

# Dimensions of irreducible modules for partition algebras and tensor power multiplicities for symmetric and alternating groups

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## Abstract

The partition algebra  $P_k(n)$  and the symmetric group  $S_n$  are in Schur-Weyl duality on the  $k$ -fold tensor power  $M_n^{\otimes k}$  of the permutation module  $M_n$  of  $S_n$ , so there is a surjection  $P_k(n) \rightarrow Z_k(n) := \text{End}_{S_n}(M_n^{\otimes k})$ , which is an isomorphism when  $n \geq 2k$ . We prove a dimension formula for the irreducible modules of the centralizer algebra  $Z_k(n)$  in terms of Stirling numbers of the second kind. Via Schur-Weyl duality, these dimensions equal the multiplicities of the irreducible  $S_n$ -modules in  $M_n^{\otimes k}$ . Our dimension expressions hold for any  $n \geq 1$  and  $k \geq 0$ . Our methods are based on an analog of Frobenius reciprocity that we show holds for the centralizer algebras of arbitrary finite groups and their subgroups acting on a finite-dimensional module. This enables us to generalize the above result to various analogs of the partition algebra including the centralizer algebra for the alternating group acting on  $M_n^{\otimes k}$  and the quasi-partition algebra corresponding to tensor powers of the reflection representation of  $S_n$ .

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# 1 Introduction

The partition algebras  $P_k(\xi)$ ,  $\xi \in \mathbb{C}$ , were introduced by Martin ([M1],[M2],[M3]) to study the Potts lattice model of interacting spins in statistical mechanics. As shown by Jones [J], there is a Schur-Weyl duality between the partition algebra  $P_k(n)$  and the symmetric group  $S_n$  acting as centralizers of each other on the  $k$ -fold tensor power  $M_n^{\otimes k}$  of the  $n$ -dimensional permutation module  $M_n$  for  $S_n$  over  $\mathbb{C}$ . The surjective algebra homomorphism given in [J] (see also [HR, Thm. 3.6]),

$$P_k(n) \rightarrow Z_k(n) := \text{End}_{S_n}(M_n^{\otimes k}) = \{T \in \text{End}(M_n^{\otimes k}) \mid T(\sigma.u) = \sigma.T(u) \quad \forall \sigma \in S_n, u \in M_n^{\otimes k}\},$$

is an isomorphism when  $n \geq 2k$ .

The partition algebra  $P_k(\xi)$  for  $k \geq 1$  has a basis over  $\mathbb{C}$  indexed by set partitions of the set  $\{1, 2, \dots, 2k\}$  into disjoint nonempty blocks. An example of such a set partition for  $k = 7$  is  $\{1, 9, 11 \mid 2, 13 \mid 3 \mid 4, 7, 8 \mid 5, 6, 12, 14\}$ , which has 5 blocks. The Stirling number  $\left\{ \begin{smallmatrix} 2k \\ r \end{smallmatrix} \right\}$  counts the number of ways to partition  $2k$  objects into  $r$  nonempty disjoint blocks, so it follows that

$$\dim P_k(\xi) = \sum_{r=1}^{2k} \left\{ \begin{smallmatrix} 2k \\ r \end{smallmatrix} \right\} = B(2k), \quad (\text{the } (2k)\text{th Bell number}).$$

In  $P_{k+1}(\xi)$ , the basis elements indexed by set partitions which have  $k+1$  and  $2(k+1)$  in the same block form a subalgebra  $P_{k+\frac{1}{2}}(\xi)$  with  $\dim P_{k+\frac{1}{2}}(\xi) = B(2k+1)$ . If we regard  $M_n$  as a module for the symmetric group  $S_{n-1}$  by restriction, there is a surjective algebra homomorphism  $P_{k+\frac{1}{2}}(n) \rightarrow Z_{k+\frac{1}{2}}(n) := \text{End}_{S_{n-1}}(M_n^{\otimes k})$ , which is an isomorphism if  $n \geq 2k$ . These intermediate algebras play an important role in understanding the structure and representation theory of partition algebras (see for example, [MR, HR]), and they are a crucial component of the work in this paper.

The irreducible modules for  $P_k(n)$  and  $P_{k+\frac{1}{2}}(n)$  are labeled by partitions  $\nu$  of  $r$ , where  $r$  is an integer satisfying  $0 \leq r \leq k$ . Since the irreducible modules  $S_n^\lambda$  for  $S_n$  are indexed by partitions  $\lambda$  of  $n$ , Schur-Weyl duality implies that the irreducible modules for  $Z_k(n)$  are also indexed by partitions  $\lambda$  of  $n$ , and for  $Z_{k+\frac{1}{2}}(n)$ , by partitions  $\mu$  of  $n-1$ . The modules  $S_n^\lambda$  (resp.  $S_{n-1}^\mu$ ) occurring in  $M_n^{\otimes k}$  are indexed by partitions with the property that the partition  $\nu = \lambda^\#$  (resp.  $\nu = \mu^\#$ ) that results from deleting the largest part of  $\lambda$  (resp. of  $\mu$ ) must satisfy  $0 \leq |\nu| \leq k$ , where  $|\nu|$  is the sum of the parts of  $\nu$ .

In this paper, we

- establish general restriction/induction results for centralizer algebras, proving in Theorem 2.7 that an analog of Frobenius reciprocity for groups holds for their centralizer algebras;
- give restriction/induction Bratteli diagrams for the symmetric group-subgroup pair  $(S_n, S_{n-1})$  and for the alternating group-subgroup pair  $(A_n, A_{n-1})$ ;
- use the reciprocity results to determine expressions for the dimensions of the irreducible modules for  $Z_k(n)$ , and  $Z_{k+\frac{1}{2}}(n)$  (in Theorem 5.5(a) and (b)), and for  $P_k(\xi)$ ,  $P_{k+\frac{1}{2}}(\xi)$  (in Corollary 5.14);
- determine the dimensions of the centralizer algebras  $Z_k(n)$  and  $Z_{k+\frac{1}{2}}(n)$  (in Theorem 5.5(c) and (d));

