Linear transformations and strong q-log-concavity for certain combinatorial triangle *

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Abstract

It is well-known that the binomial transformation preserves the log-concavity property and log-convexity property. Let $\binom{a+n}{b+k}$ be the binomial coefficients and $\binom{n,k}{j}$ be defined by $(b_0 + b_1 x + \cdots + b_k x^k)^n := \sum_{j=0}^{kn} \binom{n,k}{j} x^j$, where the sequence $(b_i)_{0 \le i \le k}$ is log-concave. In this paper, we prove that the linear transformation

$$y_n(q) = \sum_{k=0}^{n} \binom{a+n}{b+k} x_k(q)$$

preserves the strong q-log-concavity property for any fixed nonnegative integers a and b, which strengthens and gives a simple proof of results of Ehrenborg and Steingrimsson, and Wang, respectively, on linear transformations preserving the log-concavity property. We also show that the linear transformation

$$y_n = \sum_{i=0}^{kn} \binom{n,k}{j} x_i$$

not only preserves the log-concavity property, but also preserves the log-convexity property, which extends the results of Ahmia and Belbachir about the s-triangle transformation preserving the log-convexity property and log-concavity property. Let $[A_{n,k}(q)]_{n,k \ge 0}$ be an infinite lower triangular array of polynomials in q with nonnegative coefficients satisfying the recurrence

$$A_{n,k}(q) = f_{n,k}(q) A_{n-1,k-1}(q) + g_{n,k}(q) A_{n-1,k}(q) + h_{n,k}(q) A_{n-1,k+1}(q),$$

for $n \ge 1$ and $k \ge 0$, where $A_{0,0}(q) = 1$, $A_{0,k}(q) = A_{0,-1}(q) = 0$ for $k > 0$. We present criterions for the strong q-log-concavity of the sequences in each row of $[A_{n,k}(q)]_{n,k \ge 0}$. As applications, we get the strong q-log-concavity or the log-concavity of the sequences in each row of many well-known triangular arrays, such as the Bell polynomials triangle, the Eulerian polynomials triangle and the Narayana polynomials triangle in a unified approach.

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1 Introduction

Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative real numbers. It has no *internal zeros* , *i.e.*, $a_j \neq 0$ if $a_i a_k \neq 0$ for $i < j < k$. Thoughout this paper, all sequences have no internal zeros. We call it *log-concave* if $a_{k-1}a_{k+1} \leq a_k^2$ for all $k \geq 1$. The log-concave sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated, see Stanley [27] and Brenti [9] for details.

For two polynomials with real coefficients $f(q)$ and $g(q)$, denote $f(q) \geq_q g(q)$ if the difference $f(q) - g(q)$ has only nonnegative coefficients. For a polynomial sequence $(f_n(q))_{n\geq 0}$, it is called *q-log-concave* suggested by Stanley if

$$f_n(q)^2 \geq_q f_{n+1}(q)f_{n-1}(q),$$

for $n \geq 1$ and is called *strongly q-log-concave* introduced by Sagan if

$$f_{n+1}(q)f_{m-1}(q) \geq_q f_n(q)f_m(q),$$

for any $m \geq n \geq 1$. Obviously, the strong q-log-concavity implies the q-log-concavity. The q-log-concavity of polynomials have been extensively studied, see Butler [10], Krattenthaler [20], Leroux [21], Sagan [25, 26], and Su, Wang and Yeh [29] for instance. If we reverse the inequalities in above definitions, then we have the concepts for log-convexity, q-log-convexity and strong q-log-convexity, respectively. Reader can refer to Chen *et al.* [13, 14, 15], Liu and Wang [22], Zhu [32, 33], and Zhu and Sun [34] for the strong q-log-convexity.

The linear transformations are often used to study the log-concavity and log-convexity. For instance, it was proved that the binomial transformation $b_n = \sum_{k=0}^{n} \binom{n}{k}a_k$ for $n \geq 0$ preserves the log-concavity property, *i.e.*, log-concavity of $(a_n)_{n\geq 0}$ implies that of $(b_n)_{n\geq 0}$ (see [8, Theorem 2.5.7] or [19, Theorem 7.3] for instance) and log-convexity property [22]. More generally, the log-convexity property and log-concavity property are preserved under the binomial convolution, see Davenport and Pólya [18] and Wang and Yeh [31], respectively. Using certain positivity method, Ehrenborg and Steingrimsson [17] also showed that the linear transformation

$$y_n = \sum_{k=0}^{n} \binom{n+1}{k} x_k$$

preserves the log-concavity property. Furthermore, using the algebraical method, Wang [30] proved one more general result that the linear transformation

$$y_n = \sum_{k=0}^{n} \binom{a+n}{b+k} x_k$$

preserves the log-concavity property for fixed nonnegative integers a and b. Motivated by these, we prove the following stronger result.

