# Refinable functions with PV dilations.

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Abstract. A PV number is an algebraic integer $\alpha$ of degree $d \geq 2$ all of whose Galois conjugates other than itself have modulus less than 1. Erdös [**8**] proved that the Fourier transform $\widehat{\varphi}$, of a nonzero compactly supported scalar valued function satisfying the refinement equation $\varphi(x) = \frac{|\alpha|}{2}\varphi(\alpha x) + \frac{|\alpha|}{2}\varphi(\alpha x - 1)$ with $PV$ dilation $\alpha$, does not vanish at infinity so by the Riemann-Lebesgue lemma $\varphi$ is not integrable. Dai, Feng and Wang [**5**] extended his result to scalar valued solutions of $\varphi(x) = \sum_k a(k)\varphi(\alpha x - \tau(k))$ where $\tau(k)$ are integers and $a$ has finite support and sums to $|\alpha|$. In ([**22**], Conjecture 4.2) we conjectured that their result holds under the weaker assumption that $\tau$ has values in the ring of polynomials in $\alpha$ with integer coefficients. This paper formulates a stronger conjecture and provides support for it based on a solenoidal representation of $\widehat{\varphi}$, and deep results of Erdös and Mahler [**9**];Odoni [**26**] that give lower bounds for the asymptotic density of integers represented by integral binary forms of degree $> 2$;degree $= 2$, respectively. We also construct an integrable vector valued refinable function with PV dilation.

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## 1. Introduction

In this paper $\mathbb{Z}, \mathbb{N} = \{1, 2, 3, ...\}, \mathbb{Q}, \mathbb{A}, \mathcal{O}, \mathbb{R}, \mathbb{C}$ are the integer, natural, rational, algebraic, algebraic integer, real, and complex numbers. For a ring $R$, $R[X]; R[X, X^{-1}]$ is the ring of polynomials; Laurent polynomials with coefficients in $R$ in the indeterminant $X$. If $\alpha \in \mathbb{A}$ then $\mathbb{Q}[\alpha]$ equals the algebraic number field generated by $\alpha$ and we define $\mathcal{O}_\alpha = \mathcal{O} \bigcap \mathbb{Q}[\alpha]$, the degree function $d : \mathbb{A} \rightarrow \mathbb{N}$, and the trace and norm functions $T; N : \mathbb{A} \rightarrow \mathbb{Q}$. Their restrictions to $\mathcal{O}$ are integer valued. For $\alpha \in \mathbb{A}$, $P_\alpha(X) \in \mathbb{Q}[X]$ is its minimal degree monic polynomial and $L(\alpha)$ is the least common multiple of the denominators of the coefficients of $P_\alpha(X)$. $\mathcal{O} \bigcap \mathbb{Q} = \mathbb{Z}$, $\alpha \in \mathbb{A} \Rightarrow L(\alpha)\alpha \in \mathcal{O}$, and $\alpha \in \mathcal{O} \Rightarrow P_\alpha(X) \in \mathbb{Z}[X]$. There exists $B(\alpha) \in \mathbb{N}$ with

$$\mathbb{Z}[\alpha] = \mathbb{Z} + \alpha\mathbb{Z} \cdots + \alpha^{d(\alpha)-1}\mathbb{Z} \subseteq \mathcal{O}_\alpha \subseteq \frac{1}{B(\alpha)}\mathbb{Z}[\alpha], \tag{1.1}$$

and hence, since $N(\alpha)\alpha^{-1} \in \mathcal{O}_\alpha$,

$$N(\alpha)B(\alpha)\alpha^{-1} \in \mathbb{Z}[\alpha]. \tag{1.2}$$

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EXAMPLE 1.1. *If $\alpha \in \mathbb{A}$ and $P_\alpha(X) = X^3 - X^2 - 2X - 8 = 0$ Dedekind showed ([**7**], pp. 30-32), ([**25**], pp. 64) that $\{1, \alpha, \alpha(\alpha+1)/2\}$ is an integral basis for $\mathcal{O}_\alpha$. For this (1.1) holds with $B(\alpha) = 2$ and both inclusions are proper.*

$\mathbb{T} = \mathbb{R}/\mathbb{Z}; \mathbb{T}_c = \{w \in \mathbb{C} : |w| = 1\}$ is the circle group represented additively; multiplicatively. For $x \in \mathbb{R}$ we define $||x|| = \min_{k\in\mathbb{Z}} |x - k| \in [0, \frac{1}{2}]$ and observe that $||x + y|| \leq ||x|| + ||y||$. Since $x + \mathbb{Z} = y + \mathbb{Z} \Rightarrow ||x|| = ||y||$, we can define $|| \; || : \mathbb{T} \to [0, \frac{1}{2}]$ by $||x + \mathbb{Z}|| = ||x||$. For $\alpha \in \mathbb{R} \setminus [-1, 1]$ define its Pisot set

$$\Lambda_\alpha = \{\, \lambda \in \mathbb{R} \setminus \{0\} \, : \, \lim_{j\to\infty} ||\lambda\alpha^j|| = 0 \,\}. \tag{1.3}$$

A Pisot-Vijayaraghavan (PV) number [**3, 27**] is $\alpha = \alpha_1 \in \mathcal{O}$ with $d(\alpha) \geq 2$ whose Galois conjugates $\alpha_2, ..., \alpha_d$ have moduli $< 1$. The Golden Mean $\frac{1+\sqrt{5}}{2} \approx 1.6180$ has Galois congugate $\frac{1-\sqrt{5}}{2} \approx -0.6180$ so it is a PV number.