- give a combinatorial proof in Section 5.3 of the dimension formula in Theorem 5.5(a) by exhibiting a bijection between paths in the Bratteli diagram (aka vacillating tableaux) and pairs  $(P, T)$  consisting of set partitions  $P$  of  $\{1, \dots, k\}$  and semistandard tableaux  $T$  whose entries depend on  $P$  (this bijection holds for all  $k \geq 0$  and  $n \geq 1$  and extends the one in [CDDSY], which was valid for  $n \geq 2k$ );
- apply the restriction/induction results to the pair  $(S_n, A_n)$  (resp.  $(S_{n-1}, A_{n-1})$ ) to determine the dimensions of the irreducible modules for the centralizer algebras  $\widehat{Z}_k(n) := \text{End}_{A_n}(M_n^{\otimes k})$  and  $\widehat{Z}_{k+\frac{1}{2}}(n) := \text{End}_{A_{n-1}}(M_n^{\otimes k})$  (in Theorem 6.1 (a) and (b));
- determine dimension formulas for the centralizer algebras  $\widehat{Z}_k(n)$  and  $\widehat{Z}_{k+\frac{1}{2}}(n)$  (Theorem 6.1);
- compute the dimensions of the irreducible modules for the centralizer algebras  $\text{QZ}_k(n) := \text{End}_{S_n}(R_n^{\otimes k})$ , where  $R_n := S_n^{[n-1, 1]}$  is the  $(n-1)$ -dimensional irreducible reflection module of  $S_n$  corresponding to the partition  $[n-1, 1]$  of  $n$ , and for their relatives  $\text{QZ}_{k+\frac{1}{2}}(n)$ ,  $\widehat{\text{QZ}}_k(n)$ , and  $\widehat{\text{QZ}}_{k+\frac{1}{2}}(n)$  (in Theorem 7.1) and give Bratteli diagrams corresponding to  $R_n$  for the group-subgroup pairs  $(S_n, S_{n-1})$  and  $(A_n, A_{n-1})$ .

By Schur-Weyl duality, the dimension of an irreducible module for the centralizer algebra equals the multiplicity of the corresponding irreducible module for the group. Consequently, our dimension formulas also

- determine the multiplicities of irreducible modules for  $S_n, S_{n-1}, A_n$ , and  $A_{n-1}$  in  $M_n^{\otimes k}$  and in  $R_n^{\otimes k}$  for all  $n, k \in \mathbb{Z}_{>0}$  (in Theorems 5.5 (a),(b), 6.1, and 7.1).

A preliminary version of this paper [BH1], posted on the arXiv by the first two authors, established dimension formulas for centralizer algebras as alternating sums of expressions involving Stirling numbers of the second kind and the number of standard tableaux compatible with certain  $r$ -sequences of partitions for  $\lambda$ . Upon seeing this result, the third author suggested the approach that we adopt in this paper for Theorem 5.5 (a). As a consequence of this alternate way of computing the dimensions of the irreducible modules for  $Z_k(n)$ , we are able in this work to express all of the dimension formulas as positive sums using Stirling numbers of the second kind and Kostka numbers. It remains an open question to determine the relation between these two approaches.

The dimensions of the centralizer algebras  $Z_k(n)$  and  $\widehat{Z}_k(n)$  were determined previously and can be found in [HR] and [B11, B12], respectively. In this work, they are direct consequences of the dimension formulas for the irreducible modules. This is a general phenomenon: If  $Z_k(G) := \text{End}_G(X^{\otimes k})$  for a self-dual module  $X$  of a group  $G$ , then  $\dim Z_k(G) = \dim (X^{\otimes 2k})^G$ , where  $(X^{\otimes 2k})^G$  is the space of  $G$ -invariants in  $X^{\otimes 2k}$ . Therefore,  $\dim Z_k(G)$  is the multiplicity of the trivial  $G$ -module  $G^\bullet$  in  $X^{\otimes 2k}$ ; equivalently, by Schur-Weyl duality, it is the dimension of the irreducible module associated to  $G^\bullet$  for the centralizer algebra  $Z_{2k}(G)$  (see Section 2 for details).

Motivated by the work of Goupil and Chauve [GC] on Kronecker tableaux and Kronecker coefficients, Daugherty and Orellana in [DO] introduced the *quasi-partition algebras*  $\text{QP}_k(\xi)$ ,  $\xi \in \mathbb{C}$ , and showed that there is a surjection  $\text{QP}_k(n) \rightarrow \text{QZ}_k(n) = \text{End}_{S_n}(R_n^{\otimes k})$  for  $R_n = S_n^{[n-1, 1]}$ , which is an isomorphism when  $n \geq 2k$ . The dimensions for the irreducible modules for  $\text{QP}_k(\xi)$ , with  $\xi$  generic, are the same as the dimensions for  $n \geq 2k$ , and so are given by the dimension formulas in

Section 7 below. These expressions differ from the ones that appear in [DO], which were based on results in [GC] and hold whenever  $n \geq 2k$ , as the ones in Section 7 are valid for all  $k$  and  $n$ .

Using exponential generating functions from [GC], Ding [D] derived a formula for the multiplicity of the irreducible  $S_n$ -module  $S_n^\lambda$  indexed by the partition  $\lambda = [\lambda_1, \dots, \lambda_n]$  in  $M_n^{\otimes k}$  when  $1 \leq k \leq n - \lambda_2$  and used that to obtain an expression for the multiplicity of  $S_n^\lambda$  in tensor powers  $R_n^{\otimes k}$  of its reflection module  $R_n = S_n^{[n-1,1]}$ . The first is a special case of part (a) of Theorem 5.5 below and the second a special case of Theorem 7.1. As shown in [D, Sec. 3], when  $1 \leq k \leq n - \lambda_2$ , these multiplicity formulas can be used to bound the mixing time of a Markov chain on  $S_n$ .

## 2 Restriction/Induction and Dimensions

We begin with some general results on restriction and induction for centralizer algebras and then apply these results to the group-subgroup pairs  $(S_n, S_{n-1})$ ,  $(S_n, A_n)$ , and  $(A_n, A_{n-1})$  acting on the  $k$ -fold tensor power of the  $n$ -dimensional permutation module  $M_n$ . This will enable us to determine the dimension of the centralizer algebras and their irreducible modules.

Suppose  $G$  is a finite group and  $H$  is a subgroup of  $G$ . Assume  $\{G^\lambda\}_{\lambda \in \Lambda_G}$  and  $\{H^\alpha\}_{\alpha \in \Lambda_H}$  are the corresponding sets of irreducible modules for these groups over  $\mathbb{C}$ . We suppose that the restriction from  $G$  to  $H$  on  $G^\lambda$  is given by

$$\text{Res}_H^G(G^\lambda) = \bigoplus_{\alpha \in \Lambda_H} c_\alpha^\lambda H^\alpha. \quad (2.1)$$

Then by Frobenius reciprocity, induction from  $H$  to  $G$  is given by

$$\text{Ind}_H^G(H^\alpha) = \bigoplus_{\lambda \in \Lambda_G} c_\alpha^\lambda G^\lambda. \quad (2.2)$$

