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# $q$ -Quasiadditive Functions

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In this paper, we introduce the notion of  $q$ -quasiadditivity of arithmetic functions, as well as the related concept of  $q$ -quasimultiplicativity, which generalises strong  $q$ -additivity and  $q$ -multiplicativity, respectively. We show that there are many natural examples for these concepts, which are characterised by functional equations of the form  $f(q^{k+r}a + b) = f(a) + f(b)$  or  $f(q^{k+r}a + b) = f(a)f(b)$  for all  $b < q^k$  and a fixed parameter  $r$ . In addition to some elementary properties of  $q$ -quasiadditive and  $q$ -quasimultiplicative functions, we prove characterisations of  $q$ -quasiadditivity and  $q$ -quasimultiplicativity for the special class of  $q$ -regular functions. The final main result provides a general central limit theorem that includes both classical and new examples as corollaries.

**Keywords:**  $q$ -additive function,  $q$ -quasiadditive function,  $q$ -regular function, central limit theorem

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## 1 Introduction

Arithmetic functions based on the digital expansion in some base  $q$  have a long history (see, e.g., [3–8, 11]). The notion of a  $q$ -additive function is due to [11]: an arithmetic function (defined on nonnegative integers) is called  $q$ -additive if

$$f(q^k a + b) = f(q^k a) + f(b)$$

whenever  $0 \leq b < q^k$ . A stronger version of this concept is *strong* (or *complete*)  $q$ -additivity: a function  $f$  is said to be strongly  $q$ -additive if we even have

$$f(q^k a + b) = f(a) + f(b)$$

whenever  $0 \leq b < q^k$ . The class of (strongly)  $q$ -multiplicative functions is defined in an analogous fashion. Loosely speaking, (strong)  $q$ -additivity of a function means that it can be evaluated by breaking up the base- $q$  expansion. Typical examples of strongly  $q$ -additive functions are the  $q$ -ary sum of digits and the number of occurrences of a specified nonzero digit.

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There are, however, many simple and natural functions based on the  $q$ -ary expansion that are not  $q$ -additive. A very basic example of this kind are *block counts*: the number of occurrences of a certain block of digits in the  $q$ -ary expansion. This and other examples provide the motivation for the present paper, in which we define and study a larger class of functions with comparable properties.

*Definition.* An arithmetic function (a function defined on the set of nonnegative integers) is called  $q$ -quasiadditive if there exists some nonnegative integer  $r$  such that

$$f(q^{k+r}a + b) = f(a) + f(b) \quad (1)$$

whenever  $0 \leq b < q^k$ . Likewise,  $f$  is said to be  $q$ -quasimultiplicative if it satisfies the identity

$$f(q^{k+r}a + b) = f(a)f(b) \quad (2)$$

for some fixed nonnegative integer  $r$  whenever  $0 \leq b < q^k$ .

We remark that the special case  $r = 0$  is exactly strong  $q$ -additivity, so strictly speaking the term “strongly  $q$ -quasiadditive function” might be more appropriate. However, since we are not considering a weaker version (for which natural examples seem to be much harder to find), we do not make a distinction. As a further caveat, we remark that the term “quasiadditivity” has also been used in [1] for a related, but slightly weaker condition.

In the following section, we present a variety of examples of  $q$ -quasiadditive and  $q$ -quasimultiplicative functions. In Section 3, we give some general properties of such functions. Since most of our examples also belong to the related class of  $q$ -regular functions, we discuss the connection in Section 4. Finally, we prove a general central limit theorem for  $q$ -quasiadditive and -multiplicative functions that contains both old and new examples as special cases.

## 2 Examples of $q$ -quasiadditive and $q$ -quasimultiplicative functions

Let us now back up the abstract concept of  $q$ -quasiadditivity by some concrete examples.

### *Block counts*

As mentioned in the introduction, the number of occurrences of a fixed nonzero digit is a typical example of a  $q$ -additive function. However, the number of occurrences of a given block  $B = \epsilon_1\epsilon_2 \cdots \epsilon_\ell$  of digits in the expansion of a nonnegative integer  $n$ , which we denote by  $c_B(n)$ , does not represent a  $q$ -additive function. The reason is simple: the  $q$ -ary expansion of  $q^k a + b$  is obtained by joining the expansions of  $a$  and  $b$ , so occurrences of  $B$  in  $a$  and occurrences of  $B$  in  $b$  are counted by  $c_B(a) + c_B(b)$ , but occurrences that involve digits of both  $a$  and  $b$  are not.

