On the Number of Singular Vector Tuples of Hyper-Cubical Tensors

By Shalosh B. EKHAD and Doron ZEILBERGER

A week ago, Bernd Sturmfels [St] gave a fascinating Colloquium talk, here at Rutgers, where, among many other interesting facts, he mentioned the following theorem of Shmuel Friedland and Giorgio Ottaviani ([FO]).

Theorem ([F0], Theorem 1). The number of simple singular vector tuples of a generic $m_1 \times \cdots \times m_d$ (d-dimensional) tensor equals the coefficient of $\prod_{i=1}^{d} t_i^{m_i-1}$ in the polynomial

$$\prod_{i=1}^{d} \frac{\hat{t_i}^{m_i} - t_i^{m_i}}{\hat{t_i} - t_i} \quad , \quad \hat{t_i} = \left(\sum_{j=1}^{d} t_j \right) - t_i \quad .$$

Let's call this number $c(m_1, \ldots, m_d)$.

We first observe that the generating function of this d-dimensional multi-sequence is a nice rational function.

Proposition 1. Let $e_i(x_1, \ldots, x_d)$ be the elementary symmetric function of the indeterminates $x_1, \ldots, x_d$, of degree i. We have:

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_d=0}^{\infty} c(m_1, \ldots, m_d) x_1^{m_1} \ldots x_d^{m_d} = \prod_{i=1}^{d} x_i \left(\prod_{i=1}^{d} (1 - x_i) \right)^{-1} \left(1 - \sum_{i=2}^{d} (i-1) e_i(x_1, \ldots, x_d) \right)^{-1} \quad .$$

Proof: Since

$$\frac{\hat{t_i}^{m_i} - t_i^{m_i}}{\hat{t_i} - t_i} = \sum_{k_i=0}^{m_i-1} \hat{t_i}^{k_i} t_i^{m_i-1-k_i} \quad ,$$

$c(m_1, \ldots, m_k)$ is the coefficient of $\prod_{i=1}^{d} t_i^{m_i-1}$ in

$$\sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} \prod_{i=1}^{d} \hat{t_i}^{k_i} t_i^{m_i-1-k_i} \quad .$$

Hence

$$c(m_1, \ldots, m_d) = \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} ConstantTermOf \prod_{i=1}^{d} \hat{t_i}^{k_i} t_i^{-k_i} \quad .$$

Let

$$f(k_1, \ldots, k_d) := ConstantTermOf \prod_{i=1}^{d} \hat{t_i}^{k_i} t_i^{-k_i} = CoeffOf \prod_{i=1}^{d} t_i^{k_i} \quad in \quad \prod_{i=1}^{d} \left(\sum_{j=1}^{i-1} t_j + \sum_{j=i+1}^{d} t_j \right)^{k_i} \quad .$$

By the celebrated **MacMahon Master Theorem** ([M], Section III, Chapter II, p. 93ff) (with the $d \times d$ matrix that is all 1's except 0 in the diagonal), we have

$$\sum_{k_1=0}^{\infty} \dots \sum_{k_d=0}^{\infty} f(k_1, \dots, k_d) x_1^{k_1} \dots x_d^{k_d} = \left(1 - \sum_{i=2}^{d} (i-1) e_i(x_1, \dots, x_d)\right)^{-1} \quad .$$

Since

$$c(m_1, \dots, m_d) = \sum_{k_1=0}^{m_1-1} \dots \sum_{k_d=0}^{m_d-1} f(k_1, \dots, k_d) \quad ,$$

the proposition follows by straightforward generatingfunctionology.

The fact that the generating function of $c(m_1, \dots, m_d)$ is a rational function is equivalent to it satisfying a certain partial linear recurrence with constant coefficients, easily deduced from the generating function. Combined with the fact that both $c(m_1, \dots, m_d)$ and $f(k_1, \dots, k_d)$ are symmetric, enabled us to efficiently compute many values. It also follows (for example using Wilf-Zeilberger algorithmic proof theory, efficiently implemented in [AZ]) that the diagonal sequences

$$C_d(n) := c(n, \dots, n) \qquad [n \quad repeated \quad d \quad times] \quad ,$$

are **holonomic**, alias **P-recursive**, that means that for each d, the sequence $C_d(n)$ satisfies *some* homogeneous linear recurrence with polynomial coefficients. While one can use the method of [AZ], it is more efficient, since we know *a priori* that such a recurrence exists, to generate sufficiently many terms and then **guess** the recurrence. Using this method, we got the following proposition.