Assume now that  $X$  is a finite-dimensional  $G$ -module, and consider the centralizer algebra  $Z_X(G) = \text{End}_G(X) = \{T \in \text{End}(X) \mid T(g.x) = g.T(x), \forall g \in G, x \in X\}$ . Regarding  $X$  as a module for the subgroup  $H$  of  $G$  by restriction, we have reverse inclusion of the centralizer algebras  $Z_X(G) \subseteq Z_X(H) = \text{End}_H(X)$ . Let  $\Lambda_{X,G}$  (resp.  $\Lambda_{X,H}$ ) denote the subset of  $\Lambda_G$  (resp. of  $\Lambda_H$ ) corresponding to the irreducible  $G$ -modules (resp.  $H$ -modules) which occur in  $X$  with multiplicity at least one. Then *Schur-Weyl duality* implies the following:

- the irreducible  $Z_X(G)$ -modules  $Z_{X,G}^\lambda$  are in bijection with the elements of  $\lambda \in \Lambda_{X,G}$ ;
- the decomposition of  $X$  into irreducible  $G$ -modules is given by

$$X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} d_{X,G}^\lambda G^\lambda, \quad \text{where } d_{X,G}^\lambda = \dim Z_{X,G}^\lambda; \quad (2.3)$$

- the decomposition of  $X$  into irreducible  $Z_X(G)$ -modules is given by

$$X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} d_{G^\lambda} Z_{X,G}^\lambda, \quad \text{where } d_{G^\lambda} = \dim G^\lambda; \quad (2.4)$$

- as a bimodule for  $G \times Z_X(G)$ ,

$$X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} (G^\lambda \otimes Z_{X,G}^\lambda); \quad (2.5)$$

- $Z_X(G)$  is a finite-dimensional semisimple associative algebra and

$$\dim Z_X(G) = \sum_{\lambda \in \Lambda_{X,G}} (\dim Z_{X,G}^\lambda)^2 = \sum_{\lambda \in \Lambda_{X,G}} (d_{X,G}^\lambda)^2. \quad (2.6)$$

There is a corresponding Frobenius reciprocity for centralizer algebras of the group-subgroup pair  $(G, H)$ , as indicated in the next result.

**Theorem 2.7.** *For a finite-dimensional  $G$ -module  $X$ , let  $Z_X(G) = \text{End}_G(X)$  and  $Z_X(H) = \text{End}_H(X)$ . Let  $\Lambda_{X,G}$  (resp.  $\Lambda_{X,H}$ ) be the set of indices  $\lambda \in \Lambda_G$  (resp.  $\alpha \in \Lambda_H$ ) such that  $G^\lambda$  (resp.  $H^\alpha$ ) occurs in  $X$  with multiplicity  $\geq 1$ , and let  $Z_{X,G}^\lambda$  (resp.  $Z_{X,H}^\alpha$ ) denote the corresponding irreducible  $Z_X(G)$ -module (resp.  $Z_X(H)$ -module). Assume  $c_\alpha^\lambda$  is as in (2.1) and (2.2) above. Then the following hold:*

- (a)  $\text{Res}_{Z_X(G)}^{Z_X(H)}(Z_{X,H}^\alpha) = \bigoplus_{\lambda \in \Lambda_{X,G}} c_\alpha^\lambda Z_{X,G}^\lambda.$
- (b) For  $\alpha \in \Lambda_{X,H}$ ,  $d_{X,H}^\alpha = \sum_{\lambda \in \Lambda_{X,G}} c_\alpha^\lambda d_{X,G}^\lambda$ , where  $d_{X,H}^\alpha := \dim Z_{X,H}^\alpha$  and  $d_{X,G}^\lambda := \dim Z_{X,G}^\lambda.$
- (c)  $\text{Ind}_{Z_X(G)}^{Z_X(H)}(Z_{X,G}^\lambda) := Z_X(H) \otimes_{Z_X(G)} Z_{X,G}^\lambda = \bigoplus_{\alpha \in \Lambda_{X,H}} c_\alpha^\lambda Z_{X,H}^\alpha.$
- (d) As a  $Z_X(G)$ -module (via multiplication on the left),

$$Z_X(H) = \bigoplus_{\lambda \in \Lambda_{X,G}} \left( \sum_{\alpha \in \Lambda_{X,H}} c_\alpha^\lambda d_{X,H}^\alpha \right) Z_{X,G}^\lambda.$$

- (e) Assume  $Y$  is an  $H$ -module and set  $X = \text{Ind}_H^G(Y)$ . Let  $Z_Y(H) = \text{End}_H(Y)$ , and let  $Z_{Y,H}^\alpha$ ,  $\alpha \in \Lambda_{Y,H}$ , be the irreducible  $Z_Y(H)$ -modules. Then for  $\lambda \in \Lambda_{X,G}$ ,

$$\dim Z_{X,G}^\lambda = \sum_{\alpha \in \Lambda_{Y,H}} c_\alpha^\lambda \dim Z_{Y,H}^\alpha.$$

*Proof.* (a) and (b): By Schur-Weyl duality,  $X \cong \bigoplus_{\lambda \in \Lambda_{X,G}} (G^\lambda \otimes Z_{X,G}^\lambda)$  as a  $(G \times Z_X(G))$ -bimodule. Therefore, as an  $(H \times Z_X(G))$ -bimodule,

$$X \cong \sum_{\lambda \in \Lambda_{X,G}} \left( \sum_{\alpha \in \Lambda_H} c_\alpha^\lambda H^\alpha \right) \otimes Z_{X,G}^\lambda \cong \sum_{\alpha \in \Lambda_H} H^\alpha \otimes \left( \sum_{\lambda \in \Lambda_{X,G}} c_\alpha^\lambda Z_{X,G}^\lambda \right). \quad (2.8)$$

This says that the  $H$ -module  $H^\alpha$  occurs as a summand of  $X$  with multiplicity equal to  $\sum_{\lambda \in \Lambda_{X,G}} c_\alpha^\lambda \dim Z_{X,G}^\lambda$ . But from the decomposition

$$X \cong \bigoplus_{\alpha \in \Lambda_{X,H}} \left( H^\alpha \otimes Z_{X,H}^\alpha \right), \quad (2.9)$$

we know that the only  $H$ -summands occurring in  $X$  are those with  $\alpha \in \Lambda_{X,H}$ , and  $H^\alpha$  has multiplicity  $\dim Z_{X,H}^\alpha$  in  $X$ . Therefore, the sum in (2.8) must be over  $\alpha \in \Lambda_{X,H}$ , and we have  $\dim Z_{X,H}^\alpha = \sum_{\lambda \in \Lambda_{X,G}} c_\alpha^\lambda \dim Z_{X,G}^\lambda$ , as claimed in (b). Moreover, the restriction of the  $(H \times Z_X(H))$ -decomposition of  $X$  in (2.9) to  $H \times Z_X(G)$  gives  $X \cong \bigoplus_{\alpha \in \Lambda_{X,H}} H^\alpha \otimes \text{Res}_{Z_X(G)}^{Z_X(H)} \left( Z_{X,H}^\alpha \right)$ . Since the decomposition of  $X$  as a  $(H \times Z_X(G))$ -bimodule is unique,  $\text{Res}_{Z_X(G)}^{Z_X(H)} \left( Z_{X,H}^\alpha \right) = \bigoplus_{\lambda \in \Lambda_{X,G}} c_\alpha^\lambda Z_{X,G}^\lambda$  must hold, as asserted in (a). Note that part (b) is just the dimension version of this relation.