Theorem 1.1. *Let a and b be two nonnegative integers. Then the linear transformation*

$$y_n(q) = \sum_{k=0}^{n} \binom{a+n}{b+k} x_k(q)$$

preserves the strong q-log-concavity property. In particular, it preserves the log-concavity property of sequences.

The s-triangle $\binom{n}{j}_s$ is a generalization of the Pascal triangle, which is given by the ordinary multinomial coefficients [11]:

$$(1 + x + \cdots + x^s)^n := \sum_{j=0}^{sn} \binom{n}{j}_s x^j.$$

In [1, 2], Ahmia and Belbachir also demonstrated that the log-convexity property and log-concavity property are preserved under the s-triangle transformation. Motivated by this, we will consider a more general triangle in the following. Let k be a positive integer. Assume that the nonnegative sequence $(b_i)_{0 \le i \le k}$ is log-concave. Define a more generalized triangle $\binom{n,k}{j}$:

$$(b_0 + b_1 x + \cdots + b_k x^k)^n := \sum_{j=0}^{kn} \binom{n,k}{j} x^j.$$

It is obvious that the triangle $[\binom{n,k}{j}]_{n,j \ge 0}$ has the following recurrence relations:

$$\binom{n,k}{j} = \sum_{i=0}^{k} \binom{n-1,k}{j-i} b_i, \tag{1.1}$$

$$\binom{n,k}{j} = \sum_{i=0}^{sk} \binom{n-s,k}{i} \binom{s,k}{j-i}. \tag{1.2}$$

In fact, the triangle $[\binom{n,k}{j}]_{n,j \ge 0}$ generalizes many famous triangles. For instance,

(i) If both $k = b_0 = b_1 = 1$, then the triangle $[\binom{n,k}{j}]_{n,j \ge 0}$ turns out be the Pascal triangle.

(ii) If $b_0 = b_1 = \ldots = b_k = 1$, then the triangle $[\binom{n,k}{j}]_{n,j \ge 0}$ turns out be the s-triangles given by the ordinary multinomials, see [24, A027907 for s = 2, A008287 for s = 3 and A035343 for s = 4].

We also prove the next stronger result in a unified approach.

Theorem 1.2. *Let k be any fixed positive integer and $\binom{n,k}{j}$ be as above. Then the linear transformation*

$$y_n = \sum_{i=0}^{kn} \binom{n,k}{j} x_i$$

not only preserves the log-concavity property, but also preserves the log-convexity property.

In [32], Zhu defined a triangular array as follows. Let $[A_{n,k}(q)]_{n,k \ge 0}$ be an infinite lower triangular array defined by the recurrence

$$A_{n,k}(q) = f_k(q) A_{n-1,k-1}(q) + g_k(q) A_{n-1,k}(q) + h_k(q) A_{n-1,k+1}(q) \tag{1.3}$$

for $n \ge 1$ and $k \ge 0$, where $A_{0,0}(q) = 1$, $A_{0,k}(q) = A_{0,-1}(q) = 0$ for $k > 0$. This triangular array consists of polynomials in q and it will turn out to be the array with entries be real numbers for any fixed $q \ge 0$. In fact, this triangular array (1.3) unifies many well-known combinatorial triangles. The following are some basic examples.

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Example 1.3. *(1) The Catalan triangle of Aigner [4] is*

$$
C = [C_{n,k}]_{n,k \geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 3 & 1 & & \\ 5 & 9 & 5 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},
$$

where $C_{n+1,k} = C_{n,k-1} + 2\,C_{n,k} + C_{n,k+1}$ and $C_{n+1,0} = C_{n,0} + C_{n,1}$. The numbers in the 0th column are the Catalan numbers C_n.

(2) The Catalan triangle of Shaprio [28] is

$$
C' = [C'_{n,k}]_{n,k \geq 0} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 14 & 14 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},
$$

where $C'_{n+1,k} = C'_{n,k-1} + 2\,C'_{n,k} + C'_{n,k+1}$ for $k \geq 0$. The numbers in the 0th column are the Catalan numbers C_n.

(3) The Motzkin triangle [3, 4] is

$$
M = [M_{n,k}]_{n,k \geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 4 & 5 & 3 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},
$$

where $M_{n+1,k} = M_{n,k-1} + M_{n,k} + M_{n,k+1}$ and $M_{n+1,0} = M_{n,0} + M_{n,1}$. The numbers in the 0th column are the Motzkin numbers M_n.

(4) The large Schröder triangle [16] is

$$
s = [s_{n,k}]_{n,k \geq 0} = \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 6 & 4 & 1 & & \\ 22 & 16 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},
$$

where $s_{n+1,k} = s_{n,k-1} + 2\,s_{n,k} + 2s_{n,k+1}$ and $s_{n+1,0} = s_{n,0} + 2s_{n,1}$. The numbers in the 0th column are the large Schröder numbers S_n.

(5) The Bell triangle [5] is

$$
B = [B_{n,k}]_{n,k \geq 0} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 3 & 1 & & \\ 5 & 10 & 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},
$$

where $B_{n+1,k} = B_{n,k-1} + (1+k) B_{n,k} + (1+k) B_{n,k+1}$ and $B_{n+1,0} = B_{n,0} + B_{n,1}$. The numbers in the 0th column are the Bell numbers.

