

THE DESCENT STATISTIC ON SIGNED SIMSUN PERMUTATIONS

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ABSTRACT. In this paper we study the generating polynomials obtained by enumerating signed simsun permutations by number of the descents. Properties of the polynomials, including the recurrence relations and generating functions are studied.

Keywords: Signed simsun permutations; Even-signed simsun permutations; Descents

1. INTRODUCTION

Let \mathfrak{S}_n denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \dots, n\}$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *descent* in π is an element $\pi(i)$ such that $\pi(i) > \pi(i+1)$, where $i \in [n-1]$. We say that $\pi \in \mathfrak{S}_n$ has no *double descents* if there is no index $i \in [n-2]$ such that $\pi(i) > \pi(i+1) > \pi(i+2)$. A permutation $\pi \in \mathfrak{S}_n$ is called *simsun* if for all k , the subword of π restricted to $[k]$ (in the order they appear in π) contains no double descents. For example, 35142 is simsun, but 35241 is not. Let \mathcal{RS}_n be the set of simsun permutations of length n . Let $|C|$ denote the cardinality of a set C . Simion and Sundaram [14, p. 267] discovered that $|\mathcal{RS}_n| = E_{n+1}$, where E_n is the n th Euler number, which also is the number alternating permutations in \mathfrak{S}_n (see [13] for instance). Simsun permutations are a variant of the André permutations of Foata and Schützenberger [9]. We refer the reader to [3, 5, 6, 10, 11] for some recent results related to simsun permutations.

There have been extensive studies of the descent polynomials for simsun permutations (see [5, 11] for instance). Let $S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\text{des}_A(\pi)}$, where

$$\text{des}_A(\pi) = |\{i \in [n-1] : \pi(i) > \pi(i+1)\}|.$$

It follows from [14, Theorem 3.2] that the polynomials $S_n(x)$ satisfy the recurrence relation

$$S_{n+1}(x) = (1 + nx)S_n(x) + x(1 - 2x)S'_n(x),$$

with $S_0(x) = 1$. Let $RS(x, z) = \sum_{n \geq 0} S_n(x) \frac{z^n}{n!}$. Chow and Shiu [5, Theorem 2.1] obtained that

$$RS(x, z) = \left(\frac{\sqrt{2x-1} \sec\left(\frac{z}{2}\sqrt{2x-1}\right)}{\sqrt{2x-1} - \tan\left(\frac{z}{2}\sqrt{2x-1}\right)} \right)^2.$$

Recall that the classical *Eulerian polynomials* of type A are defined by $A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}_A(\pi)}$. From [9, Proposition 2.7], we have

$$S_n\left(\frac{2x}{(1+x)^2}\right) = \frac{A_{n+1}(x)}{(1+x)^n}.$$

A *left peak* in π is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where we take $\pi(0) = 0$. Let $\text{lpk}(\pi)$ denote the number of left peaks in π . For example, $\text{lpk}(21435) = 2$. It is clear that every descent of a simsun permutation is a left peak. Hence $S_n(x) = \sum_{\pi \in \mathcal{RS}_n} x^{\text{lpk}(\pi)}$. Let $\widehat{W}_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{lpk}(\pi)}$. There is a close connection between $S_n(x)$ and $\widehat{W}_n(x)$ (see [11, Eq. (6)]):

$$S_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} \widehat{W}_k(2x) \widehat{W}_{n-k}(2x).$$

It is now well known that simsun permutations and signed simsun permutations are useful in computing the *cd*-index of the Boolean algebra and the cubical lattice, respectively (see [1, 7]). Let B_n be the hyperoctahedral group of rank n . Elements π of B_n are signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i , where $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$. Throughout this paper, we always identify a signed permutation $\pi = \pi(1) \cdots \pi(n)$ with the word $\pi(0)\pi(1) \cdots \pi(n)$, where $\pi(0) = 0$. A double descent of $\pi \in B_n$ is an index $i \in [n-1]$ such that $\pi(i-1) > \pi(i) > \pi(i+1)$. Let $R(\pm[k])$ be the set of signed permutations of B_k with no double descents. Following [7, Section 7], a *signed simsun permutation* π of length n is a permutation of B_n such that for all $0 \leq k \leq n$, if we remove the k entries $\pm n, \pm(n-1), \dots, \pm(n-k+1)$ from π , the resulting permutation belongs to $R(\pm[n-k])$. Let \mathcal{RB}_n denote the set of signed simsun permutations of B_n . For example, $01(-3)2(-5)4$ is a signed simsun permutations, while $\pi' = 01(-3)2(-6)(-4)(-5)$ is not, since when we remove ± 6 from π' , the resulting permutation $01(-3)2(-4)(-5)$ contains a double descent. In particular,