THEOREM 1.1. (*Pisot, Vijayaraghavan*) *If $\alpha \in \mathbb{A}\setminus[-1, 1]$ has degree $d \geq 2$ and $\Lambda_\alpha \neq \phi$ then $\alpha$ is a PV number and*

$$\Lambda_\alpha = \{\, \alpha^m\mu \, : m \in \mathbb{Z},\, \mu \in \mathbb{Q}[\alpha] \setminus \{0\},\, T(\mu\alpha^j) \in \mathbb{Z},\, j = 0, ..., d-1 \,\}. \tag{1.4}$$

*Furthermore, for $\lambda \in \Lambda_\alpha$, $||\lambda\alpha^j|| \to 0$ exponentially fast.*

PROOF. Cassels ([**3**], Chapter VIII, Theorem 1) gives a simplified version, based on the properties of recursive sequences, of Pisot's proof in [**27**]. We relaxed the assumption that $\alpha$ is positive since $\alpha$ is a PV number iff $-\alpha$ is a PV number. The sequence $s(j) = T(\mu\alpha^j), j \geq 0$ satisfies $s(j) = -c_{d-1}s(j-1) - \cdots - c_0 s(j-d),\ j \geq d$, where $P_\alpha(X) = X^d + c_{d-1}X^{d-1} + \cdots + c_0$. Then (1.4) implies that $s$ has values in $\mathbb{Z}$. If $\lambda = \alpha^m\mu$ and $\mu = \mu_1, \mu_2, ..., \mu_d$ are the Galois conjugates of $\mu$, then

$$||\lambda\alpha^j|| \leq |\mu\alpha^{j+m} - T(\mu\alpha^{j+m})| \leq \sum_{k=2}^{d} |\mu_k|\, |\alpha_k|^{j+m}, \quad j \geq -m, \tag{1.5}$$

converges to 0 exponentially fast as $j \to \infty$ since $|\alpha_k| < 1, k = 2, ..., d$. $\square$

This paper studies *refinable functions*, nonzero complex scalar or vector valued distributions $\varphi$ satisfying a refinement equation

$$\varphi(x) = \sum_{k=1}^{\infty} a(k)\varphi(\alpha x - \tau(k)), \tag{1.6}$$

and whose Fourier transform $\widehat{\varphi}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\imath ixy}dx$ is continuous at $y = 0$ and $\widehat{\varphi}(0) \neq 0$. Here the *dilation* $\alpha \in \mathbb{R} \setminus [-1, 1]$, the coefficient sequence $a$, which is matrix valued for vector valued refinable functions, decays exponentially fast, and $\tau$ takes values in $\mathbb{Z}[\alpha, \alpha^{-1}]$. Refinable functions constructed from integer $\alpha \geq 2$ and integer valued $\tau$ include Daubechies' scaling functions used to construct orthonormal wavelet bases [**6**], basis functions constructed by Cavaretta, Dahmen, and Michelli from stationary subdivision algorithms [**4**], and multiwavelets constructed from vector valued refinable functions [**15**]. (1.6) is equivalent to

$$\widehat{\varphi}(y) = \widehat{a}(y\alpha^{-1})\widehat{\varphi}(y\alpha^{-1}) \tag{1.7}$$

$$\widehat{a}(y) = |\alpha|^{-1} \sum_{k=1}^{\infty} a(k)e^{-2\pi i\tau(k)y}. \tag{1.8}$$

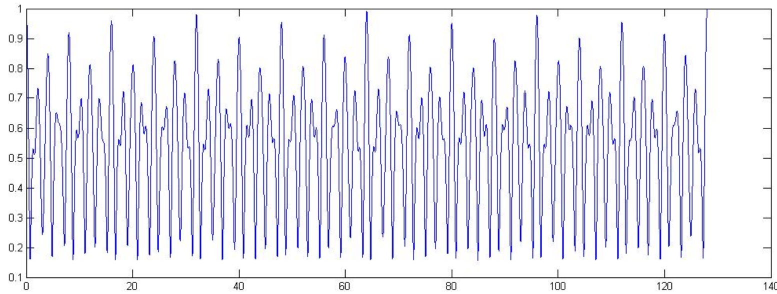


Figure 1. Fourier Transform of the Dyadic Refinement Sequence

For scalar valued $\varphi$, $\widehat{a}(0) = 1$, for vector valued $\varphi$, $\widehat{a}(0)\,\widehat{\varphi}(0) = \widehat{\varphi}(0)$, and

$$\widehat{\varphi}(y\alpha^J) = \left(\prod_{j<J} \widehat{a}(y\alpha^j)\right)\widehat{\varphi}(0), \quad J \in \mathbb{Z}, \tag{1.9}$$

where for matrices the factors move right with decreasing $j$.