For part (c), we use the following standard result. Assume  $A$  is an algebra and  $B$  is a subalgebra of  $A$ . Let  $W$  be an  $A$ -module and  $V$  be a  $B$ -module. Then

$$\text{Hom}_A(A \otimes_B V, W) = \text{Hom}_B(V, \text{Res}_B^A(W)). \quad (2.10)$$

Now suppose  $A = Z_X(H)$ ,  $B = Z_X(G)$ ,  $V = Z_{X,G}^\lambda$ , and  $W = Z_{X,H}^\alpha$ . Then

$$\text{Hom}_A(\text{Ind}_B^A(Z_{X,G}^\lambda), Z_{X,H}^\alpha) = \text{Hom}_B(Z_{X,G}^\lambda, \text{Res}_B^A(Z_{X,H}^\alpha)) = \text{Hom}_B \left( Z_{X,G}^\lambda, \bigoplus_{\mu \in \Lambda_{X,G}} c_\mu^\alpha Z_{X,G}^\mu \right).$$

Taking dimensions on both sides shows that  $\dim \text{Hom}_A(\text{Ind}_B^A(Z_{X,G}^\lambda), Z_{X,H}^\alpha) = c_\alpha^\lambda$ , and thus,  $\text{Ind}_{Z_X(G)}^{Z_X(H)}(Z_{X,G}^\lambda) = \bigoplus_{\alpha \in \Lambda_{X,H}} c_\alpha^\lambda Z_{X,H}^\alpha$ .

(d) Since  $Z_X(H)$  is a semisimple algebra, Wedderburn theory tells us  $Z_X(H) = \bigoplus_{\alpha \in \Lambda_{X,H}} d_{X,H}^\alpha Z_{X,H}^\alpha$ , where  $d_{X,H}^\alpha = \dim Z_{X,H}^\alpha$ . Restricting to  $Z_X(G)$  gives

$$\text{Res}_{Z_X(G)}^{Z_X(H)}(Z_X(H)) = \bigoplus_{\alpha \in \Lambda_{X,H}} d_{X,H}^\alpha \text{Res}_{Z_X(G)}^{Z_X(H)}(Z_{X,H}^\alpha) = \bigoplus_{\lambda \in \Lambda_{X,G}} \left( \bigoplus_{\alpha \in \Lambda_{X,H}} c_\alpha^\lambda d_{X,H}^\alpha \right) Z_{X,G}^\lambda,$$

by part (a).

(e) The proof here is similar in spirit to that in parts (a) and (b). With  $Y$  an  $H$ -module, suppose  $Y = \bigoplus_{\alpha \in \Lambda_{Y,H}} y_\alpha H^\alpha$ , and assume  $X := \text{Ind}_H^G(Y) = \bigoplus_{\lambda \in \Lambda_{X,G}} x_\lambda G^\lambda$ . Then

$$X = \text{Ind}_H^G(Y) = \sum_{\alpha \in \Lambda_{Y,H}} y_\alpha \text{Ind}_H^G(H^\alpha) = \sum_{\alpha \in \Lambda_{Y,H}} y_\alpha \left( \sum_{\lambda \in \Lambda_G} c_\alpha^\lambda G^\lambda \right) = \sum_{\lambda \in \Lambda_G} \left( \sum_{\alpha \in \Lambda_{Y,H}} c_\alpha^\lambda y_\alpha \right) G^\lambda,$$

so that the sum must be over  $\lambda \in \Lambda_{X,G}$ , and

$$\dim Z_{X,G}^\lambda = x_\lambda = \sum_{\alpha \in \Lambda_{Y,H}} c_\alpha^\lambda y_\alpha = \sum_{\alpha \in \Lambda_{Y,H}} c_\alpha^\lambda \dim Z_{Y,H}^\alpha$$

for all  $\lambda \in \Lambda_{X,G}$ . □

The following proposition will be used in Section 7 to relate multiplicities in the tensor power of the reflection module of the symmetric group to multiplicities in tensor powers of the permutation module. Assume  $G$  is a finite group and  $W$  is a  $G$ -module over  $\mathbb{C}$ . Let  $V = G^\bullet \oplus W$  be the extension of  $W$  by the trivial  $G$ -module  $G^\bullet$ . Define  $Z_k(G) = \text{End}_G(V^{\otimes k})$  and  $\text{QZ}_k(G) = \text{End}_G(W^{\otimes k})$ , and let  $\Lambda_{k,G} \subseteq \Lambda(G)$  (resp.,  $\text{q}\Lambda_{k,G} \subseteq \Lambda(G)$ ) index the irreducible  $G$ -modules that appear in  $V^{\otimes k}$  (resp., in  $W^{\otimes k}$ ) with multiplicity at least one. Let  $Z_k^\lambda$  (resp.,  $\text{QZ}_k^\lambda$ ) denote the irreducible  $Z_k(G)$ -module (resp.,  $\text{QZ}_k(G)$ -module) indexed by  $\lambda \in \Lambda_{k,G}$  (resp.,  $\lambda \in \text{q}\Lambda_{k,G}$ ).

**Proposition 2.11.** *With notation as in the previous paragraph,*

$$(a) \dim \text{QZ}_k^\lambda = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim Z_\ell^\lambda,$$

$$(b) \text{ If } W \text{ is a self-dual } G\text{-module, then } \dim \text{QZ}_k(G) = \dim \text{QZ}_{2k}^\bullet = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \dim Z_\ell^\bullet,$$

where  $\text{QZ}_{2k}^\bullet$  is the irreducible  $\text{QZ}_k(G)$ -module corresponding to  $G^\bullet$ ; equivalently, the space of  $G$ -invariants in  $W^{\otimes 2k}$ .

*Proof.* Let  $\chi_V, \chi_\bullet, \chi_W$  denote the characters of  $V, G^\bullet$ , and  $W$ , respectively, so that  $\chi_V = \chi_\bullet + \chi_W$ .