(6) The Bell polynomials triangle $B = [B_{n,k}]_{n,k \geq 0}$ [6] is

$$\begin{bmatrix} 1 & & & & \\ q & 1 & & & \\ q^2 + q & 2q+1 & 1 & & \\ q^3 + 3q^2 + q & 3q^2 + 6q + 1 & 3q + 3 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $B_{n+1,k} = B_{n,k-1} + (q+k) B_{n,k} + q(1+k) B_{n,k+1}$. The numbers in the 0th column are the Bell polynomials. In particular, it reduces to the Bell triangle [5] for $q = 1$.

(7) The Eulerian polynomials triangle $E = [E_{n,k}]_{n,k \geq 0}$ [6] is

$$\begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ q+1 & q+3 & 1 & & \\ q^2 + 4q + 1 & q^2 + 10q + 7 & 3q + 6 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $E_{n+1,k} = E_{n,k-1} + (kq + k + 1) E_{n,k} + (k+1)^2 q E_{n,k+1}$. The numbers in the 0th column are the Eulerian polynomials.

(8) The Narayana polynomials triangle $N = [N_{n,k}]_{n,k \geq 0}$ [6] is

$$\begin{bmatrix} 1 & & & & \\ q & 1 & & & \\ q^2 + q & 2q+1 & 1 & & \\ q^3 + 3q^2 + q & 3q^2 + 5q + 1 & 3q + 2 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

where $E_{n+1,k} = E_{n,k-1} + (q+1) E_{n,k} + q E_{n,k+1}$ and $E_{n+1,0} = qE_{n,0} + qE_{n,1}$. The numbers in the 0th column are the Narayana polynomials [6].

If $f_k \equiv 1$ for $k \geq 0$, then it can be reduced to the recursive matrix and $A_{n,0}(q)$ are called the Catalan-like numbers, see Aigner [4, 6]. Many combinatorial and algebraic properties of the triangular array $[A_{n,k}(q)]_{n,k \geq 0}$ have been found. For instance, Aigner [3, 4, 6, 7] researched various combinatorial properties of recursive matrices and Hankel matrices of the Catalan-like numbers. Chen, Liang and Wang [12] considered the total positivity of recursive matrices. Zhu [32] gave a criterion for the strong q-log-convexity of the first column of $[A_{n,k}(q)]_{n,k \geq 0}$. However, there is no result for the strong q-log-concavity of the sequence in each row of $[A_{n,k}(q)]_{n,k \geq 0}$. This is our another motivation. In §4, we will present some criterions for the strong q-log-concavity of the sequence in each row of $[A_{n,k}(q)]_{n,k \geq 0}$.

The remainder part of this paper is arranged as follows. In §2 and §3, we give the proofs of Theorem 1.1 and 1.2, respectively. In §4 we present sufficient conditions for the strong q-log-concavity of rows of the certain triangle. As applications, we show the

log-concavity and the strong q-log-concavity of sequences of rows in many well-known triangular arrays, such as the Catalan triangles of Aigner and Shaprio, the large Schröder triangle, the Motzkin triangle, the Bell polynomials triangle, the Eulerian polynomials triangle and the Narayana polynomials triangle, and so on, in a unified approach.

2 Proof of Theorem 1.1

Proof. Since a nonnegative sequence $(a_n)_{n \geq 0}$ with no internal zeros is log-concave if and only if $a_i a_j \geq a_{i+1} a_{j-1}$ for $i \geq j$, the sequence sequence $(a_n)_{n \geq 0}$ is strongly q-log-concave. Thus, the second part immediately follows from the first part for $q = 0$. So we only need to show the first part, *i.e.* the linear transformation

$$y_n(q) = \sum_{k=0}^{n} \binom{a+n}{b+k} x_k(q)$$

preserves the strong q-log-concavity property. We first need the next fact.

Fact 2.1. *If $(x_n(q))_{n \geq 0}$ is strongly q-log-concave, then so is $(x_n(q) + x_{n+1}(q))_{n \geq 0}$.*

Proof. Since the sequence $(x_n(q))_{n \geq 0}$ is strongly q-log-concave, by the definition, we have $x_i(q) x_j(q) \geq_q x_{i-1}(q) x_{j+1}(q)$ for any $j \geq i \geq 0$, which implies that

$$
\begin{aligned}
& [x_{j-1}(q) + x_{j-2}(q)] [x_{i+1}(q) + x_i(q)] - [x_j(q) + x_{j-1}(q)] [x_i(q) + x_{i-1}(q)] \\
= \ & [x_{j-1}(q) x_{i+1}(q) - x_j(q) x_i(q)] + [x_{j-2}(q) x_i(q) - x_{j-1}(q) x_{i-1}(q)] \\
& + [x_{j-2}(q) x_{i+1}(q) - x_j(q) x_{i-1}(q)] \\
\leq_q \ & 0
\end{aligned}
$$