$$\mathcal{RB}_2 = \{012, 01(-2), 021, 0(-2)1, 0(-1)2, 02(-1), 0(-2)(-1)\}.$$

Denote by D_n the set of even-signed permutations of B_n , i.e., for any $\pi \in D_n$, the set $\{\pi(1), \pi(2), \dots, \pi(n)\}$ contains an even number of negative terms. For $n \geq 2$, D_n forms a normal subgroup of B_n of index 2. Set $T_n = B_n \setminus D_n$. Following [4], the descent statistic of type B is defined by

$$\text{des}_B(\pi) = |\{i \in \{0, 1, 2, \dots, n-1\} : \pi(i) > \pi(i+1)\}|.$$

Let

$$E(n, k) = |\{\pi \in D_n : \text{des}_B(\pi) = k\}|, \quad \widetilde{E}(n, k) = |\{\pi \in T_n : \text{des}_B(\pi) = k\}|.$$

Very recently, Borowiec and Mlotkowski [2] studied the type D Eulerian numbers $E(n, k)$ and $\widetilde{E}(n, k)$. In particular, they discovered a remarkable formula (see [2, Proposition 4.3]):

$$E(n, k) - \widetilde{E}(n, k) = (-1)^k \binom{n}{k}. \quad (1)$$

Let \mathcal{RD}_n and \mathcal{RT}_n denote the sets of simsun permutations of D_n and T_n , respectively. Let

$$D(n, k) = |\{\pi \in \mathcal{RD}_n : \text{des}_B(\pi) = k\}|, \quad T(n, k) = |\{\pi \in \mathcal{RT}_n : \text{des}_B(\pi) = k\}|.$$

As a correspondence of 1, we recently observed the following formula:

$$D(n, k) - T(n, k) = (-1)^k \binom{n-k+1}{k}, \quad (2)$$

which is implied by Theorem 8 of Section 3. Motivated by (2), we shall study the polynomials obtained by enumerating permutations of $\mathcal{RB}_n, \mathcal{RD}_n$ and \mathcal{RT}_n by number of the descents.

2. ON SIGNED SIMSUN PERMUTATIONS OF B_n

Let $\mathcal{RB}_n^+ = \{\pi \in \mathcal{RB}_n : \pi(1) > 0\}$ and $\mathcal{RB}_n^- = \{\pi \in \mathcal{RB}_n : \pi(1) < 0\}$. We define

$$R_n^+(x) = \sum_{\pi \in \mathcal{RB}_n^+} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} R^+(n, k)x^k,$$

$$R_n^-(x) = \sum_{\pi \in \mathcal{RB}_n^-} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} R^-(n, k)x^k,$$

$$R_n(x) = \sum_{\pi \in \mathcal{RB}_n} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} R(n, k)x^k.$$

It is clear that $R(n, k) = R^+(n, k) + R^-(n, k)$. The following lemma is a fundamental result.

Lemma 1. *For $n \geq 2$, we have*

$$R^+(n, k) = 2kR^+(n-1, k) + (2n-4k+2)R^+(n-1, k-1) + R(n-1, k), \quad (3)$$

$$R^-(n, k) = 2kR^-(n-1, k) + (2n-4k+3)R^-(n-1, k-1) + R(n-1, k-1), \quad (4)$$

$$R(n, k) = (2k+1)R(n-1, k) + (2n-4k+3)R(n-1, k-1) + R^-(n-1, k-1). \quad (5)$$

Proof. Define

$$\mathcal{RB}_{n,k}^+ = \{\pi \in \mathcal{RB}_n^+ : \text{des}(\pi) = k\},$$

$$\mathcal{RB}_{n,k}^- = \{\pi \in \mathcal{RB}_n^- : \text{des}(\pi) = k\}.$$

Firstly, we prove (3). In order to get a permutation $\pi' \in \mathcal{RB}_{n,k}^+$ from a permutation $\pi \in \mathcal{RB}_{n-1}$, we distinguish among the following cases:

- (i) If $\pi \in \mathcal{RB}_{n-1,k}^+$, then we can insert the entry n right after a descent or insert the entry $-n$ right after a descent of π . Moreover, we can put the entry n at the end of π . This accounts for $(2k+1)R^+(n-1, k)$ possibilities.
- (ii) If $\pi \in \mathcal{RB}_{n-1,k-1}^+$, then we cannot insert the entry n immediately before or right after each descent of π and we cannot insert the entry n at the end of π . Moreover, we cannot insert the entry $-n$ immediately before or right after each descent bottom and we also cannot insert $-n$ right after $\pi(0)$, where a descent bottom is an entry $\pi(i)$ such that $\pi(i-1) > \pi(i)$ and $1 \leq i \leq n-1$. Hence there are $n-2k+1$ positions could be inserted the entry n or $-n$. This accounts for $(2n-4k+2)R^+(n-1, k-1)$ possibilities.
- (iii) If $\pi \in \mathcal{RB}_{n-1,k}^-$, then we have to put n right after $\pi(0)$. This accounts for $R^-(n-1, k)$ possibilities.

Therefore,

$$\begin{aligned} R^+(n, k) &= (2k+1)R^+(n-1, k) + (2n-4k+2)R^+(n-1, k-1) + R^-(n-1, k) \\ &= 2kR^+(n-1, k) + (2n-4k+2)R^+(n-1, k-1) + R(n-1, k). \end{aligned}$$

Secondly, we prove (4). In order to get a permutation $\pi' \in \mathcal{RB}_{n,k}^-$ from a permutation $\pi \in \mathcal{RB}_{n-1}$, we also distinguish among the following cases:

- (i) If $\pi \in \mathcal{RB}_{n-1,k}^-$, then we can insert the entry n right after any descent except the first descent or insert the entry $-n$ right after any descent. Moreover, we can insert n at the end of π . This accounts for $2kR^-(n-1, k)$ possibilities.

- (ii) If $\pi \in \mathcal{RB}_{n-1, k-1}^-$, we cannot insert the entry n immediately before or right after any descent and we cannot insert the entry $-n$ immediately before or right after each descent bottom. Moreover, we cannot put n at the end of π . Note that $\pi(0)$ is a descent. Hence there are $n - 1 - 1 - 2(k - 2) = n - 2k + 2$ positions could be inserted the entry n and there are $n - 2(k - 1) = n - 2k + 2$ positions could be inserted the entry $-n$. This accounts for $(2n - 4k + 4)R^-(n - 1, k - 1)$ possibilities.
- (iii) If $\pi \in \mathcal{RB}_{n-1, k-1}^+$, then we have to put the entry $-n$ right after $\pi(0)$. This accounts for $R^+(n - 1, k - 1)$ possibilities.

Therefore,

$$\begin{aligned} R^-(n, k) &= 2kR^-(n - 1, k) + (2n - 4k + 4)R^-(n - 1, k - 1) + R^+(n - 1, k - 1) \\ &= 2kR^-(n - 1, k) + (2n - 4k + 3)R^-(n - 1, k - 1) + R(n - 1, k - 1). \end{aligned}$$

Finally, combining (3) and (4), we immediately get (5). \square

So the following proposition is immediate.

Proposition 2. *The polynomials $R_n^+(x)$, $R_n^-(x)$ and $R_n(x)$ satisfy the following recurrence relations*

$$\begin{aligned} R_{n+1}^+(x) &= 2nxR_n^+(x) + 2x(1 - 2x)\frac{d}{dx}R_n^+(x) + R_n(x), \\ R_{n+1}^-(x) &= (2n + 1)xR_n^-(x) + 2x(1 - 2x)\frac{d}{dx}R_n^-(x) + xR_n(x), \\ R_{n+1}(x) &= (1 + (2n + 1)x)R_n(x) + 2x(1 - 2x)\frac{d}{dx}R_n(x) + xR_n^-(x), \end{aligned}$$

with the initial conditions $R_0^+(x) = R_0(x) = 1$ and $R_0^-(x) = 0$.