Then  $\chi_{V^{\otimes k}} = \chi_V^k = (\chi_\bullet + \chi_W)^k = \sum_{\ell=0}^k \binom{k}{\ell} \chi_W^\ell$ , and the binomial inverse of this statement is

$$\chi_{W^{\otimes k}} = \chi_W^k = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \chi_V^\ell = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \chi_{V^{\otimes \ell}}. \quad (2.12)$$

By Schur-Weyl duality (2.3) we have

$$W^{\otimes k} = \bigoplus_{\lambda \in \text{q}\Lambda_k} \text{qd}_k^\lambda G^\lambda, \quad \text{where } \text{qd}_k^\lambda = \dim \text{QZ}_k^\lambda. \quad (2.13)$$

Computing the character of (2.13) and equating it with (2.12) gives

$$\chi_{W^{\otimes k}} = \chi_W^k = \sum_{\lambda \in \text{q}\Lambda_k} \text{qd}_k^\lambda \chi_\lambda = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left( \sum_{\lambda \in \Lambda_\ell} \text{d}_\ell^\lambda \chi_\lambda \right),$$

where  $\text{d}_\ell^\lambda = \dim Z_\ell^\lambda$ , and  $\chi_\lambda$  is the character of  $G^\lambda$ . Equating the coefficient of  $\chi_\lambda$  (working in the ring of class functions on  $G$ ) gives part (a). Since  $W$  is isomorphic to its dual as a  $G$ -module, part (b) is the special case of part (a) with  $\lambda = \bullet$  (the index of the trivial module):

$$\dim \text{QZ}_k(G) = \dim \text{QZ}_{2k}^\bullet = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \dim Z_\ell^\bullet. \quad \square$$

### 3 Irreducible modules for symmetric and alternating groups and their centralizer algebras

The irreducible  $S_n$ -modules are labeled by partitions of  $n$ , so that  $\Lambda_{S_n} = \{\lambda \mid \lambda \vdash n\}$ . When writing  $\lambda = [\lambda_1, \dots, \lambda_n] \vdash n$ , we always assume that the parts of the partition  $\lambda$  are arranged so that  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , and  $|\lambda| = n$  (the sum of the parts). We identify a partition with its Young diagram, so for  $\lambda = [6, 4, 3, 2^2] \vdash 17$ , we have

$$\lambda = \begin{array}{cccccc} \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & & \\ \square & \square & \square & & & \\ \square & \square & & & & \\ \square & \square & & & & \end{array} .$$

The *hook length*  $h(b)$  of a box  $b$  in the diagram is 1 plus the number of boxes below  $b$  in the same column plus the number of boxes to the right of  $b$  in the same row, and  $h(b) = 1 + 3 + 2 = 6$  for the shaded box above. The dimension of the irreducible  $S_n$ -module  $S_n^\lambda$ , which we denote  $f^\lambda$ , can be easily computed by the well-known hook-length formula

$$f^\lambda = \frac{n!}{\prod_{b \in \lambda} h(b)}, \quad (3.1)$$

where the denominator is the product of the hook lengths as  $b$  ranges over the boxes in the Young diagram of  $\lambda$ . This is equal to the number of standard Young tableaux of shape  $\lambda$ , where a standard Young tableau  $T$  is a filling of the boxes in the Young diagram of  $\lambda$  with the numbers  $\{1, \dots, n\}$  such that the entries increase in every row from left to right and in every column from top to bottom.

The restriction and induction rules for irreducible symmetric group modules  $S_n^\lambda$  are well known (see for example [JK, Thm. 2.43]):

$$\text{Res}_{S_{n-1}}^{S_n}(S_n^\lambda) = \bigoplus_{\mu=\lambda-\square} S_{n-1}^\mu \quad \text{Ind}_{S_n}^{S_{n+1}}(S_n^\lambda) = \bigoplus_{\kappa=\lambda+\square} S_{n+1}^\kappa, \quad (3.2)$$

where the first sum is over all partitions  $\mu$  of  $n - 1$  obtained from  $\lambda$  by removing a box from the end of a row of the diagram of  $\lambda$ , and the second sum is over all partitions  $\kappa$  of  $n + 1$  obtained by adding a box to the end of a row of  $\lambda$ .

Assume  $\lambda = [\lambda_1, \dots, \lambda_{\ell(\lambda)}]$  is a partition of  $n$  and  $\gamma = [\gamma_1, \dots, \gamma_n]$  is a weak composition of  $n$ . The *Kostka number*  $K_{\lambda, \gamma}$  counts the number of semistandard tableaux  $T$  of shape  $\lambda$  and type  $\gamma$ , where  $T$  is a filling of the boxes of the Young diagram of  $\lambda$  with numbers from  $\{1, \dots, n\}$  such that  $j$  occurs  $\gamma_j$  times, and the entries of  $T$  weakly increase across the rows from left to right and strictly increase down the columns. If  $\gamma = [\gamma_1, \gamma_2, \dots, \gamma_{\ell(\gamma)}]$  is a partition, then  $K_{\lambda, \gamma} = 0$ , unless  $\lambda \geq \gamma$  in the dominance order, which is to say that for the first part where  $\lambda$  and  $\gamma$  differ  $\lambda_j > \gamma_j$ . Assume  $\mathcal{M}^\gamma$  is the *permutation module* obtained from inducing the trivial module for the Young subgroup  $S_{\gamma_1} \times S_{\gamma_2} \times \dots \times S_{\gamma_{\ell(\gamma)}}$  to  $S_n$ . The irreducible  $S_n$ -module  $S_n^\lambda$  occurs with multiplicity  $K_{\lambda, \gamma}$  in the decomposition of  $\mathcal{M}^\gamma$  into irreducible  $S_n$ -summands.

For  $\lambda \vdash n$ , let  $\lambda^*$  be the conjugate (transpose) partition. Since  $S_n^{\lambda^*} \cong S_n^{[1^n]} \otimes S_n^\lambda$ , where  $S_n^{[1^n]}$  is the one-dimensional irreducible  $S_n$ -module indexed by the partition of  $n$  into  $n$  parts of size one, which is the sign representation,  $S_n^{\lambda^*} \cong S_n^\lambda$  as  $A_n$ -modules. Thus, we may assume that  $\lambda \geq \lambda^*$  in



the dominance order. Then by Clifford theory, it is known that

$$\text{Res}_{A_n}^{S_n}(S_n^\lambda) = \begin{cases} A_n^\lambda \cong \text{Res}_{A_n}^{S_n}(S_n^{\lambda^*}) & \text{if } \lambda \neq \lambda^*, \\ A_n^{\lambda^+} \oplus A_n^{\lambda^-} & \text{if } \lambda = \lambda^*, \end{cases} \quad (3.3)$$

where in the first case,  $A_n^\lambda$  is irreducible as an  $A_n$ -module; while in the second case,  $S_n^\lambda$  decomposes into the direct sum of two irreducible  $A_n$ -modules,  $S_n^\lambda = A_n^{\lambda^+} \oplus A_n^{\lambda^-}$ , such that  $\dim A_n^{\lambda^+} = \dim A_n^{\lambda^-} = \frac{1}{2} \dim S_n^\lambda = \frac{1}{2} f^\lambda$ . Moreover,

$$\Lambda_{A_n} = \{\lambda \mid \lambda \vdash n, \lambda > \lambda^*\} \cup \{\lambda^\pm \mid \lambda \vdash n, \lambda = \lambda^*\}.$$