Example 1.2. *We call* $\varphi = \chi_{[0,1]}$ *the boxcar function. It satisfies* (*1.6*) *with* $\alpha = 2$, $a(0) = a(1) = 1$, $\tau(0) = 0$, $\tau(1) = 1$. *Then* $\widehat{a}(y) = e^{\pi i y}\cos \pi y$ *and*

$$\widehat{\varphi}(y2^J) = e^{\pi i y 2^J}\frac{\sin \pi y 2^J}{\pi y 2^J} = e^{\pi y 2^J}\prod_{j<J}\cos(\pi y 2^J). \tag{1.10}$$

$\varphi$ *is integrable and* $\widehat{\varphi}$ *vanishes at infinity, despite the fact that* $\widehat{\varphi}(\pi\lambda 2^j)$ *converges to* 1 *for every* $\lambda \in \mathbb{Z}[2, \frac{1}{2}]$ (*the set of dyadic rational numbers*)*, because for every such* $\lambda$ *there exists* $j \in \mathbb{Z}$ *such that* $\cos(\pi\lambda 2^j) = 0$.

Example 1.3. $\alpha = 2$, $a(k) = \tau(k) = 2^{1-k}, k \in \mathbb{N}$ *gives the dyadic function and* $\widehat{a}(y) = \sum_{k\in\mathbb{N}} 2^{-k}\exp(-2\pi i y 2^{1-k})$ *is Bohr almost periodic* [**2**] *since*

$$|\widehat{a}(y + 2^{L-1}) - \widehat{a}(y)| \le 2^{-L}, \quad y \in \mathbb{R},\ L \in \mathbb{N}. \tag{1.11}$$

*Furthermore,* $\inf_{y\in\mathbb{R}} |\widehat{a}(y)| > 0$, *and* $\lim_{j\to\infty}\widehat{a}(\lambda 2^j) = 1$ *for every* $\lambda \in \mathbb{Z}[2, \frac{1}{2}]$. *Figure 1 shows the modulus of* $\widehat{a}$ *over* $[0, 128]$. $\widehat{\varphi}$ *does not vanish at infinity since* $\lim_{j\to\infty}\widehat{\varphi}(2^j) \approx 0.2578 + 0.0692i$.

Example 1.4. *If* $\alpha = \frac{1+\sqrt{5}}{2}$ *is the Golden Mean then* $\varphi = \begin{bmatrix} \chi_{[0,\alpha^{-1}]} \\ \chi_{[0,1]} \end{bmatrix}$ *satisfies* (*1.6*) *with* $a(1) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $a(2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\tau(1) = 0, \tau(2) = 1$.

$$\widehat{a}(y) = \alpha^{-1}\begin{bmatrix} 0 & 1 \\ e^{-2\pi i y} & 1 \end{bmatrix}, \quad \widehat{\varphi}(0) = \begin{bmatrix} \alpha^{-1} \\ 1 \end{bmatrix}. \tag{1.12}$$

$\widehat{a}(y)$ *is never zero and satisfies* $\lim_{j\to\infty}\widehat{a}(\lambda\alpha^j) = \widehat{a}(0)$ *for all* $\lambda \in \mathbb{Z}[\alpha, \alpha^{-1}]$, *and*

$$\widehat{\varphi}(y\alpha^J) = \begin{bmatrix} e^{-\pi i y \alpha^{J-1}}\frac{\sin(\pi y \alpha^{J-1})}{\pi y \alpha^J} \\ e^{-\pi i y \alpha^J}\frac{\sin(\pi y \alpha^J)}{\pi y \alpha^J} \end{bmatrix} = \left(\prod_{j<J}\alpha^{-1}\begin{bmatrix} 0 & 1 \\ e^{-2\pi i y \alpha^j} & 1 \end{bmatrix}\right)\widehat{\varphi}(0). \tag{1.13}$$

This example is related to the wavelets with PV dilations constructed by Gazeau, Patera and Spiridinov [**13**], [**14**] and multiresolution analyses on quasicrystals [**22**]. For $\lambda \in \mathbb{Z}[\alpha, \alpha^{-1}]$, $\widehat{\varphi}(\lambda \alpha^j)$ is close to the eigenspace of $\widehat{a}(0)$ with eigenvalue $\alpha; -\frac{1}{\alpha}$ for $j \approx -\infty; \infty$. This fact makes $\phi$ integrable despite $\widehat{a}$ never vanishing, in contrast to scalar refinable functions with PV dilations discussed below.

Erdös [**8**] proved that if $\alpha$ is a PV number then the refinable function satisfying

$$\varphi(x) = \frac{|\alpha|}{2}\varphi(\alpha x) + \frac{|\alpha|}{2}\varphi(\alpha x - 1) \tag{1.14}$$

(a Bernoulli convolution) is not integrable by showing that

$$\widehat{\varphi}(\alpha^J) = e^{-\pi i \frac{\alpha^J}{\alpha - 1}} \prod_{j<J} \cos(\pi \alpha^j) \tag{1.15}$$