By Proposition 2, it is easy to verify that $\deg R_n^+(x) = \lfloor n/2 \rfloor$ and $\deg R_n^-(x) = \deg R_n(x) = \lceil n/2 \rceil$. For convenience, we list the first few terms of $R_n^+(x)$, $R_n^-(x)$ and $R_n(x)$:

$$\begin{aligned} R_1^+(x) &= 1, R_2^+(x) = 1 + 3x, R_3^+(x) = 1 + 16x, R_4^+(x) = 1 + 61x + 41x^2; \\ R_1^-(x) &= x, R_2^-(x) = 3x, R_3^-(x) = 7x + 9x^2, R_4^-(x) = 15x + 80x^2; \\ R_1(x) &= 1 + x, R_2(x) = 1 + 6x, R_3(x) = 1 + 23x + 9x^2, R_4(x) = 1 + 76x + 121x^2. \end{aligned}$$

Define

$$\begin{aligned} R^+(x; t) &= \sum_{n \geq 0} R_n^+(x) \frac{t^n}{n!}; \\ R^-(x; t) &= \sum_{n \geq 0} R_n^-(x) \frac{t^n}{n!}; \\ R(x; t) &= \sum_{n \geq 0} R_n(x) \frac{t^n}{n!}. \end{aligned}$$

Clearly, $R(x; t) = R^+(x; t) + R^-(x; t)$. We can now conclude the first main result of this paper.

Theorem 3. Let $u(x, t) = t\sqrt{2x-1} - \arctan(\sqrt{2x-1})$. The generating functions $R^+(x; t)$ and $R^-(x; t)$ are respectively given by

$$R^+(x; t) = \frac{\sqrt{2x-1}}{2\sqrt{x}p(x)} (p^2(x)F_2(u(x, t)) + F_1(u(x, t))),$$

$$R^-(x; t) = \frac{-\sqrt{2x-1}}{2p(x)} (p^2(x)F_2(u(x, t)) - F_1(u(x, t))),$$

where $p(x) = \left(\frac{\sqrt{2x+1}}{\sqrt{2x-1}}\right)^{\frac{\sqrt{2}}{4}}$ and

$$F_1(x) = \frac{-1}{\sqrt{2}\sin(x)} \left(\frac{-\sin(x)}{1+\cos(x)}\right)^{-\frac{1}{\sqrt{2}}}, \quad F_2(x) = \frac{-1}{\sqrt{2}\sin(x)} \left(\frac{-\sin(x)}{1+\cos(x)}\right)^{\frac{1}{\sqrt{2}}}.$$

Proof. Note that $R(x; t) = R^+(x; t) + R^-(x; t)$. By writing the statement of Proposition 2 in terms of generating functions $R^+(x; t)$ and $R^-(x; t)$, we obtain

$$(1-2xt)\frac{d}{dt}R^+(x; t) = 2x(1-2x)\frac{d}{dx}R^+(x; t) + R^+(x; t) + R^-(x; t),$$

$$(1-2xt)\frac{d}{dt}R^-(x; t) = 2x(1-2x)\frac{d}{dx}R^-(x; t) + xR^+(x; t) + 2xR^-(x; t).$$

In order to solve this system of partial differential equations, let us define

$$M(x; t) = a(x)R^+(x; t) + b(x)R^-(x; t),$$

where $a(x)$ and $b(x)$ are functions on x . It follows that $M(x; t)$ satisfies

$$(1-2xt)\frac{d}{dt}M(x; t) = 2x(1-2x)\frac{d}{dx}M(x; t)$$

$$+ \left(a(x) + xb(x) - 2x(1-2x)\frac{d}{dx}a(x)\right)R^+(x; t)$$

$$+ \left(2xb(x) + a(x) - 2x(1-2x)\frac{d}{dx}b(x)\right)R^-(x; t).$$

Now, let us assume that the functions $a(x)$ and $b(x)$ satisfies the following system of differential equations:

$$a(x) + xb(x) - 2x(1-2x)\frac{d}{dx}a(x) = 0,$$

$$2xb(x) + a(x) - 2x(1-2x)\frac{d}{dx}b(x) = 0.$$

The system has the following general solution

$$a(x) = \frac{\sqrt{x}}{\sqrt{2x-1}}(c/p(x) + dp(x)), \quad b(x) = \frac{1}{\sqrt{2x-1}}(-c/p(x) + dp(x)),$$