for any $i \geq j \geq 0$, as desired. This proves the fact. $\qquad\square$

In the following, we will prove the desired result by induction on n.
If $0 \leq n \leq 3$, then we have

$$
\begin{align}
y_0(q) &= x_0(q) \binom{a}{b}, \tag{2.1} \\
y_1(q) &= x_0(q) \binom{a+1}{b} + x_1(q) \binom{a+1}{b+1}, \tag{2.2} \\
y_2(q) &= x_0(q) \binom{a+2}{b} + x_1(q) \binom{a+2}{b+1} + x_2(q) \binom{a+2}{b+2}, \tag{2.3} \\
y_3(q) &= x_0(q) \binom{a+3}{b} + x_1(q) \binom{a+3}{b+1} + x_2(q) \binom{a+3}{b+2} + x_3(q) \binom{a+3}{b+3}, \tag{2.4}
\end{align}
$$

since $y_n(q) = \sum_{k=0}^{n} \binom{n+a}{k+b} x_k(q)$. Hence, by (2.1)-(2.3) we obtain that

$$
\begin{aligned}
y_1^2(q) - y_2(q)y_0(q) &= x_0^2(q)\left[\binom{a+1}{b}^2 - \binom{a}{b}\binom{a+2}{b}\right] \\
&\quad + \left[x_1^2(q)\binom{a+1}{b+1}^2 - x_0(q)x_2(q)\binom{a}{b}\binom{a+2}{b+2}\right] \\
&\quad + x_0(q)x_1(q)\left[2\binom{a+1}{b}\binom{a+1}{b+1} - \binom{a}{b}\binom{a+2}{b+1}\right]. \quad (2.5)
\end{aligned}
$$

Thus we deduce for $a < b$ that

$$
y_1^2(q) - y_2(q)y_0(q) \geq_q 0.
$$

On the other hand, by (2.5) we also have

$$
y_1^2(q) - y_2(q)y_0(q) \geq_q 0
$$

for $a \geq b$ since

$$
\binom{a+1}{b}^2 - \binom{a}{b}\binom{a+2}{b} = \binom{a+1}{b-1}\binom{a}{b},
$$

$$
x_1^2(q)\binom{a+1}{b+1}^2 - x_0(q)x_2(q)\binom{a}{b}\binom{a+2}{b+2}
$$

$$
\geq_q x_1^2(q)\left[\binom{a+1}{b+1}^2 - \binom{a}{b}\binom{a+2}{b+2}\right] = \frac{x_1^2(q)}{b+1}\binom{a+1}{b+2}\binom{a}{b},
$$

$$
2\binom{a+1}{b}\binom{a+1}{b+1} - \binom{a}{b}\binom{a+2}{b+1} = \frac{a}{a-b+1}\binom{a+1}{b+1}\binom{a}{b}.
$$

In the following, we will prove $y_1(q)y_2(q) - y_3(q)y_0(q) \geq_q 0$. Note for $a \leq b$ that $y_0(q) = 0$.

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So it suffices to consider the remaining case $a \geq b$. It follows from (2.1)-(2.4) we have

$$
\begin{aligned}
& y_1(q)y_2(q) - y_3(q)y_0(q) \\
= & \left[x_0(q)\binom{a+1}{b} + x_1(q)\binom{a+1}{b+1} \right] \left[x_0(q)\binom{a+2}{b} + x_1(q)\binom{a+2}{b+1} + x_2(q)\binom{a+2}{b+2} \right] \\
& - x_0(q)\binom{a}{b} \left[x_0(q)\binom{a+3}{b} + x_1(q)\binom{a+3}{b+1} + x_2(q)\binom{a+3}{b+2} + x_3(q)\binom{a+3}{b+3} \right] \\
= & \; x_0^2(q) \left[\binom{a+1}{b}\binom{a+2}{b} - \binom{a}{b}\binom{a+3}{b} \right] + x_1^2(q)\binom{a+1}{b+1}\binom{a+2}{b+1} \\
& + x_0(q)x_1(q) \left[\binom{a+1}{b}\binom{a+2}{b+1} + \binom{a+1}{b+1}\binom{a+2}{b} - \binom{a}{b}\binom{a+3}{b+1} \right] \\
& + x_0(q)x_2(q) \left[\binom{a+1}{b}\binom{a+2}{b+2} - \binom{a}{b}\binom{a+3}{b+2} \right] \\
& + x_1(q)x_2(q)\binom{a+1}{b+1}\binom{a+2}{b+2} - x_0(q)x_3(q)\binom{a}{b}\binom{a+3}{b+3} \\
\geq_q & \; x_0^2(q) \left[\binom{a+1}{b}\binom{a+2}{b} - \binom{a}{b}\binom{a+3}{b} \right] \\
& + x_0(q)x_1(q) \left[\binom{a+1}{b}\binom{a+2}{b+1} + \binom{a+1}{b+1}\binom{a+2}{b} - \binom{a}{b}\binom{a+3}{b+1} \right] \\
& + x_0(q)x_2(q) \left[\binom{a+1}{b+1}\binom{a+2}{b+1} + \binom{a+1}{b}\binom{a+2}{b+2} - \binom{a}{b}\binom{a+3}{b+2} \right] \\
& + x_1(q)x_2(q) \left[\binom{a+1}{b+1}\binom{a+2}{b+2} - \binom{a}{b}\binom{a+3}{b+3} \right] \\
\geq_q & \; 0
\end{aligned}
$$