However, if  $B$  is a block different from  $00 \cdots 0$ , then  $c_B$  is  $q$ -quasiadditive: note that the representation of  $q^{k+\ell} a + b$  is of the form

$$\underbrace{a_1 a_2 \cdots a_\mu}_{\text{expansion of } a} \underbrace{00 \cdots 0}_\ell \underbrace{b_1 b_2 \cdots b_\nu}_{\text{expansion of } b}$$

whenever  $0 \leq b < q^k$ , so occurrences of the block  $B$  have to belong to either  $a$  or  $b$  only. This implies that  $c_B(q^{k+\ell} a + b) = c_B(a) + c_B(b)$ , with one small caveat: if the block starts and/or ends with a sequence of zeros, then the count needs to be adjusted by assuming the digital expansion of a nonnegative integer to be padded with zeros on the left and on the right.

For example, let  $B$  be the block 0101 in base 2. The binary representations of 469 and 22 are 111010101 and 10110, respectively, so we have  $c_B(469) = 2$  and  $c_B(22) = 1$  (note the occurrence of 0101 at the beginning of 10110 if we assume the expansion to be padded with zeros), as well as

$$c_B(240150) = c_B(2^9 \cdot 469 + 22) = c_B(469) + c_B(22) = 3.$$

Indeed, the block  $B$  occurs three times in the expansion of 240150, which is 111010101000010110.

### *The number of runs and the Gray code*

The number of ones in the Gray code of a nonnegative integer  $n$ , which we denote by  $h_{\text{GRAY}}(n)$ , is also equal to the number of runs (maximal sequences of consecutive identical digits) in the binary representations of  $n$  (counting the number of runs in the representation of 0 as 0); the sequence defined by  $h_{\text{GRAY}}(n)$  is A005811 in Sloane's On-Line Encyclopedia of Integer Sequences [17]. An analysis of its expected value is performed in [10]. The function  $h_{\text{GRAY}}$  is 2-quasiadditive up to some minor modification: set  $f(n) = h_{\text{GRAY}}(n)$  if  $n$  is even and  $f(n) = h_{\text{GRAY}}(n) + 1$  if  $n$  is odd. The new function  $f$  can be interpreted as the total number of occurrences of the two blocks 01 and 10 in the binary expansion (considering binary expansions to be padded with zeros at both ends), so the argument of the previous example applies again and shows that  $f$  is 2-quasiadditive.

### *The nonadjacent form and its Hamming weight*

The nonadjacent form (NAF) of a nonnegative integer is the unique base-2 representation with digits 0, 1,  $-1$  ( $-1$  is usually represented as  $\bar{1}$  in this context) and the additional requirement that there may not be two adjacent nonzero digits, see [18]. For example, the NAF of 27 is  $100\bar{1}0\bar{1}$ . It is well known that the NAF always has minimum Hamming weight (i.e., the number of nonzero digits) among all possible binary representations with this particular digit set, although it may not be unique with this property (compare, e.g., [18] with [15]).

The Hamming weight  $h_{\text{NAF}}$  of the nonadjacent form has been analysed in some detail [13, 20], and it is also an example of a 2-quasiadditive function. It is not difficult to see that  $h_{\text{NAF}}$  is characterised by the recursions  $h_{\text{NAF}}(2n) = h_{\text{NAF}}(n)$ ,  $h_{\text{NAF}}(4n + 1) = h_{\text{NAF}}(n) + 1$ ,  $h_{\text{NAF}}(4n - 1) = h_{\text{NAF}}(n) + 1$  together with the initial value  $h_{\text{NAF}}(0) = 0$ . The identity

$$h_{\text{NAF}}(2^{k+2}a + b) = h_{\text{NAF}}(a) + h_{\text{NAF}}(b)$$

can be proved by induction. In Section 4, this example will be generalised and put into a larger context.

### *The number of optimal $\{0, 1, -1\}$ -representations*

As mentioned above, the NAF may not be the only representation with minimum Hamming weight among all possible binary representations with digits 0, 1,  $-1$ . The number of optimal representations of a given nonnegative integer  $n$  is therefore a quantity of interest in its own right. Its average over intervals of the form  $[0, N)$  was studied by Grabner and Heuberger [12], who also proved that the number  $r_{\text{OPT}}(n)$  of optimal representations of  $n$  can be obtained in the following way:

**Lemma 1** (Grabner–Heuberger [12]). *Let sequences  $u_i$  ( $i = 1, 2, \dots, 5$ ) be given recursively by*

$$u_1(0) = u_2(0) = \dots = u_5(0) = 1, \quad u_1(1) = u_2(1) = 1, \quad u_3(1) = u_4(1) = u_5(1) = 0,$$

and

$$\begin{aligned}
u_1(2n) &= u_1(n), & u_1(2n+1) &= u_2(n) + u_4(n+1), \\
u_2(2n) &= u_1(n), & u_2(2n+1) &= u_3(n), \\
u_3(2n) &= u_2(n), & u_3(2n+1) &= 0, \\
u_4(2n) &= u_1(n), & u_4(2n+1) &= u_5(n+1), \\
u_5(2n) &= u_4(n), & u_5(2n+1) &= 0.
\end{aligned}$$

The number  $r_{\text{OPT}}(n)$  of optimal representations of  $n$  is equal to  $u_1(n)$ .