where

$$p(x) = \left(\frac{\sqrt{2x+1}}{\sqrt{2x-1}}\right)^{\frac{\sqrt{2}}{4}}.$$

From now, let us assume that the pair of the functions $(a(x), b(x))$ are either $(a_1(x), b_1(x)) = \left(\frac{\sqrt{x}p(x)}{\sqrt{2x-1}}, \frac{p(x)}{\sqrt{2x-1}}\right)$ or $(a_2(x), b_2(x)) = \left(\frac{\sqrt{x}}{\sqrt{2x-1}p(x)}, \frac{-1}{\sqrt{2x-1}p(x)}\right)$. Hence, the generating function $M(x; t) = M_{a_j, b_j}(x; t)$ for a pair $(a_j(x), b_j(x))$, $j = 1, 2$, satisfies

$$(1 - 2xt) \frac{d}{dt} M_{a_j, b_j}(x; t) = 2x(1 - 2x) \frac{d}{dx} M_{a_j, b_j}(x; t) = 0,$$

which implies that there exist functions F_1 and F_2 such that

$$M_{a_j, b_j}(x; t) = F_j(t\sqrt{2x-1} - \arctan(\sqrt{2x-1})).$$

Since $R^+(x; 0) = R_0^+(x) = 1$ and $R^-(x; 0) = R_0^-(x) = 0$, we see that $M_{a_j, b_j}(x; 0) = a_j(x) = F_j(-\arctan(\sqrt{2x-1}))$. Thus,

$$F_1(x) = -\frac{p((\tan^2(x) + 1)/2)}{\sqrt{2} \sin(x)} = \frac{-1}{\sqrt{2} \sin(x)} \left(\frac{-\sin(x)}{1 + \cos(x)} \right)^{\frac{-1}{\sqrt{2}}},$$

$$F_2(x) = -\frac{p((\tan^2(x) + 1)/2)}{\sqrt{2} \sin(x)} = \frac{-1}{\sqrt{2} \sin(x)} \left(\frac{-\sin(x)}{1 + \cos(x)} \right)^{\frac{1}{\sqrt{2}}}.$$

Therefore,

$$a_1(x)R^+(x; t) + b_1(x)R^-(x; t) = F_1(u(x, t)), \quad a_2(x)R^+(x; t) + b_2(x)R^-(x; t) = F_2(u(x, t)),$$

which implies

$$R^+(x; t) = \frac{\sqrt{2x-1}}{2\sqrt{xp(x)}} (p^2(x)F_2(u(x, t)) + F_1(u(x, t))),$$

$$R^-(x; t) = \frac{-\sqrt{2x-1}}{2p(x)} (p^2(x)F_2(u(x, t)) - F_1(u(x, t))).$$

This completes the proof. \square

Here we give two examples as applications of Theorem 3.

Example 4. Let $r_n = |\mathcal{RB}_n| = \sum_{k \geq 0} R(n, k)$. In order to get the first few terms of r_n , we need to expand the generating function $R(1; t)$. Let $\alpha = \sqrt{2}$. Then for $x = 1$, we get

$$\begin{aligned} R(1; t) &= \left(\frac{\alpha + 1}{\alpha - 1} \right)^{-\alpha/4} \frac{1}{\cos t - \sin t} \left(\frac{\sin(t - \pi/4)}{\cos(t - \pi/4) - 1} \right)^{\alpha/2} \\ &= \frac{(\cos t - \sin t)^{\alpha/2-1}}{(\alpha - \sin t - \cos t)^{\alpha/2} (1 + \alpha)^{\alpha/2}} \\ &= \frac{1}{(\alpha^2 - 1)^{\alpha/2}} \left(1 + \frac{\alpha^2 - 4\alpha + 2}{2(1 - \alpha)} t + \frac{\alpha^4 - 12\alpha^3 + 34\alpha^2 - 32\alpha + 12}{8(1 - \alpha)^2} t^2 + \dots \right) \\ &= 1 + 2t + 7\frac{t^2}{2!} + 33\frac{t^3}{3!} + 198\frac{t^4}{4!} + 1439\frac{t^5}{5!} + 12291\frac{t^6}{6!} + 120622\frac{t^7}{7!} + \dots \end{aligned}$$