The restriction rules for alternating groups are the following (see [R, Thm. 6.1], or [Mb] which surveys how to derive these rules using Mackey's theorem and Clifford theory):

$$\begin{aligned} \text{Res}_{A_{n-1}}^{A_n}(A_n^\lambda) &= \left( \bigoplus_{\substack{\mu=\lambda-\square \\ \mu > \mu^*}} A_{n-1}^\mu \right) \oplus \left( \bigoplus_{\substack{\mu=\lambda-\square \\ \mu=\mu^*}} (A_{n-1}^{\mu^+} \oplus A_{n-1}^{\mu^-}) \right) & \text{if } \lambda > \lambda^*, \\ \text{Res}_{A_{n-1}}^{A_n}(A_n^{\lambda^\pm}) &= \left( \bigoplus_{\substack{\mu=\lambda-\square \\ \mu > \mu^*}} A_{n-1}^\mu \right) \oplus \left( \bigoplus_{\substack{\mu=\lambda-\square \\ \mu=\mu^*}} A_{n-1}^{\mu^\pm} \right) & \text{if } \lambda = \lambda^*. \end{aligned} \quad (3.4)$$

Now let  $M_n$  be the  $n$ -dimensional permutation module for  $S_n$ , and set  $X = M_n^{\otimes k}$  for  $k \geq 0$  in applying the results of Section 2, where  $M_n^{\otimes 0} = S_n^{[n]}$  (the trivial  $S_n$ -module). Since  $M_n$  will be fixed throughout, it is convenient here to adopt the shorthand notation in Table 1. For all  $k \in \mathbb{Z}_{\geq 0}$ ,  $\Lambda_{k, S_n}$  (resp.  $\Lambda_{k, A_n}$ ) is the set of indices for the irreducible  $Z_k(n)$ -summands (resp.  $\widehat{Z}_k(n)$ -summands) in  $M_n^{\otimes k}$  with multiplicity at least one; similarly  $\Lambda_{k, S_{n-1}}$  (resp.  $\Lambda_{k, A_{n-1}}$ ) is the set of indices for the irreducible  $Z_{k+\frac{1}{2}}(n)$ -summands (resp.  $\widehat{Z}_{k+\frac{1}{2}}(n)$ -summands) in  $M_n^{\otimes k}$  with multiplicity at least one.

centralizer algebra	irreducible modules
$Z_k(n) := \text{End}_{S_n}(M_n^{\otimes k})$	$Z_{k,n}^\lambda, \lambda \vdash n, \lambda \in \Lambda_{k, S_n} \subseteq \Lambda_{S_n}$
$Z_{k+\frac{1}{2}}(n) := \text{End}_{S_{n-1}}(M_n^{\otimes k})$	$Z_{k+\frac{1}{2},n}^\mu, \mu \vdash n-1, \mu \in \Lambda_{k, S_{n-1}} \subseteq \Lambda_{S_{n-1}}$
$\widehat{Z}_k(n) := \text{End}_{A_n}(M_n^{\otimes k})$	$\widehat{Z}_{k,n}^\lambda, \lambda \vdash n, \lambda > \lambda^*, \lambda \in \Lambda_{k, A_n} \subseteq \Lambda_{A_n}$ $\widehat{Z}_{k,n}^{\lambda^\pm}, \lambda \vdash n, \lambda = \lambda^*, \lambda^\pm \in \Lambda_{k, A_n} \subseteq \Lambda_{A_n}$
$\widehat{Z}_{k+\frac{1}{2}}(n) := \text{End}_{A_{n-1}}(M_n^{\otimes k})$	$\widehat{Z}_{k+\frac{1}{2},n}^\mu, \mu \vdash n-1, \mu > \mu^*, \mu \in \Lambda_{k+\frac{1}{2}, A_{n-1}} \subseteq \Lambda_{A_{n-1}}$ $\widehat{Z}_{k+\frac{1}{2},n}^{\mu^\pm}, \mu \vdash n, \mu = \mu^*, \mu^\pm \in \Lambda_{k+\frac{1}{2}, A_{n-1}} \subseteq \Lambda_{A_{n-1}}$

Table 1: Notation for the centralizer algebras and modules associated with the tensor product  $M_n^{\otimes k}$  of the permutation module  $M_n \cong S_n^{[n]} \oplus S_n^{[n-1,1]}$  of  $S_n$  and its restriction to  $S_{n-1}$ ,  $A_n$ , and  $A_{n-1}$ .

Theorem 2.7(b) together with (3.3) imply the following:

**Proposition 3.5.** *Assume  $\lambda \vdash n$ ,  $\lambda \in \Lambda_{k,A_n}$ , and  $\lambda \geq \lambda^*$ . Then*

$$\begin{aligned} \dim \widehat{Z}_{k,n}^\lambda &= \dim Z_{k,n}^\lambda + \dim Z_{k,n}^{\lambda^*}, & \text{if } \lambda > \lambda^*, \\ \dim \widehat{Z}_{k,n}^{\lambda^+} &= \dim \widehat{Z}_{k,n}^{\lambda^-} = \dim Z_{k,n}^\lambda, & \text{if } \lambda = \lambda^*. \end{aligned} \quad (3.6)$$

**Example 3.7.** For  $S_4$ , we have  $M_4^{\otimes 3} = 5S_4^{[4]} \oplus 10S_4^{[3,1]} \oplus 5S_4^{[2^2]} \oplus 6S_4^{[2,1^2]} \oplus S_4^{[1^4]}$ , and for  $A_4$ ,  $M_4^{\otimes 3} = 6A_4^{[4]} \oplus 16A_4^{[3,1]} \oplus 5A_4^{[2^2]^+} \oplus 5A_4^{[2^2]^-}$ , as can be seen in row  $\ell = 3$  of Figures 1 and 2, where

$$\begin{aligned} \dim \widehat{Z}_{3,4}^{[4]} &= \dim Z_{3,4}^{[4]} + \dim Z_{3,4}^{[1^4]} = 5 + 1 = 6, \\ \dim \widehat{Z}_{3,4}^{[3,1]} &= \dim Z_{3,4}^{[3,1]} + \dim Z_{3,4}^{[2,1^2]} = 10 + 6 = 16, \\ \dim \widehat{Z}_{3,4}^{[2^2]^\pm} &= \dim Z_{3,4}^{[2^2]} = 5. \end{aligned}$$