fails to converge to 0 as $J \to \infty$ We give his proof. Since $||\alpha^j||$ converges to 0 exponentially fast as $|j| \to \infty$, $\cos \pi \alpha^j$ converges to $\pm 1$ exponentially fast and hence $\varphi(\alpha^J)$ converges to 0 iff there exists $j \in \mathbb{Z}$ such that $\cos \pi \alpha^j = 0$, or equivalently if $\alpha^j \in \frac{1}{2} + \mathbb{Z}$. This is impossible because if $\alpha^j \in \mathbb{Q}$ then the Galois conjugates of $\alpha$ would be multiples of each other by roots of a cyclotomic polynomial and thus have identical moduli. Dai, Feng and Wang [**5**] extended Erdös' result to scalar valued refinable functions $\varphi$ that satisfy (1.6) where $a$ has finite support, $\alpha$ is a PV number, and $\tau$ is integer valued. We give their proof. Since $\tau$ is integer valued, $\widehat{a}$ in (1.8) has period 1. Since $a$ has finite support, $\widehat{a}$ is real analytic so the set of its zeros in $[0, 1)$ is a finite set $F$, so the set of zeros of $\widehat{a}$ equals $F + \mathbb{Z}$. If $\widehat{\varphi}$ vanished at infinity then for every $m \in \mathbb{N}$, $\lim_{J\to\infty} \widehat{\varphi}(m\alpha^J) = 0$ and since $||m\alpha^j||$ converges to 0 exponentially fast as $|j| \to \infty$, the same argument used to obtain Erdös' result implies that there exists $j_m \in \mathbb{Z}$ such that $m\alpha^{j_m} \in F + \mathbb{Z}$. Let $d = d(\alpha)$. (1.1) implies that there exist unique $k_m \in \mathbb{Z}$ and $\beta_m \in \alpha\mathbb{Z} + \cdots + \alpha^{d-1}\mathbb{Z}$ with

$$m\alpha^{j_m} = mk_m + m\beta_m, \quad j_m \geq 0 \tag{1.16}$$

and (1.2) implies that

$$m\alpha^{j_m} = (N(\alpha)B(\alpha)))^{j_m} mk_m + (N(\alpha)B(\alpha)))^{j_m} m\beta_m, \quad j_m < 0. \tag{1.17}$$

Since $\alpha^{j_m} \notin \mathbb{Q}$, $\beta_m \neq 0$, choosing a prime $p$ that does not divide $N(\alpha)B(\alpha)$ and letting $m = p^n, n \in \mathbb{N}$ produces a set of values of $(N(\alpha)B(\alpha)))^{j_m} m\beta_m$ that are infinite modulo $\mathbb{Z}$ (infinite modulo $\mathbb{Q}$ since $1, \alpha, ..., \alpha^{d-1}$ are rationally independent), contradicting the fact that $F$ is finite.

This argument holds under the weaker assumption that $a$ decays exponentially fast because then $\widehat{a}$ is a real analytic function with period 1. The remainder of this paper provides support for the following extension of these results:

Conjecture 1.1. *If $\phi$ is a scalar valued refinable function satisfying (1.6,) $\alpha$ is a PV number, $a$ decays exponentially fast, and $\tau$ has values in $\mathbb{Z}[\alpha, \alpha^{-1}]$, then $\widehat{\varphi}$ does not vanish at infinity and hence $\varphi$ is not integrable.*

## 2. Solenoidal Representation

Let $\mathbb{T}^{\mathbb{Z}}$ be the group of functions $g : \mathbb{Z} \to \mathbb{T}$ under pointwise addition and the product topology. Tychonoff's theorem [**31**] implies that $\mathbb{T}^{\mathbb{Z}}$ is compact. Define the *shift automorphism* $\sigma : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^{\mathbb{Z}}$ by $(\sigma g)(j) = g(j + 1), \quad g \in \mathbb{T}^{\mathbb{Z}}, \ j \in \mathbb{Z}$, and homomorphisms $\rho_n : \mathbb{T}^{\mathbb{Z}} \to \mathbb{T}^n$ by $\rho_n(g) = [g(0), ..., g(n-1)]^T, \;\; g \in \mathbb{T}^{\mathbb{Z}}, \; n \in \mathbb{N}$. Let $\alpha$ be a PV number of degree $d = d(\alpha)$. Its Galois conjugates $\alpha = \alpha_1, ...\alpha_d$ are the roots of its minimal polynomial $P_\alpha(X) = X^d + c_{d-1}X^{d-1} + \cdots + c_0 \in \mathbb{Z}[X]$. Define the Vandermonde $V$, Frobenius Companion $C$, and Diagonal $D$, matrices

$$V = \begin{bmatrix} 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_d \\ \vdots & \dots & \vdots \\ \alpha_1^{d-1} & \dots & \alpha_d^{d-1} \end{bmatrix} C = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \\ -c_0 & \dots & \dots & -c_{d-1} \end{bmatrix} D = \begin{bmatrix} \alpha_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_d \end{bmatrix}$$

$\det V = \prod_{i<j}(\alpha_j - \alpha_i) \neq 0 \Rightarrow V$ is invertible, $CV = VD$, and $V^{-1}C = DV^{-1}$. Let $S_- = \{[0, s_2, ..., s_d]^T : s_j \in \mathbb{C}, \; \alpha_j = \overline{\alpha}_k \Rightarrow s_j = \overline{s}_k\}$, $S_+ = \{[s_1, 0, ..., 0]^T : s_1 \in \mathbb{R}\}$, $S = S_+ + S_-$. These are real subspaces of $\mathbb{C}^d$ with dimensions $d-1, 1, d$. Construct homomorphisms $\vartheta : S \to \mathbb{T}^{\mathbb{Z}}$ and $\theta : \mathbb{R} \to \mathbb{T}^{\mathbb{Z}}$,

$$\vartheta(s)(j) = \sum_{k=1}^{d} s_k \, \alpha_k^j + \mathbb{Z}, \quad s \in S, \; j \in \mathbb{Z}, \tag{2.1}$$