A straightforward calculation shows that

$$\begin{aligned}
u_1(8n) &= u_2(8n) = \dots = u_5(8n) = u_1(8n+1) = u_2(8n+1) = u_1(n), \\
u_3(8n+1) &= u_4(8n+1) = u_5(8n+1) = 0.
\end{aligned} \tag{3}$$

This gives us the following result (see the full version of this extended abstract for a detailed proof):

**Lemma 2.** *The number of optimal  $\{0, 1, -1\}$ -representations of a nonnegative integer is a 2-quasimultiplicative function. Specifically, for any three nonnegative integers  $a, b, k$  with  $b < 2^k$ , we have*

$$r_{\text{OPT}}(2^{k+3}a + b) = r_{\text{OPT}}(a)r_{\text{OPT}}(b).$$

In Section 4, we will show that this is also an instance of a more general phenomenon.

### The run length transform and cellular automata

The *run length transform* of a sequence is defined in a recent paper of Sloane [19]: it is based on the binary representation, but could in principle also be generalised to other bases. Given a sequence  $s_1, s_2, \dots$ , its run length transform is obtained by the rule

$$t(n) = \prod_{i \in \mathcal{L}(n)} s_i,$$

where  $\mathcal{L}(n)$  is the multiset of run lengths of  $n$  (lengths of blocks of consecutive ones in the binary representation). For example, the binary expansion of 1910 is 11101110110, so the multiset  $\mathcal{L}(n)$  of run lengths would be  $\{3, 3, 2\}$ , giving  $t(1910) = s_2 s_3^2$ .

A typical example is obtained for the sequence of Jacobsthal numbers given by the formula  $s_n = \frac{1}{3}(2^{n+2} - (-1)^n)$ . The associated run length transform  $t_n$  (sequence A071053 in the OEIS [17]) counts the number of odd coefficients in the expansion of  $(1 + x + x^2)^n$ , and it can also be interpreted as the number of active cells at the  $n$ -th generation of a certain cellular automaton. Further examples stemming from cellular automata can be found in Sloane's paper [19].

The argument that proved  $q$ -quasiadditivity of block counts also applies here, and indeed it is easy to see that the identity

$$t(2^{k+1}a + b) = t(a)t(b),$$

where  $0 \leq b < 2^k$ , holds for the run length transform of any sequence, meaning that any such transform is 2-quasimultiplicative. In fact, it is not difficult to show that every 2-quasimultiplicative function with parameter  $r = 1$  is the run length transform of some sequence.

### 3 Elementary properties

Now that we have gathered some motivating examples for the concepts of  $q$ -quasiadditivity and  $q$ -quasimultiplicativity, let us present some simple results about functions with these properties. First of all, let us state an obvious relation between  $q$ -quasiadditive and  $q$ -quasimultiplicative functions:

**Proposition 3.** *If a function  $f$  is  $q$ -quasiadditive, then the function defined by  $g(n) = c^{f(n)}$  for some positive constant  $c$  is  $q$ -quasimultiplicative. Conversely, if  $f$  is a  $q$ -quasimultiplicative function that only takes positive values, then the function defined by  $g(n) = \log_c f(n)$  for some positive constant  $c \neq 1$  is  $q$ -quasiadditive.*

The next proposition deals with the parameter  $r$  in the definition of a  $q$ -quasiadditive function:

**Proposition 4.** *If the arithmetic function  $f$  satisfies  $f(q^{k+r}a + b) = f(a) + f(b)$  for some fixed nonnegative integer  $r$  whenever  $0 \leq b < q^k$ , then it also satisfies  $f(q^{k+s}a + b) = f(a) + f(b)$  for all nonnegative integers  $s \geq r$  whenever  $0 \leq b < q^k$ .*

*Proof.* If  $a, b$  are nonnegative integers with  $0 \leq b < q^k$ , then clearly also  $0 \leq b < q^{k+s-r}$  if  $s \geq r$ , and thus

$$f(q^{k+s}a + b) = f(q^{(k+s-r)+r}a + b) = f(a) + f(b).$$

□

**Corollary 5.** *If two arithmetic functions  $f$  and  $g$  are  $q$ -quasiadditive functions, then so is any linear combination  $\alpha f + \beta g$  of the two.*

*Proof.* In view of the previous proposition, we may assume the parameter  $r$  in (1) to be the same for both functions. The statement follows immediately. □

Finally, we observe that  $q$ -quasiadditive and  $q$ -quasimultiplicative functions can be computed by breaking the  $q$ -ary expansion into pieces. A detailed proof can be found in the full version:

**Lemma 6.** *If  $f$  is a  $q$ -quasiadditive ( $q$ -quasimultiplicative) function, then*

- $f(0) = 0$  ( $f(0) = 1$ , respectively, unless  $f$  is identically 0),
- $f(qa) = f(a)$  for all nonnegative integers  $a$ .