## 4 Bratteli diagrams

Let  $(G, H)$  be a pair consisting of a finite group  $G$  and a subgroup  $H \subseteq G$ . As in Section 2, let  $\{G^\lambda\}_{\lambda \in \Lambda_G}$  and  $\{H^\alpha\}_{\alpha \in \Lambda_H}$  be the irreducible modules of  $G$  and  $H$  over  $\mathbb{C}$  with restriction and induction rules given by

$$\text{Res}_H^G(G^\lambda) = \bigoplus_{\alpha \in \Lambda_H} c_\alpha^\lambda H^\alpha \quad \text{and} \quad \text{Ind}_H^G(H^\alpha) = \bigoplus_{\lambda \in \Lambda_G} c_\alpha^\lambda G^\lambda. \quad (4.1)$$

Let  $U^0 = G^\bullet$ , the trivial  $G$ -module, and assume for  $k \in \mathbb{Z}_{\geq 0}$  that the  $G$ -module  $U^k$  has been defined. Let  $U^{k+\frac{1}{2}}$  be the  $H$ -module defined by  $U^{k+\frac{1}{2}} := \text{Res}_H^G(U^k)$ , and then let  $U^{k+1}$  be the  $G$ -module specified by  $U^{k+1} := \text{Ind}_H^G(U^{k+\frac{1}{2}})$ . In this way,  $U^\ell$  is defined inductively for all  $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ , and  $U^k = (\text{Ind}_H^G \text{Res}_H^G)^k(U^0)$  for all  $k \in \mathbb{Z}_{\geq 0}$ . The module  $V := \text{Ind}_H^G(\text{Res}_H^G(U^0)) = U^1$  is isomorphic to  $G/H$  as a  $G$ -module, where  $G$  acts on the left cosets of  $G/H$  by multiplication.

For a  $G$ -module  $X$  and an  $H$ -module  $Y$ , the ‘‘tensor identity’’ says that  $\text{Ind}_H^G(\text{Res}_H^G(X) \otimes Y) \cong X \otimes \text{Ind}_H^G(Y)$  (see for example [HR, (3.18)] for an explicit isomorphism). Hence, when  $X = U^k$  and  $Y = \text{Res}_H^G(U^0)$ , this gives

$$\text{Ind}_H^G(\text{Res}_H^G(U^k)) \cong \text{Ind}_H^G(\text{Res}_H^G(U^k) \otimes \text{Res}_H^G(U^0)) \cong U^k \otimes \text{Ind}_H^G(\text{Res}_H^G(U^0)) = U^k \otimes V. \quad (4.2)$$

By induction, we have the following isomorphisms for all  $k \in \mathbb{Z}_{\geq 0}$ :

$$V^{\otimes k} \cong U^k \quad (\text{as } G\text{-modules}) \quad \text{and} \quad \text{Res}_H^G(V^{\otimes k}) \cong U^{k+\frac{1}{2}} \quad (\text{as } H\text{-modules}). \quad (4.3)$$

It follows that there are centralizer algebras isomorphisms:

$$\begin{aligned} Z_k(G) &:= \text{End}_G(V^{\otimes k}) \cong \text{End}_G(U^k), \\ Z_{k+\frac{1}{2}}(H) &:= \text{End}_H(\text{Res}_H^G(V^{\otimes k})) \cong \text{End}_H(U^{k+\frac{1}{2}}). \end{aligned} \quad (4.4)$$

Suppose for  $k \in \mathbb{Z}_{\geq 0}$  that

- $\Lambda_{k,G} \subseteq \Lambda_G$  indexes the irreducible  $G$ -modules, and hence also the irreducible  $Z_k(G)$ -modules, in  $U^k \cong V^{\otimes k}$ ;
- $\Lambda_{k+\frac{1}{2},H} \subseteq \Lambda_H$  indexes the irreducible  $H$ -modules, and hence also the irreducible  $Z_{k+\frac{1}{2}}(H)$ -modules, in  $U^{k+\frac{1}{2}} \cong \text{Res}_H^G(V^{\otimes k})$ .

The *restriction-induction Bratteli diagram* for the pair  $(G, H)$  is an infinite, rooted tree  $\mathcal{B}(G, H)$  whose vertices are organized into rows labeled by half integers  $\ell$  in  $\frac{1}{2}\mathbb{Z}_{\geq 0}$ . For  $\ell = k \in \mathbb{Z}_{\geq 0}$ , the vertices on row  $k$  are the elements of  $\Lambda_{k,G}$ , and the vertices on row  $\ell = k + \frac{1}{2}$  are the elements of  $\Lambda_{k+\frac{1}{2},H}$ . The vertex on row 0 is the root, the label of the trivial  $G$ -module, and the vertex on row  $\frac{1}{2}$  is the label of the trivial  $H$ -module. For the pair  $(S_n, S_{n-1})$  (or  $(A_n, A_{n-1})$ ), the labels on rows 0 and  $\frac{1}{2}$  are the partitions  $[n], [n-1]$  having just one part.

The edges of  $\mathcal{B}(G, H)$  are given by drawing  $c_\alpha^\lambda$  edges from  $\lambda \in \Lambda_{k,G}$  to  $\alpha \in \Lambda_{k+\frac{1}{2},H}$ , where  $c_\alpha^\lambda$  is as in (4.1). Similarly, there are  $c_\beta^\kappa$  edges from  $\beta \in \Lambda_{k+\frac{1}{2},H}$  to  $\kappa \in \Lambda_{k+1,G}$ . The Bratteli diagram is constructed in such a way that

- the number of paths from the root at level 0 to  $\lambda \in \Lambda_{k,G}$  equals the multiplicity of  $G^\lambda$  in  $U^k \cong V^{\otimes k}$  and thus also equals the dimension of the irreducible  $Z_k(G)$ -module  $Z_{U^k,G}^\lambda$  (these numbers are computed in Pascal-triangle-like fashion and are placed below each vertex);
- the number of paths from the root at level 0 to  $\alpha \in \Lambda_{k+\frac{1}{2},H}$  equals the multiplicity of  $H^\alpha$  in  $U^{k+\frac{1}{2}}$  and thus also equals the dimension of the  $Z_{k+\frac{1}{2}}(H)$ -module  $Z_{U^{k+\frac{1}{2},H}}^\alpha$  (and is indicated beneath each vertex);
- the sum of the squares of the labels on row  $k$  (resp. row  $k + \frac{1}{2}$ ) equals  $\dim Z_k(G)$  (resp.  $\dim Z_{k+\frac{1}{2}}(H)$ ).