$$\theta(y)(j) = y \, \alpha^j + \mathbb{Z}, \quad y \in \mathbb{R}, \; j \in \mathbb{Z}, \tag{2.2}$$

and $\sigma$ invariant subgroups $M_\pm = \vartheta(S_\pm)$, $M = M_+ + M_-$, and $G = \overline{M}$. Then $DS_\pm = S_\pm$, $S = V^{-1}R^d$, $M = \vartheta(S)$, $\rho_d(M) = \mathbb{T}^d$, $\rho_d(G) = \mathbb{T}^d$, and hence $G$ is a compact, connected, abelian group with dimension $\geq d$, the restriction $\sigma : G \to G$ gives a dynamical system $(G, \sigma)$ that is an *extension* [**11**] of the dynamical system $(\mathbb{T}^d, C)$ since $C \circ \rho_d = \rho_d \circ \sigma$. $0 \in G$ is an equilibrium point of $\sigma$ and $M_+; M_-$ is the unstable;stable manifold of 0. Define $K =$ kernel of $\rho_d : G \to \mathbb{T}^d$, $K_- = \bigcup_{j \in \mathbb{Z}} \sigma^j(K)$, $G_+ = M_+$, $G_- = M_- + K_-$, and $H = G_+ \cap G_-$. Then $G = M + K = G_+ + G_-$ and the orbits of points in $H$ are *homoclinic* since $h \in H \Rightarrow \lim_{j \to \pm\infty} \sigma^j h = 0$.

Remark 2.1. *Following Lind and Ward* ([**23**], *p. 411*) *we define a solenoid to be a compact, connected, dimension $n < \infty$ abelian group. Its dual group* (*see Section 4*) *is a discrete, torsion free, rank $n$ abelian group, or equivalently, a subgroup of $\mathbb{Q}^n$. The dynamical system $(G, \sigma)$ is expansive, as defined by Lam* [**19**], *if there exists a neighborhood $U$ of 0 such that $\bigcap_{j \in \mathbb{Z}} \sigma^j(U) = \{0\}$. In* [**20**] *we used topological entropy and Pontryagin duality to prove that every group that admits an expansive automorphism is a solenoid. Schmidt (*[**29**], *Chap. III) used algebraic methods to characterize more general expansive transformation groups.*

Theorem 2.1. *The assumptions above imply that:*

1. *$G$ is isomorphic to a group extension of $\mathbb{T}^d$ by $K$.*
2. *If $|c_0| = 1$ then $K = \{0\}$ and $G$ is isomorphic to $\mathbb{T}^d$.*
3. *If $|c_0| > 1$ then $K$ is homeomorphic to Cantor's set.*
4. *$G$ is a $d$-dimensional solenoid and $(G, \sigma)$ is expansive.*
5. *$\theta(\mathbb{R}) = \vartheta(S_+)$ is dense in $G$.*
6. *$\theta(\Lambda_\alpha) = H$.*

Proof. (1) holds since $\rho_d : G \to \mathbb{T}^d$ is surjective. (2-3) hold since (2.1) $\Rightarrow$ every $g \in G_0$, hence every $g \in G$, satisfies

$$(P_\alpha(\sigma)g)(k) = c_0 g(k) + \cdots + c_{d-1} g(k + d - 1) + g(k + d) = 0, \quad k \in \mathbb{Z},$$

so $K = \{ g \in \mathbb{T}^{\mathbb{Z}} : g(k) = 0, k \geq 0 \text{ and } c_0^{-k} g(k) = 0, k < 0\}$. (4) holds since $G = G_+ + G_-$. (5) holds since $(\sigma\theta(x))(j) = \theta(x)(j+1) = \theta(x\alpha) \Rightarrow \theta(\mathbb{R})$ is $\sigma$-invariant and $\rho_d(\theta(\mathbb{R})) = R[1, \alpha, ...., \alpha^{d-d}]^T + \mathbb{Z}^d$ is dense in $\mathbb{T}^d$ by the Kronecker-Weyl theorem [**33**],(Appendix, Theorem 4.1) since the entries of $[1, \alpha, ..., \alpha^{s-1}]^T$ are rationally independent. (6) holds since $\theta(s_1) \in H \Leftrightarrow \theta(s_1) \in M_- + K_-$ iff there exist $-[0, s_2, ..., s_d]^T \in S_-, m \in \mathbb{Z}$ with $\theta(s_1) + \vartheta([0, s_2, ..., s_d]^T) \in \sigma^m K \Leftrightarrow \rho_d(\sigma^{-m}\vartheta(s)) = 0$ where $s = [s_1, s_2, ..., s_d]^T$. Theorem 1.1 implies that this condition holds iff $VD^{-m}s = [T(\alpha^{-m}s_1), ..., T(\alpha^{-m+d-1}s_1)]^T \in \mathbb{Z}^d \Leftrightarrow s_1 \in \Lambda_\alpha$. $\square$

Assume that $\alpha$ is a PV number of degree $d = d(\alpha)$, $a : \mathbb{N} \to \mathbb{C}$ decays exponentially fast, $\sum_{k\in\mathbb{N}} a(k) = 1$, and $\tau : \mathbb{N} \to \mathbb{Z}[\alpha, \alpha^{-1}]$. For $k \in \mathbb{N}$ define $\tau_k : \mathbb{Z} \to \mathbb{Z}$ such that $\tau(k) = \sum_{j\in\mathbb{Z}} \tau_k(j)\alpha^j$. Each $\tau_k$ is finitely supported. Construct $A : G \to \mathbb{C}$

$$A(g) = |\alpha|^{-1} \sum_{k\in\mathbb{N}} a(k) \exp\left(-2\pi i \sum_{j\in\mathbb{Z}} \tau_k(j) g(k)\right), \quad g \in G. \tag{2.3}$$