because $(x_n(q))_{n\geq 0}$ is strongly q-log-concave and the following equalities

$$
\begin{aligned}
\binom{a+1}{b}\binom{a+2}{b} - \binom{a}{b}\binom{a+3}{b} &= \frac{2b}{(a+1-b)(a+3-b)}\binom{a}{b}\binom{a+2}{b}, \\
\binom{a+1}{b}\binom{a+2}{b+1} + \binom{a+1}{b+1}\binom{a+2}{b} - \binom{a}{b}\binom{a+3}{b+1} &= \frac{a(a-b)+a+b}{(a+1-b)(a+2-b)}\binom{a}{b}\binom{a+2}{b+1}, \\
\binom{a+1}{b+1}\binom{a+2}{b+1} + \binom{a+1}{b}\binom{a+2}{b+2} - \binom{a}{b}\binom{a+3}{b+2} &= \frac{ab+2a-b}{(1+b)(2+b)}\binom{a}{b}\binom{a+2}{b+1}, \\
\binom{a+1}{b+1}\binom{a+2}{b+2} - \binom{a}{b}\binom{a+3}{b+3} &= \frac{2(a-b)}{(b+1)(b+3)}\binom{a}{b}\binom{a+2}{b+2}.
\end{aligned}
$$

Thus, we obtain that $y_0(q), y_1(q), y_2(q), y_3(q)$ is strongly q-log-concave. So we proceed to the inductive step ($n \geq 4$).

Note that

$$
\begin{aligned}
y_n(q) &= \sum_{k=0}^{n} \binom{a+n}{b+k} x_k(q) \\
&= \sum_{k=0}^{n-1} \binom{a+n-1}{b+k} [x_k(q) + x_{k+1}(q)].
\end{aligned}
$$

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Thus by the induction hypothesis and the strong q-log-concavity of $(x_k(q) + x_{k+1}(q))_{k \geq 0}$ by Fact 2.1, we have $y_0(q), y_1(q), y_2(q), \ldots, y_n(q)$ is strongly q-log-concave. This completes the proof. $\quad \square$

Remark 2.1. If we only consider the log-concavity, we don't need to prove $y_1 y_2 - y_0 y_3 \geq 0$ in induction base. Thus, it is obvious that our proof is much simpler.

3 Proof of Theorem 1.2

Proof. The proof for the second part is similar to that of the first. Therefore, for brevity, we only show that the linear transformation

$$y_n = \sum_{i=0}^{kn} \binom{n,k}{j} x_i$$

preserves the log-concavity property. We first prove the next fact.

Fact 3.1. *If $(x_n)_{n \geq 0}$ is log-concave, then so is $\left(\sum_{i=0}^{k} b_i x_{n+i}\right)_{n \geq 0}$.*

Proof. Assume that $z_n = \sum_{i=0}^{k} b_i x_{n+i}$. Let

$$B = [b_{j-i}]_{i,j \geq 0} = \begin{bmatrix} b_0 & b_1 & b_2 & \ldots & b_k & 0 & \ldots \\ 0 & b_0 & b_1 & \ldots & b_{k-1} & b_k & \ldots \\ 0 & 0 & b_0 & \ldots & b_{k-2} & b_{k-1} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix}$$

and

$$X = [x_{j+i}]_{i,j \geq 0} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & \ldots \\ x_1 & x_2 & x_3 & x_4 & x_5 & \ldots & \ldots \\ x_2 & x_3 & x_4 & x_5 & \ldots & \ldots & \ldots \\ x_3 & x_4 & x_5 & \ldots & \ldots & \ldots & \ldots \\ x_4 & x_5 & \ldots & \ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{bmatrix}.$$

Let $Z = [z_{j+i}]_{i,j \geq 0}$. It is clear that $Z = BX$. Note that B is TP_2 since $(b_i)_{0 \leq i \leq k}$ is log-concave and every minor of order 2 of X is nonpositive because $(x_n)_{n \geq 0}$ is log-concave. So the classical Cauchy-Binet Theorem shows that every minor of order 2 of the product Z of B and X is nonpositive. Hence $(z_n)_{n \geq 0}$ is log-concave. This completes the proof. $\quad \square$

In the following, we will prove that the linear transformation

$$y_n = \sum_{i=0}^{kn} \binom{n,k}{i} x_i$$

preserves the log-concavity property by induction on n.