Example 5. Let $s_n = \sum_{k \geq 0} kR(n, k)$. In order to get the first few terms of s_n , we need to expand the generating function $R'(1; t) = \frac{d}{dx} R(x; t) |_{x=1}$. Let $\alpha = \sqrt{2}$. Then by Theorem 3, we

obtain

$$\begin{aligned} R'(1;t) &= \left(\frac{\alpha-1}{\alpha+1}\right)^{\alpha/4} \frac{(3+4t)\cos t + (4t-7)\sin t + 4t-2}{4(1-\sin(2t))} \left(\frac{\cos t - \sin t}{\alpha + \sin t + \cos t}\right)^{-\alpha/2} \\ &\quad + \left(\frac{\alpha+1}{\alpha-1}\right)^{\alpha/4} \frac{\sin t - \cos t}{4(1-\sin(2t))} \left(\frac{\cos t - \sin t}{\alpha + \sin t + \cos t}\right)^{\alpha/2} \\ &= t + 6\frac{t^2}{2!} + 41\frac{t^3}{3!} + 318\frac{t^4}{4!} + 2840\frac{t^5}{5!} + 28736\frac{t^6}{6!} + 325991\frac{t^7}{7!} + \dots \end{aligned}$$

3. ON EVEN-SIGNED SIMSUN PERMUTATIONS

Let

$$\begin{aligned} \mathcal{RD}_n^+ &= \{\pi \in \mathcal{RD}_n : \pi(1) > 0\}, \quad \mathcal{RD}_n^- = \{\pi \in \mathcal{RD}_n : \pi(1) < 0\}, \\ \mathcal{RT}_n^+ &= \{\pi \in \mathcal{RT}_n : \pi(1) > 0\}, \quad \mathcal{RT}_n^- = \{\pi \in \mathcal{RT}_n : \pi(1) < 0\}. \end{aligned}$$

Define

$$\begin{aligned} D_n^+(x) &= \sum_{\pi \in \mathcal{RD}_n^+} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} D^+(n, k)x^k, \quad D_n^-(x) = \sum_{\pi \in \mathcal{RD}_n^-} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} D^-(n, k)x^k, \\ T_n^+(x) &= \sum_{\pi \in \mathcal{RT}_n^+} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} T^+(n, k)x^k, \quad T_n^-(x) = \sum_{\pi \in \mathcal{RT}_n^-} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} T^-(n, k)x^k, \\ D_n(x) &= \sum_{\pi \in \mathcal{RD}_n} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} D(n, k)x^k, \quad T_n(x) = \sum_{\pi \in \mathcal{RT}_n} x^{\text{des}_B(\pi)} = \sum_{k \geq 0} T(n, k)x^k. \end{aligned}$$

Clearly, $R^+(n, k) = D^+(n, k) + T^+(n, k)$, $R^-(n, k) = D^-(n, k) + T^-(n, k)$ and $R(n, k) = D(n, k) + T(n, k)$.

Lemma 6. *We have*

$$\begin{aligned} D^+(n, k) &= kR^+(n-1, k) + (n-2k+1)R^+(n-1, k-1) + D(n-1, k); \\ T^+(n, k) &= kR^+(n-1, k) + (n-2k+1)R^+(n-1, k-1) + T(n-1, k); \\ D^-(n, k) &= kR^-(n-1, k) + (n-2k+2)R^-(n-1, k-1) + T^+(n-1, k-1); \\ T^-(n, k) &= kR^-(n-1, k) + (n-2k+2)R^-(n-1, k-1) + D^+(n-1, k-1); \\ D(n, k) &= (1+k)D(n-1, k) + (n-2k+1)D(n-1, k-1) + kT(n-1, k) + \\ &\quad (n-2k+2)T(n-1, k-1) + D^{-1}(n-1, k-1); \\ T(n, k) &= (1+k)T(n-1, k) + (n-2k+1)T(n-1, k-1) + kD(n-1, k) + \\ &\quad (n-2k+2)D(n-1, k-1) + T^{-1}(n-1, k-1). \end{aligned}$$

Proof. Define

$$\begin{aligned} \mathcal{RD}_{n,k}^+ &= \{\pi \in \mathcal{RD}_n^+ : \text{des}(\pi) = k\}, \quad \mathcal{RD}_{n,k}^- = \{\pi \in \mathcal{RD}_n^- : \text{des}(\pi) = k\}; \\ \mathcal{RT}_{n,k}^+ &= \{\pi \in \mathcal{RT}_n^+ : \text{des}(\pi) = k\}, \quad \mathcal{RT}_{n,k}^- = \{\pi \in \mathcal{RT}_n^- : \text{des}(\pi) = k\}. \end{aligned}$$

We only prove the first recurrence relation and the others can be proved in the same way.