**Proposition 7.** *Suppose that the function  $f$  is  $q$ -quasiadditive with parameter  $r$ , i.e.,  $f(q^{k+r}a + b) = f(a) + f(b)$  whenever  $0 \leq b < q^k$ . Going from left to right, split the  $q$ -ary expansion of  $n$  into blocks by inserting breaks after each run of  $r$  or more zeros. If these blocks are the  $q$ -ary representations of  $n_1, n_2, \dots, n_\ell$ , then we have*

$$f(n) = f(n_1) + f(n_2) + \dots + f(n_\ell).$$

Moreover, if  $m_i$  is the greatest divisor of  $n_i$  which are not divisible by  $q$  for  $i = 1, \dots, \ell$ , then

$$f(n) = f(m_1) + f(m_2) + \dots + f(m_\ell).$$

Analogous statements hold for  $q$ -quasimultiplicative functions, with sums replaced by products.

*Proof.* This is obtained by a straightforward induction on  $\ell$  together with the fact that  $f(q^h a) = f(a)$ , which follows from the previous lemma.  $\square$

*Example 1.* Recall that the Hamming weight of the NAF (which is the minimum Hamming weight of a  $\{0, 1, -1\}$ -representation) is 2-quasiadditive with parameter  $r = 2$ . To determine  $h_{\text{NAF}}(314\,159\,265)$ , we split the binary representation, which is 10010101110011011000010100001, into blocks by inserting breaks after each run of at least two zeros:

$$100|101011100|110110000|1010000|1.$$

The numbers  $n_1, n_2, \dots, n_\ell$  in the statement of the proposition are now 4, 348, 432, 80, 1 respectively, and the numbers  $m_1, m_2, \dots, m_\ell$  are therefore 1, 87, 27, 5, 1. Now we use the values  $h_{\text{NAF}}(1) = 1$ ,  $h_{\text{NAF}}(5) = 2$ ,  $h_{\text{NAF}}(27) = 3$  and  $h_{\text{NAF}}(87) = 4$  to obtain

$$h_{\text{NAF}}(314\,159\,265) = 2h_{\text{NAF}}(1) + h_{\text{NAF}}(5) + h_{\text{NAF}}(27) + h_{\text{NAF}}(87) = 11.$$

*Example 2.* In the same way, we consider the number of optimal representations  $r_{\text{OPT}}$ , which is 2-quasimultiplicative with parameter  $r = 3$ . Consider for instance the binary representation of 204 280 974, namely 1100001011010001010010001110. We split into blocks:

$$110000|101101000|101001000|1110.$$

The four blocks correspond to the numbers  $48 = 16 \cdot 3$ ,  $360 = 8 \cdot 45$ ,  $328 = 8 \cdot 41$  and  $14 = 2 \cdot 7$ . Since  $r_{\text{OPT}}(3) = 2$ ,  $r_{\text{OPT}}(45) = 5$ ,  $r_{\text{OPT}}(41) = 1$  and  $r_{\text{OPT}}(7) = 1$ , we obtain  $r_{\text{OPT}}(204\,280\,974) = 10$ .

## 4 $q$ -Regular functions

In this section, we introduce  $q$ -regular functions and examine the connection to our concepts. See [2] for more background on  $q$ -regular sequences.

A function  $f$  is  $q$ -regular if it can be expressed as  $f = \mathbf{u}^t \mathbf{f}$  for a vector  $\mathbf{u}$  and a vector-valued function  $\mathbf{f}$ , and there are matrices  $M_i$ ,  $0 \leq i < q$ , satisfying

$$\mathbf{f}(qn + i) = M_i \mathbf{f}(n) \tag{4}$$

for  $0 \leq i < q$ ,  $qn + i > 0$ . We set  $\mathbf{v} = \mathbf{f}(0)$ .

Equivalently, a function  $f$  is  $q$ -regular if and only if  $f$  can be written as

$$f(n) = \mathbf{u}^t \prod_{i=0}^L M_{n_i} \mathbf{v} \tag{5}$$

where  $n_L \dots n_0$  is the  $q$ -ary expansion of  $n$ .

