find transformations that send H to L. For each $f \in F_{\text{list}}$, apply change of variables $x \mapsto f$ to H, and then find transformations (ii)+(iii) [11].

4. Transformations (ii)+(iii) only exist for the third element in F_{list}. Applying transformation (i), $x \mapsto f = \frac{(x+1)^4\left(x^2-2\,x+3\right)^2}{4(4\,x+3)x^4}$, to H produces $H_f\,:\,y'' + \frac{(10\,x^4-x^2-6\,x-9)}{x(4\,x+3)(x^2-2\,x+3)(x+1)}y' + 5\,\frac{(x+1)^2}{x^2(4\,x+3)^2}y = 0$.

5. $y(f)$ is a solution of H_f. Computing transformations (ii)+(iii) gives a solution of L:
$Y = \frac{(\frac{x+1}{x})^{1/3}(x^2-2x+3)^{1/6}}{(4x+3)^{1/12}} \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \,\middle|\, \frac{(x+1)^4\left(x^2-2\,x+3\right)^2}{4(4\,x+3)x^4}\right)$. To obtain another solution, replace $y(x)$ by another solution of H.

Our implementation [1] performs the above steps for $(e_0, e_1, e_\infty) = (0, \frac{1}{2}, \frac{1}{m})$ with $m \in \{3, 4, 6\}$, and contains various improvements: It decomposes f to obtain a smaller solution if possible, and compares exponent-differences to reduce the number of cases, see [11] for details.

A Appendix

We tabulate all Belyi$^{(1)}$ maps $O_1, \dots, O_{20}$ with $|\,E_{\infty 24}\,| = 5$, and all Belyi$^{(1)}$ maps $P_1, \dots, P_{12}$ with $|\,E_{\infty 26}\,| = 5$. See [1] for explicit expressions in $\mathbb{Q}(s)(x)$ for each of these maps.

As one can see, there is only one orbit for each branching pattern, except for (2^3), (2^3), $(1^2, 4)$ for which there is none. This means that there do not exist $g_0, g_1, g_t, g_\infty \in S_6$ for which each of g_0, g_1 is a product of 3 disjoint 2-cycles, g_t is a 2-cycle, g_∞ is a 4-cycle, $g_0 g_1 g_t g_\infty = 1$, for which $<g_0, g_1, g_t, g_\infty>$ is transitive.

n	branching pattern	name	$\lvert\mathcal{O}\rvert$	decomp.
2	(2), (1^2), (1^2)	O_1	1	
	(1^2), (1^2), (2)	O_2	1	
3	(3), $(1,2)$, (1^3)	O_3	1	
	$(1,2)$, $(1,2)$, $(1,2)$	O_4	4	
	(1^3), $(1,2)$, (3)	O_5	1	
4	$(1^2,2)$, $(1^2,2)$, (4)	O_6	4	
	(4), (2^2), (1^4)	O_7	1	$2\circ 2$
	$(1,3)$, (2^2), $(1^2,2)$	O_8	3	
	(2^2), (2^2), $(1^2,2)$	O_9	2	$2\circ 2$
	$(1^2,2)$, (2^2), $(1,3)$	O_{10}	3	
	$(1^2,2)$, (2^2), (2^2)	O_{11}	2	$2\circ 2$
5	$(1^2,3)$, $(1,2^2)$, $(1,4)$	O_{12}	10	
	$(1,2^2)$, $(1,2^2)$, $(1,4)$	O_{13}	8	
6	$(1^2,4)$, (2^3), $(1^2,4)$	O_{14}	6	
	$(1,2,3)$, (2^3), $(1^2,4)$	O_{15}	9	
	(2^3), (2^3), $(1^2,4)$	—	—	
	$(1^3,3)$, (2^3), $(2,4)$	O_{16}	2	$2\circ 3$
	$(1^2,2^2)$, (2^3), $(2,4)$	O_{17}	4	$2\circ 3$
8	$(1^4,4)$, (2^4), (4^2)	O_{18}	2	$2\circ 2\circ 2$
	$(1^3,2,3)$, (2^4), (4^2)	O_{19}	6	$2\circ 4$
	$(1^2,2^3)$, (2^4), (4^2)	O_{20}	4	$2\circ 2\circ 2$

n	branching pattern	name	$\lvert\mathcal{O}\rvert$	decomp.
2	(2), (1^2), (1^2)	P_1	1	
	(1^2), (1^2), (2)	P_2	1	
3	(3), $(1,2)$, (1^3)	P_3	1	
	$(1,2)$, $(1,2)$, $(1,2)$	P_4	4	
	(1^3), $(1,2)$, (3)	P_5	1	
4	(4), (2^2), (1^4)	P_6	1	$2\circ 2$
	$(1,3)$, (2^2), $(1^2,2)$	P_7	3	
	(2^2), (2^2), $(1^2,2)$	P_8	2	$2\circ 2$
	$(1^2,2)$, (2^2), $(1,3)$	P_9	3	
	$(1^2,2)$, (2^2), (2^2)	P_{10}	2	$2\circ 2$
	(1^4), (2^2), (4)	P_{11}	1	$2\circ 2$
6	$(1^4,2)$, (2^3), (6)	P_{12}	2	$2\circ 3$

Table 3: Belyi$^{(1)}$ with $\lvert E_{\infty 24}\rvert = 5$ resp. $\lvert E_{\infty 26}\rvert = 5$

References

[1] M. van Hoeij, V.J. Kunwar: *Classifying (almost)-Belyi maps with Five Exceptional Points*. Data, algorithms, and examples: http://www.math.fsu.edu/∼hoeij/FiveSing

[2] R. Vidunas, G. Filipuk: *A Classification of Covering yielding Heun to Hypergeometric Reductions*, Funkcialaj Ekvacioj, **56** No. 2, 271-321 (2013).