If $0 \le n \le 2$, then by $y_n = \sum_{i=0}^{kn} \binom{n,k}{i} x_i$, we obtain that

$$
\begin{aligned}
y_1^2 - y_2 y_0 &= \left[\sum_{i=0}^{k} \binom{1,k}{i} x_i \right]^2 - x_0 \sum_{i=0}^{2k} \binom{2,k}{i} x_i \\
&= \sum_{i=0}^{2k} \left[\sum_{j=0}^{i} \binom{1,k}{i-j}\binom{1,k}{j} x_j x_{i-j} \right] - x_0 \sum_{i=0}^{2k} \sum_{j=0}^{i} \binom{1,k}{i-j}\binom{1,k}{j} x_i \\
&\ge \sum_{i=0}^{2k} \left[\sum_{j=0}^{i} \binom{1,k}{i-j}\binom{1,k}{j} x_0 x_i \right] - \sum_{i=0}^{2k} \sum_{j=0}^{i} \binom{1,k}{i-j}\binom{1,k}{j} x_i x_0 \\
&\ge 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.1)
\end{aligned}
$$

because $x_j x_{i-j} \ge x_0 x_i$ from the log-concavity of $(x_n)_{n \ge 0}$.

Thus, we get that y_0, y_1, y_2 is log-concave. So we proceed to the inductive step ($n \ge 3$). Note that

$$
\begin{aligned}
y_n &= \sum_{i=0}^{kn} \binom{n,k}{i} x_i \\
&= \sum_{i=0}^{kn} \binom{n-1,k}{i} \sum_{j=0}^{k} b_j x_{i+j}.
\end{aligned}
$$

Thus by the induction hypothesis and the log-concavity of $(\sum_{j=0}^{k} b_j x_{i+j})_{i \ge 0}$ by means of Fact 3.1, we have the sequence $(y_n)_{n \ge 0}$ is log-concave. This completes the proof. $\qquad\square$

4 Strong q-log-concavity of rows in combinatorial triangles

In this section, we will give some sufficient conditions for the strong q-log-concavity of rows in certain triangular arrays.

Theorem 4.1. *Assume that the infinite lower triangular array $[A_{n,k}(q)]_{n,k \ge 0}$ satisfies the recurrence*

$$
A_{n,k}(q) = f_{n,k}(q) A_{n-1,k-1}(q) + g_{n,k}(q) A_{n-1,k}(q) + h_{n,k}(q) A_{n-1,k+1}(q) \qquad (4.1)
$$

for $n \ge 1$ and $k \ge 0$, where $A_{0,0}(q) = 1$, $A_{0,k}(q) = A_{0,-1}(q) = 0$ for $k > 0$. Assume that three sequences of polynomials with nonnegative coefficients $(f_{n,k}(q))_{k \ge 0}$, $(g_{n,k}(q))_{k \ge 0}$ and $(h_{n,k}(q))_{k \ge 0}$ are strongly q-log-concave in k, respectively. If

$$
\begin{aligned}
f_{n,l}(q)g_{n,k}(q) + f_{n,k}(q)g_{n,l}(q) &\ge_q f_{n,l+1}(q)g_{n,k-1}(q) + f_{n,k-1}(q)g_{n,l+1}(q), \\
g_{n,l}(q)h_{n,k}(q) + g_{n,k}(q)h_{n,l}(q) &\ge_q g_{n,l+1}(q)h_{n,k-1}(q) + g_{n,k-1}(q)h_{n,l+1}(q), \\
f_{n,l}(q)h_{n,k}(q) + f_{n,k}(q)h_{n,l}(q) &\ge_q f_{n,k-1}(q)h_{n,l+1}(q) + f_{n,l+1}(q)h_{n,k-1}(q)
\end{aligned}
$$

and $g_{n,k}(q)g_{n,l}(q) \ge_q f_{n,l+1}(q)h_{n,k-1}(q)$

for all $1 \le k \le l \le n$, then, for any fixed n, the row sequence $(A_{n,k}(q))_{0 \le k \le n}$ is strongly q-log-concave in k. In particular, each row sequence $(A_{n,k}(q))_{0 \le k \le n}$ is log-concave in k for any fixed $q \ge 0$.

Proof. In order to prove that $(A_{n,k}(q))_{0 \leq k \leq n}$ is strongly q-log-concave in k, we only need to prove

$$A_{n,l}(q)A_{n,k}(q) - A_{n,l+1}(q)A_{n,k-1}(q) \geq_q 0$$

for any $l \geq k$, which will be done by induction on n. It is obvious for $n = 0$. So, we assume that it follows for $n \leq m - 1$. For brevity, we write f_k (resp. g_k, h_k) for $f_{n,k}(q)$ (resp. $g_{n,k}(q)$, $h_{n,k}(q)$). Then for $n = m$ and $1 \leq k \leq l \leq m$, by (4.1), we have