In order to get a permutation $\pi' \in \mathcal{RD}_{n,k}^+$ from a permutation $\pi \in \mathcal{RB}_{n-1}$, we distinguish among the following cases:

- (i) If $\pi \in \mathcal{RD}_{n-1,k}^+$, then we can insert the entry n right after any descent. Moreover, we can also put n at the end of π . This accounts for $(k+1)D^+(n-1, k)$ possibilities.
- (ii) If $\pi \in \mathcal{RD}_{n-1,k-1}^+$, we cannot insert n immediately before or right after any descent, and we cannot put n at the end of π . Thus n can be inserted into the remaining $n-1-2(k-1) = n-2k+1$ positions. This accounts for $(n-2k+1)D^+(n-1, k-1)$ possibilities.
- (iii) If $\pi \in \mathcal{RD}_{n-1,k}^-$, then we have to insert n right after $\pi(0)$. This accounts for $D^-(n-1, k)$ possibilities.
- (iv) If $\pi \in \mathcal{RT}_{n-1,k}^+$, then we can insert $-n$ right after any descent. This accounts for $kT^+(n-1, k)$ possibilities.
- (v) If $\pi \in \mathcal{RT}_{n-1,k-1}^+$, then we cannot insert $-n$ immediately before or right after any descent bottom. Moreover, we cannot put $-n$ right after $\pi(0)$. Thus the entry $-n$ can be inserted into the remaining $n-1-2(k-1) = n-2k+1$ positions. This accounts for $(n-2k+1)T^+(n-1, k-1)$ possibilities.

Therefore,

$$\begin{aligned}
D^+(n, k) &= (k+1)D^+(n-1, k) + (n-2k+1)D^+(n-1, k-1) + D^-(n-1, k) + \\
&\quad kT^+(n-1, k) + (n-2k+1)T^+(n-1, k-1) \\
&= k(D^+(n-1, k) + T^+(n-1, k)) + (n-2k+1)(D^+(n-1, k-1) + T^+(n-1, k-1)) \\
&\quad + D^+(n-1, k) + D^-(n-1, k) \\
&= kR^+(n-1, k) + (n-2k+1)R^+(n-1, k-1) + D(n-1, k),
\end{aligned}$$

and this completes the proof. \square

By Lemma 6, we immediately get the following proposition.

Proposition 7. *Set $D_0^+(x) = D_0(x) = 1, D_0^-(x) = T_0(x) = T_0^-(x) = T_0^+(x) = 0$. Then for $n \geq 0$, we have the following recurrence relations:*

$$\begin{aligned}
D_{n+1}^+(x) &= nxR_n^+(x) + x(1-2x)\frac{d}{dx}R_n^+(x) + D_n(x); \\
T_{n+1}^+(x) &= nxR_n^+(x) + x(1-2x)\frac{d}{dx}R_n^+(x) + T_n(x); \\
D_{n+1}^-(x) &= (n+1)xR_n^-(x) + x(1-2x)\frac{d}{dx}R_n^-(x) + xT_n^+(x); \\
T_{n+1}^-(x) &= (n+1)xR_n^-(x) + x(1-2x)\frac{d}{dx}R_n^-(x) + xD_n^+(x); \\
D_{n+1}(x) &= (1+nx)D_n(x) + (n+1)xT_n(x) + x(1-2x)\frac{d}{dx}R_n(x) + xD_n^-(x); \\
T_{n+1}(x) &= (1+nx)T_n(x) + (n+1)xD_n(x) + x(1-2x)\frac{d}{dx}R_n(x) + xT_n^-(x).
\end{aligned}$$

The *Chebyshev polynomials of the second kind* may be recursively defined by $U_0(x) = 1, U_1(x) = 2x$ and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ for $n \geq 1$. A well known explicit formula is the following:

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

Now we present the second main result of this paper.