THEOREM 2.2. *These assumptions give the solenoidal representation $\widehat{a} = A \circ \theta$.*

PROOF. Follows from (1.8) and (2.2). $\square$

## 3. Zero Sets

This section develops relationships between and properties of the zero sets $\mathcal{S}(\widehat{\phi})$, $\mathcal{S}(\widehat{a})$, and $\mathcal{S}(A)$. Since $a$ has exponential decay $A$ is a real-analytic function so $\mathcal{S}(A)$ is a real-analytic set. Theorem 2.2 implies $\theta(\mathcal{S}(\widehat{a})) \subset \mathcal{S}(A)$ and (1.7) implies that

$$\mathcal{S}(\widehat{\varphi}) = \alpha\mathcal{S}(\widehat{a}) + \alpha\mathcal{S}(\widehat{\varphi}), \tag{3.1}$$

(+ of zero multiplicities) so the lower $\underline{d}$ and upper $\overline{d}$ asymptotic densities satisfy

$$\underline{d}(\mathcal{S}(\widehat{a})) = (|\alpha| - 1)\, \underline{d}(S(\widehat{\varphi})) \leq \overline{d}(\mathcal{S}(\widehat{a})) = (|\alpha| - 1)\, \overline{d}(S(\widehat{\varphi})). \tag{3.2}$$

THEOREM 3.1. *If $A$ is not zero then $\overline{d}(\mathcal{S}(\widehat{a})) < \infty$.*

PROOF. $\mathcal{H}(G) = \{$ closed subsets of $G \}$ with the Hausdorff topology is a compact space ([**10**], p. 205, 255) and $\mathcal{T}(G) = \{\theta([0, 1]) + g : g \in G\}$ is a closed subset and hence compact. If $\overline{d}(\mathcal{S}(\widehat{a})) = \infty$ then there exist $g \in G$ and a sequence $x_n \in \mathbb{R},\ n \in \mathbb{N}$ such that $A$ has at least $n$ zeros in $\theta([x_n, x_n + 1])$ and $\lim_{n\to\infty} \theta(x_n) = g$. Define $f(x) = |A(\theta(x) + g)|^2,\ x \in \mathbb{R}$. Then either $f$ has an infinite number of zeros in the interval $[0, 1]$ or Rolle's theorem implies that it has a zero of infinite order in $[0, 1]$. Since $A$ and hence $f$ is real-analytic, $f = 0$. Since $\theta(\mathbb{R})$, and hence $\theta(\mathbb{R}) + g$, is dense in $G$, $A$ is zero thus giving a contradicting. $\square$

For $m \in \mathbb{Z}$ and $\epsilon = [\epsilon_2, ..., \epsilon_d]$ satisfying $\epsilon_k > 0$ and $\alpha_j = \overline{\alpha}_k \Rightarrow \epsilon_j = \epsilon_k$ define

$$U(m, \epsilon) = \sigma^m K + \{ \vartheta(s) : s \in S_- \text{ and } |s_k| < \epsilon_k,\ k = 2, ..., d \} \subset G_-. \tag{3.3}$$

Since $U(m, \epsilon)$ decreases to $\{0\}$ as $m \to \infty$ and $\epsilon_k \to 0$ we can, and will, choose $m$ and $\epsilon$ so that $A$ never vanishes on $U(m, \epsilon)$. Also, $U(m, \epsilon)$ is $\sigma$ invariant since $\sigma U(m, \epsilon) = U(m + 1, [\epsilon_2|\alpha_2|, ..., \epsilon_d|\alpha_d|]) \subset U(m, \epsilon)$. Let $a$ be the number of real $\alpha_k, k = 2, ..., d$ and $b$ be the number of complex conjugate pairs of $\alpha_k, k = 2, ..., d$, and let $\gamma = |\det V|\, |c_0|^{-m}\, 2^a \pi^b\, \epsilon_1 \cdots \epsilon_d$. For $L > 0$ define

$$W(L) = \{ VD^{-m}[y, s_2, ..., s_d]^T : y \in (-L, L), |s_k| < \epsilon_k,\ k = 2, ..., d \},$$
$$Y(L) = \{ y \in (-L, L) : \theta(y) \in U(m, \epsilon) \}.$$

The sets $W(L)$, $L > 0$ are convex cylinders parallel to the vector $[1, \alpha, ..., \alpha^{d-1}]^T$ whose entries are rationally independent, therefore

$$\lim_{L\to\infty} \frac{1}{2L} \text{ card } W(L) \cap \mathbb{Z}^d = \frac{1}{2L} \text{vol } W(L) = \gamma. \tag{3.4}$$