Proposition 2. The sequence $C_3(n) = c(n, n, n)$ satisfies the following fifth-order linear recurrence equation with polynomial coefficients.

$$2\,(n+2)\left(245\,n^4 + 3094\,n^3 + 14447\,n^2 + 29474\,n + 22100\right)(n+1)^2\,C_3(n)$$

$$-\,(n+2)\left(21805\,n^6 + 330981\,n^5 + 2012733\,n^4 + 6230951\,n^3 + 10263446\,n^2 + 8425060\,n + 2639760\right)C_3(n+1)$$

$$+\left(-13230\,n^7 - 249641\,n^6 - 1998705\,n^5 - 8785333\,n^4 - 22847777\,n^3 - 35069178\,n^2 - 29331496\,n - 10279296\right)\cdot C_3(n+2)$$

$$+\left(21560\,n^7 + 413637\,n^6 + 3343917\,n^5 + 14735333\,n^4 + 38132651\,n^3 + 57777574\,n^2 + 47273504\,n + 16026528\right)\cdot C_3(n+3)$$

$$-\,(n+4)\left(4410\,n^6 + 70147\,n^5 + 452903\,n^4 + 1516515\,n^3 + 2769127\,n^2 + 2601986\,n + 975888\right)C_3(n+4)$$

$$+\,(n+5)(n+4)(n+3)\left(245\,n^4 + 2114\,n^3 + 6635\,n^2 + 8882\,n + 4224\right)C_3(n+5) = 0 \quad ,$$

subject to the initial conditions

$$C_3(1) = 1 \quad , \quad C_3(2) = 6 \quad , \quad C_3(3) = 37 \quad , \quad C_3(4) = 240 \quad , \quad C_3(5) = 1621 \quad .$$

Using the methods of [WZ] and [Z], we found the following asymptotic formula.

Proposition 3.

$$C_3(n) \sim \frac{2}{\sqrt{3}\,\pi}\,8^n \cdot n^{-1} \cdot$$

$$\left(1-\frac{13}{3}n^{-1}+\frac{1477}{27}n^{-2}-\frac{93707}{81}n^{-3}+\frac{8343061}{243}n^{-4}-\frac{2866730137}{2187}n^{-5}+\frac{1204239422533}{19683}n^{-6}+O(n^{-7})\right) \quad .$$

We observe that the "connective constant", 8, is *sub-dominant*. With any other initial conditions it would have been 9. This is a very rare phenomenon in combinatorics.

The sequence $C_3(n)$ is sequence $A271905$ in the On-Line Encyclopedia of Integer Sequences [Sl]. For the record, here are the first few terms:

1, 6, 37, 240, 1621, 11256, 79717, 572928, 4164841, 30553116, 225817021, 1679454816, 12556853401, 94313192616, 711189994357, 5381592930816, 40848410792017, 310909645663332, 2372280474687277, 18141232682656320, 139010366280363601, 1067160872528170536, 8206301850166625797, 63203453697218605440.

We tried to find a recurrence for $C_4(n)$, but, since 160 terms did not suffice, we gave up. Nevertheless, using numerics, it if extremely likely that

$$C_4(n) \sim \alpha\, 81^n \cdot n^{-\frac{3}{2}} \quad ,$$

for some constant α, but we are unable to conjecture its value. For the record, here are the first few terms:

1, 24, 997, 51264, 2940841, 180296088, 11559133741, 765337680384,51921457661905, 3590122671128664, 252070718210663749, 17922684123178825536, 1287832671004683373753, 93368940577497932331288, 6821632357294515590873917, 501741975445243527381995520, 37121266623211130111114816929, 2760712710223967190110979892824, 206267049696409355312012281872181.

The first few terms of $C_5(n)$ are: 1, 120, 44121, 23096640, 14346274601, 9859397817600, 7244702262723241, 5582882474985676800.

The first few terms of $C_6(n)$ are: 1, 720, 2882071, 18754813440, 153480509680141, 1435747717722810960.

Using reliable numeric estimates we are confident in making the following conjecture.

Conjecture:

$$C_d(n) \sim \alpha_d \cdot ((d-1)^d)^n \cdot n^{-(d-1)/2} \quad ,$$

for a constant α_d.

One of us (DZ) is pledging \$100 dollars to the OEIS Foundation in honor of the first prover, and an additional \$25 for an explicit expression for α_d in terms of d and π.

Readers are welcome to explore further using the Maple package `SVT.txt` available from `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/svt.html`, where there are many more terms of the sequences $C_d(n)$ for $3 \le d \le 6$.

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