Theorem 8. *For $n \geq 0$, we have*

$$D_n(x) - T_n(x) = D_{n+1}^+(x) - T_{n+1}^+(x) = \frac{1}{x} (T_{n+2}^-(x) - D_{n+2}^-(x)) = x^{\frac{n+1}{2}} U_{n+1} \left(\frac{1}{2\sqrt{x}} \right). \quad (6)$$

Proof. Using Proposition 7, we get

$$D_n(x) - T_n(x) = D_{n+1}^+(x) - T_{n+1}^+(x) = \frac{1}{x} (T_{n+2}^-(x) - D_{n+2}^-(x)),$$

and $D_n(x) - T_n(x) = (1-x)(D_{n-1}(x) - T_{n-1}(x)) - x(T_{n-1}^-(x) - D_{n-1}^-(x))$. Note that

$$x^{\frac{n+1}{2}} U_{n+1} \left(\frac{1}{2\sqrt{x}} \right) = \sum_{k \geq 0} (-1)^k \binom{n-k+1}{k} x^k.$$

If $n \leq 3$, the equality (6) is obvious, so we proceed to the inductive step. Assume that $n \geq 4$ and that the equality holds for $m \leq n-1$. Then

$$\begin{aligned} D_{m+1}(x) - T_{m+1}(x) &= (1-x)(D_m(x) - T_m(x)) - x(T_m^-(x) - D_m^-(x)) \\ &= (1-x) \sum_{k \geq 0} (-1)^k \binom{m-k+1}{k} x^k - x^2 \sum_{k \geq 0} (-1)^k \binom{m-k-1}{k} x^k \\ &= \sum_{k \geq 0} (-1)^k \left(\binom{m-k+1}{k} + \binom{m-k+2}{k-1} - \binom{m-k+1}{k-2} \right) x^k \\ &= \sum_{k \geq 0} (-1)^k \left(\binom{m-k+1}{k} + \frac{m-2k+3}{k-1} \binom{m-k+1}{k-2} \right) x^k \\ &= \sum_{k \geq 0} (-1)^k \frac{(m-k+1)!}{k!(m-2k+2)!} (m-k+2) x^k \\ &= \sum_{k \geq 0} (-1)^k \binom{m-k+2}{k} x^k, \end{aligned}$$

as desired. \square

Combining (6) and the fact that $R_n(x) = D_n(x) + T_n(x)$, so the following is immediate.

Corollary 9. *We have*

$$\begin{aligned} D(n, k) &= \frac{1}{2} R(n, k) + \frac{1}{2} \binom{n-k+1}{k} (-1)^k; \\ T(n, k) &= \frac{1}{2} R(n, k) - \frac{1}{2} \binom{n-k+1}{k} (-1)^k. \end{aligned}$$

The *Fibonacci sequence* is defined recursively by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (see [12, A000045]). A well known sum formula for F_n is the following:

$$F_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}.$$

As a special case of (6), we get the following.

Corollary 10. *Let $d_n = D_n(-1)$ and $t_n = T_n(-1)$. We have $d_n - t_n = F_{n+2}$.*

For convenience, we end this paper by providing the first few terms of the polynomials discussed in this section:

$$\begin{aligned}
 D_1^+(x) &= 1, D_2^+(x) = 1 + x, D_3^+(x) = 1 + 7x, D_4^+(x) = 1 + 29x + 21x^2; \\
 T_1^+(x) &= 0, T_2^+(x) = 2x, T_3^+(x) = 9x, T_4^+(x) = 32x + 20x^2; \\
 D_1^-(x) &= 0, D_2^-(x) = x, D_3^-(x) = 3x + 5x^2, D_4^-(x) = 7x + 41x^2; \\
 T_1^-(x) &= x, T_2^-(x) = 2x, T_3^-(x) = 4x + 4x^2, T_4^-(x) = 8x + 39x^2; \\
 D_1(x) &= 1, D_2(x) = 1 + 2x, D_3(x) = 1 + 10x + 5x^2, D_4(x) = 1 + 36x + 62x^2; \\
 T_1(x) &= x, T_2(x) = 4x, T_3(x) = 13x + 4x^2, T_4(x) = 40x + 59x^2.
 \end{aligned}$$

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