The notion of  $q$ -regular functions is a generalisation of  $q$ -additive and  $q$ -multiplicative functions. However, we emphasise that  $q$ -quasiadditive and  $q$ -quasimultiplicative functions are not necessarily  $q$ -regular: a  $q$ -regular sequence can always be bounded by  $O(n^c)$  for a constant  $c$ , see [2, Thm. 16.3.1]. In our setting however, the values of  $f(n)$  can be chosen arbitrarily for those  $n$  whose  $q$ -ary expansion does not contain  $0^r$ . Therefore a  $q$ -quasiadditive or -multiplicative function can grow arbitrarily fast.

We call  $(\mathbf{u}, (M_i)_{0 \leq i < q}, \mathbf{v})$  a *linear representation* of the  $q$ -regular function  $f$ . Such a linear representation is called *zero-insensitive* if  $M_0 \mathbf{v} = \mathbf{v}$ , meaning that in (5), leading zeros in the  $q$ -ary expansion of  $n$  do not change anything. We call a linear representation *minimal* if the dimension of the matrices  $M_i$  is minimal among all linear representations of  $f$ .

Following [9], every  $q$ -regular function has a zero-insensitive minimal linear representation.

#### 4.1 When is a $q$ -regular function $q$ -quasimultiplicative?

We now give a characterisation of  $q$ -regular functions that are  $q$ -quasimultiplicative. Proofs of the results in this and the following subsection can be found in the full version.

**Theorem 8.** *Let  $f$  be a  $q$ -regular sequence with zero-insensitive minimal linear representation (5). Then the following two assertions are equivalent:*

- *The sequence  $f$  is  $q$ -quasimultiplicative with parameter  $r$ .*
- $M_0^r = \mathbf{v} \mathbf{u}^t$ .

*Example 3* (The number of optimal  $\{0, 1, -1\}$ -representations). The number of optimal  $\{0, 1, -1\}$ -representations as described in Section 2 is a 2-regular sequence by Lemma 1. A minimal zero-insensitive linear representation for the vector  $(u_1(n), u_2(n), u_3(n), u_1(n+1), u_4(n+1), u_5(n+1))^t$  is given by

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$\mathbf{u}^t = (1, 0, 0, 0, 0, 0)$  and  $\mathbf{v} = (1, 1, 1, 1, 0, 0)^t$ .

As  $M_0^3 = \mathbf{v} \mathbf{u}^t$ , this sequence is 2-quasimultiplicative with parameter 3, which is the same result as in Lemma 2.

*Remark.* The condition on the minimality of the linear representation in Theorem 8 is necessary as illustrated by the following example:

Consider the sequence  $f(n) = 2^{s_2(n)}$  where  $s_2(n)$  is the binary sum of digits function. This sequence is 2-regular and 2-(quasi-)multiplicative with parameter  $r = 0$ . A minimal linear representation is given by  $M_0 = 1$ ,  $M_1 = 2$ ,  $\mathbf{v} = 1$  and  $\mathbf{u} = 1$ . As stated in Theorem 8, we have  $M_0^0 = \mathbf{v} \mathbf{u}^t = 1$ .

If we use the zero-insensitive non-minimal linear representation defined by  $M_0 = \begin{pmatrix} 1 & 13 \\ 0 & 2 \end{pmatrix}$ ,  $M_1 = \begin{pmatrix} 2 & 27 \\ 0 & 5 \end{pmatrix}$ ,  $\mathbf{v} = (1, 0)^t$  and  $\mathbf{u}^t = (1, 0)$  instead, we have  $\text{rank } M_0^r = 2$  for all  $r \geq 0$ . Thus  $M_0^r \neq \mathbf{v} \mathbf{u}^t$ .

#### 4.2 When is a $q$ -regular function $q$ -quasiadditive?

The characterisation of  $q$ -regular functions that are also  $q$ -quasiadditive is somewhat more complicated. Again, we consider a zero-insensitive (but not necessarily minimal) linear representation. We let  $U$  be the smallest vector space such that all vectors of the form  $\mathbf{u}^t \prod_{i \in I} M_{n_i}$  lie in the affine subspace  $\mathbf{u}^t + U^t$  ( $U^t$  is used as a shorthand for  $\{\mathbf{x}^t : \mathbf{x} \in U\}$ ). Such a vector space must exist, since  $\mathbf{u}^t$  is a vector of this form (corresponding to the empty product, where  $I = \emptyset$ ). Likewise, let  $V$  be the smallest vector space such that all vectors of the form  $\prod_{j \in J} M_{n_j} \mathbf{v}$  lie in the affine subspace  $\mathbf{v} + V$ .