LEMMA 3.1. *The $\mathbb{R}$-linear function $\xi : W(L) \cap \mathbb{Z}^d \to (-L, L)$ defined by $\xi(w) = [1, 0, ..., 0]\, D^m V^{-1} w$ is a bijection onto $Y(L)$. Therefore*

$$\lim_{L\to\infty} \frac{1}{2L} \text{ card } Y(L) = \gamma. \tag{3.5}$$

PROOF. Assume $w \in W(L) \cap \mathbb{Z}^d$. If $\xi(w) = 0$ then $w \in VD^{-m}S_-$ and hence $\lim_{\ell\to\infty} C^\ell w = 0$. Since $w \in \mathbb{Z}^d$, $w = 0$, so $\xi$ is injective. Assume that $y \in (-L, L)$. Then $\theta(y) \in U(m, \epsilon)$ iff there exist $-s_k$ that satisfy $|s_k| < \epsilon_k, k = 2, ..., d$ and

$$\theta(y) \in \sigma^j K + \vartheta([0, -s_2, ..., -s_d]^T) \Leftrightarrow \sigma^{-m}\vartheta([y, s_2, ..., s_d]^T) \in K$$
$$\Leftrightarrow \rho_d(\sigma^{-m}\vartheta([y, s_2, ..., s_d]^T)) = 0 \Leftrightarrow VD^{-m}[y, s_2, ..., s_d]^T \in \mathbb{Z}^d.$$

Since $VD^{-m}[y, s_2, ..., s_d]^T \in W(L)$, the last inclusion holds iff $y \in \xi(W(L) \cap \mathbb{Z}^d)$. This shows that $\xi$ maps $W(L) \cap \mathbb{Z}^d$ onto $Y(L)$. Then (3.4) $\Rightarrow$ (3.5). $\square$

THEOREM 3.2. *If $\widehat{\varphi}$ vanishes at infinity then $\underline{d}(\mathcal{S}(\widehat{\varphi})) \geq \gamma$.*

PROOF. (3.5) implies that it suffices to show that for every $L > 0$ it suffices to show that $\widehat{\varphi}$ equals 0 at every point in $Y(L)$. Since $\widehat{\varphi}$ vanishes at infinity,

$$0 = \lim_{J\to\infty} \widehat{\varphi}(\alpha^J y) = \lim_{J\to\infty} \widehat{\varphi}(y) \prod_{j=1}^{J} \widehat{a}(\alpha^j y).$$

Since $y \in Y(L)$ and $\jmath \geq 1$ implies $\theta(\alpha^j y) = \sigma^j \theta(y) \in U(m, \epsilon)$, and since $A$ never vanishes on $U(m, \epsilon)$, (2.2) implies that $\widehat{a}(\alpha^j y) = A(\sigma^j(\theta(y)) \neq 0$. Since $\sigma^j(\theta(y))$ converges to 0 exponentially fast, $\prod_{j=1}^{\infty} \widehat{a}(\alpha^j y) \neq 0$, and hence $\widehat{\varphi}(y) = 0$. $\square$

COROLLARY 3.1. *If $\widehat{\varphi}$ vanishes at infinity then $\underline{d}(\mathcal{S}(\widehat{a})) \geq (|\alpha| - 1)\,\gamma$. $\mathcal{S}(A)$ is a union of embedded manifolds in $G$ and has dimension $d - 1$.*

PROOF. The first assertion follows from (3.2). Since $\mathcal{S}(A)$ is an real-analytic set it is homeomorphic to a union of embedded manifolds by Lojasiewicz's structure theorem for real-analytic sets [**17**], [**21**], [**24**]. Since $\theta(\mathbb{R})$ is a uniformly distributed embedding, if the dimension of $\mathcal{S}(A)$ were less than $d - 1$ then $\underline{d}(\mathcal{S}(\widehat{a})) = 0$. $\square$

THEOREM 3.3. *If $\alpha$ is a PV number of degree $d \geq 2$ then the set of norms $N(\Lambda_\alpha)$ is a set of rational numbers whose denominators have only a finite number of prime divisors and whose numerators are values of integral forms* (*homogeneous polynomials*) *of degree $d$ in $d$ integer variables. The number of these integer values having modulus $\leq L$ is asymptotically bounded below by $O(L^{2/d})$ for $d \geq 3$ and by $O(L/(\log L)^p)$ for some $p \in (0, 1)$ for $d = 2$.*

PROOF. Let $\alpha = \alpha_1, ...\alpha_d$ be the Galois conjugates of $\alpha$. Theorem 1.1 implies that $\lambda \in \Lambda_\alpha$ iff there exist $m \in \mathbb{Z}$ and $\mu_k \in \mathbb{Q}[\alpha_k]$ such that $\lambda = \mu_1 \alpha^j$ where $[\mu_1, ..., \mu_d]^T \in V^{-1}\mathbb{Z}^d$. The elements of the $k$th row of $V^{-1}$, being the coefficients of the Lagrange interpolating polynomial $Q_k(X) = \frac{P_\alpha(X)}{(X-\alpha_k)P'(\alpha_k)}$, belong to the field $\mathbb{Q}[\alpha_k]$ and the elements in every column of $V^{-1}$ are Galois conjugates. Therefore $N(\lambda) = N(\alpha)^m\, N(\mu_1)$, and $N(\mu_1) = \prod_{k=1}^{d} \mu_k$ is a form with rational coefficients of degree $d$ in $d$ integer variables (the coordinates of $\mathbb{Z}^d$). The denominators of the coefficients of the form $N(\mu_1)$ must divide $\det V$ so the prime factors of the denominators of the numbers in $N(\Lambda_\alpha)$ must divide $N(\alpha)$ or $\det V$. Therefore a positive fraction of the numbers in $N(\Lambda_\alpha)$ have numerators that are values of an integral form of degree $d$ in $d$ integer variables. For $d \geq 3$ we obtain a binary form of degree 3 by setting all except 2 of these integer variables to 0, and obtain the lower asymptotic bound for the values of the numerators by a Theorem of Erdös and Mahler [**9**]. For $d = 2$ we obtain a binary quadratic form and we obtain a lower asymptotic bound given by a Theorem of Odoni ([**26**], Theorem S). □