$$
\begin{aligned}
&A_{m,l}(q)A_{m,k}(q) - A_{m,l+1}(q)A_{m,k-1}(q) \\
={}& [f_l A_{m-1,l-1}(q) + g_l A_{m-1,l}(q) + h_l A_{m-1,l+1}(q)] \times \\
&[f_k A_{m-1,k-1}(q) + g_k A_{m-1,k}(q) + h_k A_{m-1,k+1}(q)] - \\
&[f_{l+1} A_{m-1,l}(q) + g_{l+1} A_{m-1,l+1}(q) + h_{l+1} A_{m-1,l+2}(q)] \times \\
&[f_{k-1} A_{m-1,k-2}(q) + g_{k-1} A_{m-1,k-1}(q) + h_{k-1} A_{m-1,k}(q)] \\
={}& \underbrace{f_l f_k A_{m-1,l-1}(q) A_{m-1,k-1}(q) - f_{l+1} f_{k-1} A_{m-1,l}(q) A_{m-1,k-2}(q)} + \\
& f_l g_k A_{m-1,l-1}(q) A_{m-1,k}(q) - f_{k-1} g_{l+1} A_{m-1,l+1}(q) A_{m-1,k-2}(q) + \\
& [f_k g_l - f_{l+1} g_{k-1}] A_{m-1,l}(q) A_{m-1,k-1}(q) + \\
& \underbrace{h_l h_k A_{m-1,l+1}(q) A_{m-1,k+1}(q) - h_{l+1} h_{k-1} A_{m-1,l+2}(q) A_{m-1,k}(q)} + \\
& h_k g_l A_{m-1,l}(q) A_{m-1,k+1}(q) - h_{l+1} g_{k-1} A_{m-1,l+2}(q) A_{m-1,k-1}(q) + \\
& [h_l g_k - h_{k-1} g_{l+1}] A_{m-1,l+1}(q) A_{m-1,k}(q) + \\
& \underbrace{g_l g_k A_{m,l}(q) A_{m,k}(q) - g_{l+1} g_{k-1} A_{m-1,l+1}(q) A_{m-1,k-1}(q)} + \\
& f_l h_k A_{m-1,l-1}(q) A_{m-1,k+1}(q) - h_{l+1} f_{k-1} A_{m-1,l+2}(q) A_{m-1,k-2}(q) + \\
& [h_l f_k A_{m-1,l+1}(q) A_{m-1,k-1}(q) - f_{l+1} h_{k-1} A_{m-1,l}(q) A_{m-1,k}(q)]. \qquad (4.2)
\end{aligned}
$$

In what follows we will prove the nonnegativity of (4.2) in q.

Firstly, it follows from the strong q-log-concavities of $(A_{m-1,k}(q))_{0 \leq k \leq m-1}$ and $(f_k(q))_{k \geq 0}$ that

$$A_{m-1,l-1}(q)A_{m-1,k-1}(q) - A_{m-1,l}(q)A_{m-1,k-2}(q) \geq_q 0$$

and

$$f_l f_k - f_{l+1} f_{k-1} \geq_q 0,$$

which implies

$$f_l f_k A_{m-1,l-1}(q)A_{m-1,k-1}(q) - f_{l+1} f_{k-1} A_{m-1,l}(q)A_{m-1,k-2}(q) \geq_q 0. \qquad (4.3)$$

Similarly, we also have

$$h_l h_k A_{m-1,l+1}(q)A_{m-1,k+1}(q) - h_{l+1} h_{k-1} A_{m-1,l+2}(q)A_{m-1,k}(q) \geq_q 0. \qquad (4.4)$$

Secondly, by the strong q-log-concavity of $(A_{m-1,k}(q))_{0 \leq k \leq m-1}$ and

$$f_l g_k + f_k g_l \geq_q f_{l+1} g_{k-1} + f_{k-1} g_{l+1},$$

we get

$$
\begin{aligned}
&f_l g_k A_{m-1,l-1}(q) A_{m-1,k}(q) - f_{k-1} g_{l+1} A_{m-1,l+1}(q) A_{m-1,k-2}(q) + \\
&[f_k g_l - f_{l+1} g_{k-1}] A_{m-1,l}(q) A_{m-1,k-1}(q) \\
\geq_q\ &[f_l g_k + f_k g_l - f_{l+1} g_{k-1}] A_{m-1,l}(q) A_{m-1,k-1}(q) - f_{k-1} g_{l+1} A_{m-1,l+1}(q) A_{m-1,k-2}(q) \\
\geq_q\ &[f_l g_k + f_k g_l - f_{l+1} g_{k-1} - f_{k-1} g_{l+1}] A_{m-1,l}(q) A_{m-1,k-1}(q) \\
\geq_q\ &0.
\end{aligned} \tag{4.5}
$$