**Theorem 9.** *Let  $f$  be a  $q$ -regular sequence with zero-insensitive linear representation (5). The sequence  $f$  is  $q$ -quasiadditive with parameter  $r$  if and only if all of the following statements hold:*

- $\mathbf{u}^t \mathbf{v} = 0$ ,
- $U^t$  is orthogonal to  $(M_0^r - I)\mathbf{v}$ , i.e.,  $\mathbf{x}^t(M_0^r - I)\mathbf{v} = \mathbf{x}^t M_0^r \mathbf{v} - \mathbf{x}^t \mathbf{v} = 0$  for all  $\mathbf{x} \in U$ ,
- $V$  is orthogonal to  $\mathbf{u}^t(M_0^r - I)$ , i.e.,  $\mathbf{u}^t(M_0^r - I)\mathbf{y} = \mathbf{u}^t M_0^r \mathbf{y} - \mathbf{u}^t \mathbf{y} = 0$  for all  $\mathbf{y} \in V$ ,
- $U^t M_0^r V = 0$ , i.e.,  $\mathbf{x}^t M_0^r \mathbf{y} = 0$  for all  $\mathbf{x} \in U$  and  $\mathbf{y} \in V$ .

*Example 4.* For the Hamming weight of the nonadjacent form, a zero-insensitive (and also minimal) linear representation for the vector  $(h_{\text{NAF}}(n), h_{\text{NAF}}(n+1), h_{\text{NAF}}(2n+1), 1)^t$  is

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$\mathbf{u}^t = (1, 0, 0, 0)$  and  $\mathbf{v} = (0, 1, 1, 1)^t$ .

The three vectors  $\mathbf{w}_1 = \mathbf{u}^t M_1 - \mathbf{u}^t$ ,  $\mathbf{w}_2 = \mathbf{u}^t M_1^2 - \mathbf{u}^t$  and  $\mathbf{w}_3 = \mathbf{u}^t M_1 M_0 M_1 - \mathbf{u}^t$  are linearly independent. If we let  $W$  be the vector space spanned by those three, it is easily verified that  $M_0$  and  $M_1$  map the affine subspace  $\mathbf{u}^t + W^t$  to itself, so  $U = W$  is spanned by these vectors.

Similarly, the three vectors  $M_1 \mathbf{v} - \mathbf{v}$ ,  $M_1^2 \mathbf{v} - \mathbf{v}$  and  $M_1 M_0 M_1 \mathbf{v} - \mathbf{v}$  span  $V$ .

The first condition of Theorem 9 is obviously true. We only have to verify the other three conditions with  $r = 2$  for the basis vectors of  $U$  and  $V$ , which is done easily. Thus  $h_{\text{NAF}}$  is a 2-regular sequence that is also 2-quasiadditive, as was also proved in Section 2.

Finding the vector spaces  $U$  and  $V$  is not trivial. But in a certain special case of  $q$ -regular functions, we can give a sufficient condition for  $q$ -additivity, which is easier to check. These  $q$ -regular functions are output sums of transducers as defined in [14]: a transducer transforms the  $q$ -ary expansion of an integer  $n$  (read from the least significant to the most significant digit) deterministically into an output sequence and leads to a state  $s$ . The output sum is then the sum of this output sequence together with the final output of the state  $s$ . This defines the value of the  $q$ -regular function evaluated at  $n$ . The function  $h_{\text{NAF}}$  discussed in the example above, as well as many other examples, can be represented in this way.

**Proposition 10.** *The output sum of a connected transducer is  $q$ -additive with parameter  $r$  if the following conditions are satisfied:*

- *The transducer has the reset sequence  $0^r$  going to the initial state, i.e., reading  $r$  zeros always leads to the initial state of the transducer.*
- *For every state, the output sum along the path of the reset sequence  $0^r$  equals the final output of this state.*
- *Additional zeros at the end of the input sequence do not change the output sum.*



## 5 A central limit theorem for $q$ -quasiadditive and $-$ multiplicative functions

In this section, we prove a central limit theorem for  $q$ -quasimultiplicative functions taking only positive values. By Proposition 3, this also implies a central limit theorem for  $q$ -quasiadditive functions.