[3] M. van Hoeij, R. Vidūnas: *Belyi functions for hyperbolic hypergeometric-to-Heun transformations*, J. Algebra (2015).

[4] F. Beukers, H. Montanus: *Explicit calculation of elliptic fibrations of K3-surfaces and their Belyi-maps*, Cambridge Univ. Press, Cambridge (2008).

[5] R. Vidunas, A.V. Kitaev: *Computation of Highly Ramified Coverings*, Math. Comp. **78**, No. 268, 2371-2395 (2009).

[6] R. Vidūnas: *Transformations of Some Gauss Hypergeometric Functions*, J. Comp. & App. Mathematics, Vol 178, 473-487 (2005).

[7] J. Sijsling, J. Voight: *On computing Belyi maps*, Publications mathématiques de Besançon 73-131 (2014).

[8] J. Sijsling: *BelyiMaps: explicit determination of the permutation triples of given ramification type*, sites.google.com/site/sijsling/programs (2010).

[9] R. Debeerst, M. van Hoeij, W. Koepf: *Solving Differential Equations in Terms of Bessel Functions*, ISSAC'08 Proceedings, 39-46 (2008).

[10] V. J. Kunwar, M. van Hoeij: *Second Order Differential Equations with Hypergeometric Solutions of Degree Three*, ISSAC 2013 Proceedings, 235-242 (2013)
Implementation: www.math.fsu.edu/∼vkunwar/hypergeomdeg3

[11] V. J. Kunwar: *Hypergeometric Solutions of Linear Differential Equations with Rational Function Coefficients*, Ph.D thesis and implementation, available at www.math.fsu.edu/∼vkunwar (2014).

[12] A. V. Kitaev, *Grothendieck's Dessins d'Enfants, Their Deformations and Algebraic Solutions of the Sixth Painlevé and Gauss Hypergeometric Equations*, Algebra i Analiz 17, 224-273 (2005).

[13] A. V. Kitaev, *Remarks Towards Classification of $RS_4^2(3)$-Transformations and Algebraic Solutions of the Sixth Painlevé Equation*, Theorie asymptotiques et equations de Painlevé (Angers, Juin 2004), Eds. E. Delabaere, M. Loday, Seminaires et Congrés 14, 199-227 (2006).

[14] F. V. Andreev, A. V, Kitaev, *Transformations $RS_4^2(3)$ of the Ranks $\leq$ 4 and Algebraic Solutions of the Sixth Painlevé Equation*, Comm. Math. Phys. 228, 151-176 (2002).

[15] T. Fang: *Solving Linear Differential Equations in Terms of Hypergeometric Functions by 2-Descent*, Ph.D thesis (2012).

[16] Q. Yuan: *Finding all Bessel type solutions for Linear Differential Equations with Rational Function Coefficients*, Ph.D thesis and implementation, available at www.math.fsu.edu/∼qyuan (2012).

[17] A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J.-M. Maillard, J-A. Weil and N. Zenine: *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*. J. Phys. A: Math. Theor. **44** 045204 (2011).

[18] K. Takeuchi: *Commensurability classes of arithmetic triangle groups*, J. Fac. Sci, Univ. Tokyo, Sect.1 A, 201-212 (1977).

[19] J. Kovacic: *An algorithm for solving second order linear homogeneous equations*, J. Symbolic Computations, 2, 3-43 (1986).

[20] M. Jimbo and T. Miwa: *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II*, Physica 2D, 407448 (1981).

[21] A. Zvonkin: *Belyi functions: examples, properties, and applications*, Proceedings AAECC. Vol. 11 (2008).

[22] S. Donaldson: *Riemann surfaces: notes*. (2004).

[23] H. Volklein: *Groups as Galois Groups*. Cambridge Studies in Advanced Mathematics, **53**, Cambridge University Press (1996).

[24] S. Donaldson: *Fields of Definition of Function Fields and Hurwitz Families and; Groups as Galois Groups*, Communications in Algebra 5, 1782 (1977).

[25] S. Donaldson, P. Bailey: *Hurwitz monodromy, spin separation and higher levels of a Modular Tower*, Proceedings of Symposia in Pure Mathematics **70** (2002).

[26] D. Eisenbud, N. Elkies, J. Harris and R. Speiser: *On the Hurwitz scheme and its monodromy*, Compositio Mathematica, **77**: 95-117 (1991).

[27] K. Magaard, S. Shpectorov, and H. Volklein: *A GAP Package for Braid Orbit Computation and Applications*, Eperimental Mathematics; **12**, No. 4 385-394 (2003).

[28] T. R. Walsh: *Space-Efficient Generation of Nonisomorphic Maps and Hypermaps*, J. of Integer Sequences, **18** (2015).

[29] L. Schneps: *The Grothendieck Theory of Dessins d'Enfants*, London Math. Soc. Lecture Note Series, Cambridge U. Press (1994).

[30] S. K. Lando, A. K. Zvonkin: *Graphs on Surfaces and Their Applications*, ISBN 3-540-00203-0 Springer-Verlag Berlin Heidelberg (2004).

[31] V. Ancona, E. Strickland: *Trends in Contemporary Mathematics*, Springer INdAM Series (2014).