We refer the reader to Section 4 for a discussion of Pontryagin duality. If $\chi \in \widehat{G}$ and $c \in \mathbb{T}_c$ then the zero set $\mathcal{S}(\chi - c) = \{g \in G : \chi(g) - c = 0\}$ has dimension $d-1$. We call a real-analytic subset of $G$ *simple* if it is contained in a finite union of such sets. Lagarias and Yang conjectured [**18**] that certain real-analytic subsets of $\mathcal{T}^n$, that arise in the construction of refinable functions of several variables related to tilings and that are analogous to our set $\mathcal{S}(A)$, are simple. We used Lojasiewicz's theorem [**21**] to prove their conjecture. Thus we find the following result interesting:

THEOREM 3.4. *If $A$ is nonzero then $\mathcal{S}(A)$ is not simple.*

PROOF. The argument used in the proof of Theorem 3.2 shows that for every $\lambda \in \Lambda_\alpha$ there exists $m \in \mathbb{Z}$ such that $\lambda\alpha^m \in \mathcal{S}(\widehat{a})$. Therefore Theorem 3.3 implies that the set of norms $N(\mathcal{S}(\widehat{a}))$ contains a set of rational numbers whose numerators whose modulus is less than $L$ has asymptotic density $> O(L^{1/d})$. If $\mathcal{S}(A)$ were a proper simple subset of $G$ then all but a finite number of points in $\mathcal{S}(\widehat{a})$ would be contained in a finite union of sets having the form $\beta + \delta\mathbb{Z}$ where $\beta$ and $\delta$ are elements in $\mathbb{Q}[\alpha]$. However $N(\beta + \delta k)$ is a form of degree $d$ with rationall coefficients in single integer variable $k$ and therefore the set of numerators of the values of this form has asymptotic density $O(L^{1/d})$, thus giving a contradiction. □

REMARK 3.1. *Integral binary quadratic forms were studied by Gauss* [**12**]*, who focussed on forms having negative discriminant. Asymptotic estimates for the number of integers represented by integral binary quadratic forms with negative discriminant were obtained in the doctoral dissertation of Bernays* [**1**] *and by James* [**16**]*. Numerical studies were compiled by Sloan* [**30**]*.*

Inspired by developments in Diophantine geometry and o-minimal theory we make the following assertion whose validity proves Conjecture 1.1.

CONJECTURE 3.1. *Every real-analytic subset of $G$ that intersects every homoclinic orbit is simple.*

## 4. Appendix: Pontryagin Duality and the Kronecker-Weyl Theorem

A *character* of a locally compact abelian topological group $G$ is a continuous homomorphism $\chi : G \to \mathbb{T}_c$. The *dual group* $\widehat{G}$ consists of all characters under pointwise multiplication and the topology of uniform convergence on compact subsets. The *Pontryagin duality* theorem says that the homomorphism $\gamma : G \to \widehat{\widehat{G}}$

$$\gamma(g)(\chi) = \chi(g), \quad g \in G, \ \chi \in \widehat{G} \tag{4.1}$$

is a bijective isomorphism. This was proved for second countable groups that are either compact or discrete in 1934 by Lev Semyonovich Pontryagin [**28**] and extended to general locally compact groups in 1934 by Egbert van Kampen [**32**]. This theory shows that $G$ is compact; discrete; connected; dimension $d$ iff $\widehat{G}$ is discrete; compact; torsion free; rank $d$. For $a \in \mathbb{Z}^n$ define $\chi_a \in \widehat{\mathbb{T}^n}$ by $\chi_a(g) = \exp 2\pi i(a(1)g(1) + \cdots + a(n)g(n))$. The mapping $a \to \chi_a$ is an isomorphism. If $H$ is a closed subgroup of $G$ then the quotient $G/H$ is locally compact and we have an obvious injective homomorphism $\widehat{G/H} \to \widehat{G}$. Therefore Pontrygin duality implies that $H$ is proper iff every character on $G$ that vanishes on $H$ vanishes on $G$. This gives the following classical result [**33**].

LEMMA 4.1. *(Kronecker-Weyl Theorem) If $v \in \mathbb{R}^n$ then $\mathbb{R}v + \mathbb{Z}^n$ is dense in $\mathbb{T}^n$ iff the entries of $v$ are rationally independent.*

PROOF. The closure $H = \overline{\mathbb{R}v + \mathbb{Z}^n}$ is a closed subgroup of $\mathbb{T}^n$ and is proper iff there exists $a \in \mathbb{Z}^n \backslash \{0\}$ with $\chi_a(tv + \mathbb{Z}^n) = \exp(2\pi i t a^T v),\ t \in \mathbb{R}$, or equivalently, $a^T v = 0$. This occurs iff the entries of $v$ are rationally dependent. □

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