To this end, we define a generating function: let  $f$  be a  $q$ -quasimultiplicative function with positive values, let  $\mathcal{M}_k$  be the set of all nonnegative integers less than  $q^k$  (i.e., those positive integers whose  $q$ -ary expansion needs at most  $k$  digits), and set

$$F(x, t) = \sum_{k \geq 0} x^k \sum_{n \in \mathcal{M}_k} f(n)^t.$$

The decomposition of Proposition 7 now translates directly to an alternative representation for  $F(x, t)$ : let  $\mathcal{B}$  be the set of all positive integers not divisible by  $q$  whose  $q$ -ary representation does not contain the block  $0^r$ , let  $\ell(n)$  denote the length of the  $q$ -ary representation of  $n$ , and define the function  $B(x, t)$  by

$$B(x, t) = \sum_{n \in \mathcal{B}} x^{\ell(n)} f(n)^t.$$

We remark that in the special case where  $q = 2$  and  $r = 1$ , this simplifies greatly to

$$B(x, t) = \sum_{k \geq 1} x^k f(2^k - 1)^t. \tag{6}$$

**Proposition 11.** *The generating function  $F(x, t)$  can be expressed as*

$$F(x, t) = \frac{1}{1-x} \cdot \frac{1}{1 - \frac{x^r}{1-x} B(x, t)} \left( 1 + (1+x+\dots+x^{r-1})B(x, t) \right) = \frac{1 + (1+x+\dots+x^{r-1})B(x, t)}{1-x-x^r B(x, t)}.$$

*Proof.* The first factor stands for the initial sequence of leading zeros, the second factor for a (possibly empty) sequence of blocks consisting of an element of  $\mathcal{B}$  and  $r$  or more zeros, and the last factor for the final part, which may be empty or an element of  $\mathcal{B}$  with up to  $r - 1$  zeros (possibly none) added at the end.  $\square$

Under suitable assumptions on the growth of a  $q$ -quasiadditive or  $q$ -quasimultiplicative function, we can exploit the expression of Proposition 11 to prove a central limit theorem in the following steps (full proofs can again be found in the full version).

*Definition.* We say that a function  $f$  has *at most polynomial growth* if  $f(n) = O(n^c)$  and  $f(n) = \Omega(n^{-c})$  for a fixed  $c \geq 0$ . We say that  $f$  has *at most logarithmic growth* if  $f(n) = O(\log n)$ .

Note that our definition of at most polynomial growth is slightly different than usual: the extra condition  $f(n) = \Omega(n^{-c})$  ensures that the absolute value of  $\log f(n)$  does not grow too fast.

**Lemma 12.** *Assume that the positive,  $q$ -quasimultiplicative function  $f$  has at most polynomial growth.*

*There exist positive constants  $\delta$  and  $\epsilon$  such that*

- $B(x, t)$  has radius of convergence  $\rho(t) > \frac{1}{q}$  whenever  $|t| \leq \delta$ .

- For  $|t| \leq \delta$ , the equation  $x + x^r B(x, t) = 1$  has a complex solution  $\alpha(t)$  with  $|\alpha(t)| < \rho(t)$  and no other solutions with modulus  $\leq (1 + \epsilon)|\alpha(t)|$ .
- Thus the generating function  $F(x, t)$  has a simple pole at  $\alpha(t)$  and no further singularities of modulus  $\leq (1 + \epsilon)|\alpha(t)|$ .
- Finally,  $\alpha$  is an analytic function of  $t$  for  $|t| \leq \delta$ .

**Lemma 13.** Assume that the positive,  $q$ -quasimultiplicative function  $f$  has at most polynomial growth. With  $\delta$  and  $\epsilon$  as in the previous lemma, we have, uniformly in  $t$ ,

$$[x^k]F(x, t) = \kappa(t) \cdot \alpha(t)^{-k} (1 + O((1 + \epsilon)^{-k}))$$

for some function  $\kappa$ . Both  $\alpha$  and  $\kappa$  are analytic functions of  $t$  for  $|t| \leq \delta$ , and  $\kappa(t) \neq 0$  in this region.

**Theorem 14.** Assume that the positive,  $q$ -quasimultiplicative function  $f$  has at most polynomial growth.

Let  $N_k$  be a randomly chosen integer in  $\{0, 1, \dots, q^k - 1\}$ . The random variable  $L_k = \log f(N_k)$  has mean  $\mu k + O(1)$  and variance  $\sigma^2 k + O(1)$ , where the two constants are given by

$$\mu = \frac{B_t(1/q, 0)}{q^{2r}}$$

and

$$\begin{aligned} \sigma^2 = & -B_t(1/q, 0)^2 q^{-4r+1} (q-1)^{-1} + 2B_t(1/q, 0)^2 q^{-3r+1} (q-1)^{-1} - B_t(1/q, 0)^2 q^{-4r} (q-1)^{-1} \\ & - 4rB_t(1/q, 0)^2 q^{-4r} + B_{tt}(1/q, 0) q^{-2r} - 2B_t(1/q, 0) B_{tx}(1/q, 0) q^{-4r-1}. \end{aligned} \quad (7)$$

If  $f$  is not the constant function  $f \equiv 1$ , then  $\sigma^2 \neq 0$  and the normalised random variable  $(L_k - \mu k) / (\sigma \sqrt{k})$  converges weakly to a standard Gaussian distribution.