In a similar way, we also have

$$
\begin{aligned}
&h_k g_l A_{m-1,l}(q) A_{m-1,k+1}(q) - h_{l+1} g_{k-1} A_{m-1,l+2}(q) A_{m-1,k-1}(q) + \\
&[h_l g_k - h_{k-1} g_{l+1}] A_{m-1,l+1}(q) A_{m-1,k}(q) \\
\geq_q\ &[h_k g_l + h_l g_k - h_{k-1} g_{l+1} - h_{l+1} g_{k-1}] A_{m-1,l+1}(q) A_{m-1,k}(q) \\
\geq_q\ &0.
\end{aligned} \tag{4.6}
$$

Finally, it follows from the strong q-log-concavity of $(g_k(q))_{k \geq 0}$ that

$$
g_l g_k - g_{l+1} g_{k-1} \geq_q 0.
$$

Thus we have

$$
\begin{aligned}
&\underbrace{g_l g_k A_{m-1,l}(q) A_{m-1,k}(q) - g_{l+1} g_{k-1} A_{m-1,l+1}(q) A_{m-1,k-1}(q)} + \\
&f_l h_k A_{m-1,l-1}(q) A_{m-1,k+1}(q) - h_{l+1} f_{k-1} A_{m-1,l+2}(q) A_{m-1,k-2}(q) + \\
&[h_l f_k A_{m-1,l+1}(q) A_{m-1,k-1}(q) - f_{l+1} h_{k-1} A_{m-1,l}(q) A_{m-1,k}(q)] \\
\geq_q\ &\underbrace{g_l g_k [A_{m-1,l}(q) A_{m-1,k}(q) - A_{m-1,l+1}(q) A_{m-1,k-1}(q)]} + \\
&f_l h_k A_{m-1,l-1}(q) A_{m-1,k+1}(q) - h_{l+1} f_{k-1} A_{m-1,l+2}(q) A_{m-1,k-2}(q) + \\
&\underbrace{[f_{l+1} h_{k-1} + h_{l+1} f_{k-1} - f_l h_k]} A_{m-1,l+1}(q) A_{m-1,k-1}(q) - \\
&f_{l+1} h_{k-1} A_{m-1,l}(q) A_{m-1,k}(q)] \\
=\ &\underbrace{g_l g_k [A_{m-1,l}(q) A_{m-1,k}(q) - A_{m-1,l+1}(q) A_{m-1,k-1}(q)]} + \\
&\underbrace{f_{l+1} h_{k-1} [A_{m-1,l+1}(q) A_{m-1,k-1}(q) - A_{m-1,l}(q) A_{m-1,k}(q)]} + \\
&f_l h_k [A_{m-1,l-1}(q) A_{m-1,k+1}(q) - A_{m-1,l+1}(q) A_{m-1,k-1}(q)] + \\
&h_{l+1} f_{k-1} [A_{m-1,l+1}(q) A_{m-1,k-1}(q) - A_{m-1,l+2}(q) A_{m-1,k-2}(q)] \\
\geq_q\ &\underbrace{[g_l g_k - f_{l+1} h_{k-1}][A_{m-1,l}(q) A_{m-1,k}(q) - A_{m-1,l+1}(q) A_{m-1,k-1}(q)]} \\
\geq_q\ &0
\end{aligned} \tag{4.7}
$$

since

$$
f_l h_k + f_k h_l \geq_q f_{k-1} h_{l+1} + f_{l+1} h_{k-1}
$$

and

$$
g_k g_l \geq_q f_{l+1} h_{k-1}.
$$

Thus, by (4.2)- (4.7), we get

$$
A_{m,k}(q) A_{m,l}(q) - A_{m,l+1}(q) A_{m,k-1}(q) \geq_q 0
$$

for $1 \leq k \leq l \leq m$. The proof is complete. $\qquad\square$

The following special case related to the Riordan array [23] may be more interesting.

Proposition 4.2. *Define the matrix* $[A_{n,k}(q)]_{n,k\geq 0}$ *recursively:*

$$
\begin{aligned}
A_{0,0}(q) &= 1, \quad A_{0,k}(q) = 0 \quad (k > 0), \\
A_{n,0}(q) &= e(q)\, A_{n-1,0}(q) + h(q)\, A_{n-1,1}(q), \\
A_{n,k}(q) &= A_{n-1,k-1}(q) + g(q)\, A_{n-1,k}(q) + h(q)\, A_{n-1,k+1}(q) \quad (n, k \geq 1).
\end{aligned}
$$

If $e(q)g(q) \geq_q h(q) \geq_q 0$ *and* $g(q) \geq_q e(q) \geq_q 0$, *then each row* $(A_{n,k}(q))_{0 \leq k \leq n}$ *is strongly q-log-concave.*

Applying Theorem 4.1 to combinatorial arrays in Example 1.3, we have the following results in a unified manner.

Corollary 4.3. *Each row sequence of the Bell polynomials triangle, the Eulerian polynomials triangle and the Narayana polynomials triangle is strongly q-log-concave, respectively.*

Corollary 4.4. *Each row sequence in the Catalan triangles of Aigner and Shaprio, the Motzkin triangle, the large Schröder triangle, and the Bell triangle is log-concave, respectively.*

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