**Corollary 15.** Assume that the  $q$ -quasiadditive function  $f$  has at most logarithmic growth.

Let  $N_k$  be a randomly chosen integer in  $\{0, 1, \dots, q^k - 1\}$ . The random variable  $L_k = f(N_k)$  has mean  $\hat{\mu} k + O(1)$  and variance  $\hat{\sigma}^2 k + O(1)$ , where the two constants  $\mu$  and  $\sigma^2$  are given by the same formulas as in Theorem 14, with  $B(x, t)$  replaced by

$$\hat{B}(x, t) = \sum_{n \in \mathcal{B}} x^{\ell(n)} e^{f(n)t}.$$

If  $f$  is not the constant function  $f \equiv 0$ , then the normalised random variable  $(L_k - \hat{\mu} k) / (\hat{\sigma} \sqrt{k})$  converges weakly to a standard Gaussian distribution.

*Remark.* By means of the Cramér-Wold device (and Corollary 5), we also obtain joint normal distribution of tuples of  $q$ -quasiadditive functions.

We now revisit the examples discussed in Section 2 and state the corresponding central limit theorems. Some of them are well known while others are new. We also provide numerical values for the constants in mean and variance.

*Example 5* (see also [8, 16]). The number of blocks 0101 occurring in the binary expansion of  $n$  is a 2-quasiadditive function of at most logarithmic growth. Thus by Corollary 15, the standardised random variable is asymptotically normally distributed, the constants being  $\hat{\mu} = \frac{1}{16}$  and  $\hat{\sigma}^2 = \frac{17}{256}$ .

*Example 6* (see also [13, 20]). The Hamming weight of the nonadjacent form is 2-quasiadditive with at most logarithmic growth (as the length of the NAF of  $n$  is logarithmic). Thus by Corollary 15, the standardised random variable is asymptotically normally distributed. The associated constants are  $\hat{\mu} = \frac{1}{3}$  and  $\hat{\sigma}^2 = \frac{2}{27}$ .

*Example 7* (see Section 2). The number of optimal  $\{0, 1, -1\}$ -representations is 2-quasimultiplicative. As it is always greater or equal to 1 and 2-regular, it has at most polynomial growth. Thus Theorem 14 implies that the standardised logarithm of this random variable is asymptotically normally distributed with numerical constants given by  $\mu \approx 0.060829$ ,  $\sigma^2 \approx 0.038212$ .

*Example 8* (see Section 2). Suppose that the sequence  $s_1, s_2, \dots$  satisfies  $s_n \geq 1$  and  $s_n = O(c^n)$  for a constant  $c \geq 1$ . The run length transform  $t(n)$  of  $s_n$  is 2-quasimultiplicative. As  $s_n \geq 1$  for all  $n$ , we have  $t(n) \geq 1$  for all  $n$  as well. Furthermore, there exists a constant  $A$  such that  $s_n \leq Ac^n$  for all  $n$ , and the sum of all run lengths is bounded by the length of the binary expansion, thus

$$t(n) = \prod_{i \in \mathcal{L}(n)} s_i \leq \prod_{i \in \mathcal{L}(n)} (Ac^i) \leq (Ac)^{1+\log_2 n}.$$

Consequently,  $t(n)$  is positive and has at most polynomial growth. By Theorem 14, we obtain an asymptotic normal distribution for the standardised random variable  $\log t(N_k)$ . The constants  $\mu$  and  $\sigma^2$  in mean and variance are given by

$$\mu = \sum_{i \geq 1} (\log s_i) 2^{-i-2}$$

and

$$\sigma^2 = \sum_{i \geq 1} (\log s_i)^2 (2^{-i-2} - (2i-1)2^{-2i-4}) - \sum_{j > i \geq 1} (\log s_i)(\log s_j)(i+j-1)2^{-i-j-3}.$$

These formulas can be derived from those given in Theorem 14 by means of the representation (6), and the terms can also be interpreted easily: write  $\log t(n) = \sum_{i \geq 1} X_i(n) \log s_i$ , where  $X_i(n)$  is the number of runs of length  $i$  in the binary representation of  $n$ . The coefficients in the two formulas stem from mean, variance and covariances of the  $X_i(n)$ .

In the special case that  $s_n$  is the Jacobsthal sequence ( $s_n = \frac{1}{3}(2^{n+2} - (-1)^n)$ , see Section 2), we have the numerical values  $\mu \approx 0.429947$ ,  $\sigma^2 \approx 0.121137$ .

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