When  $(G, H) = (S_n, S_{n-1})$  or when  $(G, H) = (A_n, A_{n-1})$ , it is well known (and easy to verify) that the permutation module satisfies  $M_n \cong U^1 = \text{Ind}_{S_{n-1}}^{S_n}(\text{Res}_{S_{n-1}}^{S_n}(S_n^{[n]}))$ . Then (4.4) implies there are partition algebra surjections as  $P_k(n) \rightarrow Z_k(n) = \text{End}_{S_n}(M_n^{\otimes k}) \cong \text{End}_{S_n}(U^k)$  and  $P_{k+\frac{1}{2}}(n) \rightarrow Z_{k+\frac{1}{2}}(n) = \text{End}_{S_{n-1}}(M_n^{\otimes k}) \cong \text{End}_{S_{n-1}}(U^k)$ . Using the restriction/induction rules for  $S_n$  in (3.2) and for  $A_n$  in (3.4), we construct the Bratteli diagram for  $(S_4, S_3)$  (see Figure 1) and for  $(A_4, A_3)$  (see Figure 2). In Appendices A.1 and A.3, we construct the Bratteli diagrams for  $(S_6, S_5)$  and  $(A_6, A_5)$ .

**Remark 4.5.** Amazingly, the Bratteli diagrams in Figures 1 and 2 also appear in the Schur-Weyl duality analysis of the McKay correspondence, as discussed in [B] and [BH3]. The binary octahedral subgroup  $\mathbf{O}$  of the special unitary group  $SU_2$  is the two-fold cover of the octahedral group, which is isomorphic to the symmetric group  $S_4$ . We use that fact to show that the Bratteli diagram for tensor powers of the 2-dimensional spin module of  $\mathbf{O}$  (which is the defining module of  $\mathbf{O}$  and  $SU_2$ ) is identical to Figure 1. Similarly, the binary tetrahedral subgroup  $\mathbf{T}$  is the two-fold cover of tetrahedral group, which is isomorphic to the alternating group  $A_4$ . The Bratteli diagram for tensor powers of the 2-dimensional defining module of  $\mathbf{T}$  is identical to Figure 2.

**Remark 4.6.** The *tensor power Bratteli diagram*  $\mathcal{B}_V(G)$  is constructed using the centralizer algebras  $Z_k(G) = \text{End}_G(V^{\otimes k})$ . The vertices on level  $k$  of  $\mathcal{B}_V(G)$  are labeled by elements of  $\Lambda_{k,G}$ , and

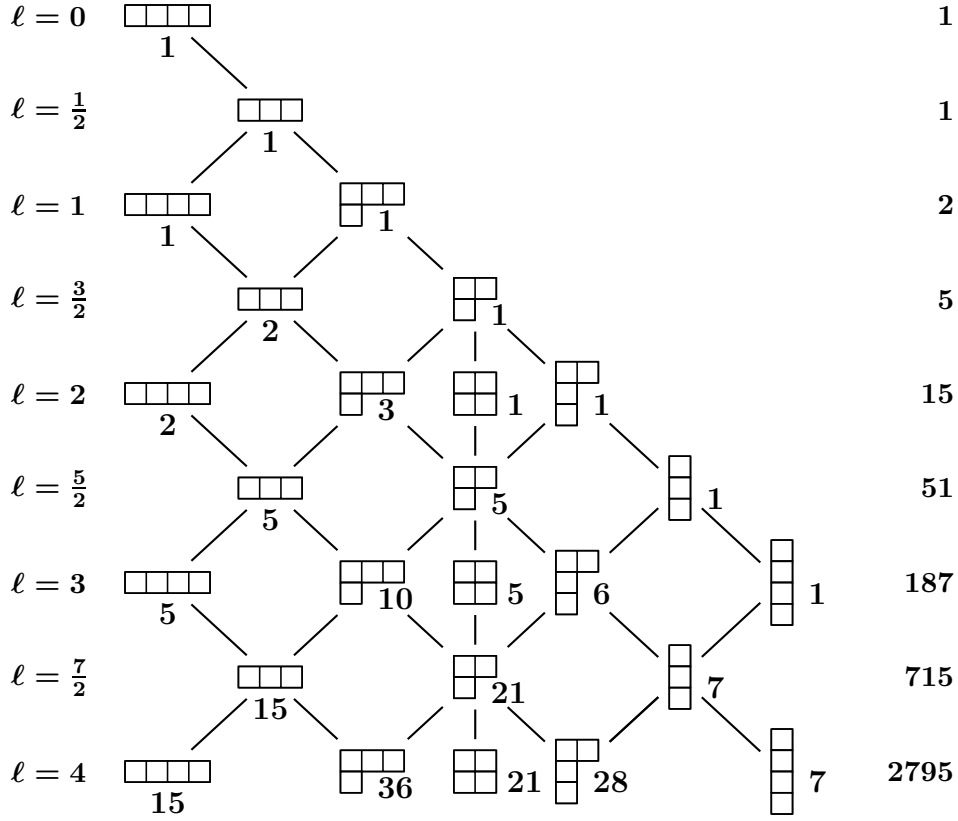


Figure 1: Levels  $\ell = 0, \frac{1}{2}, 1, \dots, \frac{7}{2}, 4$  of the Bratteli diagram for the pair  $(S_4, S_3)$ .

there are  $c_\mu^\lambda$  edges from  $\lambda \in \Lambda_{k,G}$  to  $\mu \in \Lambda_{k+1,G}$  if  $G^\lambda \otimes V \cong \bigoplus_{\mu \in \Lambda_G} c_\mu^\lambda G^\mu$ . In the special case that  $V = \text{Ind}_H^G(\text{Res}_H^G(U^0))$  and  $U^0$  is the trivial  $G$ -module,  $\mathcal{B}_V(G)$  is identical to  $\mathcal{B}(G, H)$  except that the half integer levels are missing from  $\mathcal{B}_V(G)$ . So for example, in the tensor power Bratteli diagram that corresponds to Figure 1, there are two edges from the vertex  $\begin{smallmatrix} \square & \square & \square & \square \end{smallmatrix}$  on level 1 to the vertex  $\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$  on level 2. Including the intermediate half-integer levels, which corresponds to performing restriction and then induction, results in a diagram without multiple edges between vertices when  $(G, H) = (S_n, S_{n-1})$  or  $(A_n, A_{n-1})$ , since the restriction/induction rules for those pairs are multiplicity free. The half-integer centralizer algebras have proven to be a powerful tool in studying the structure of these tensor power centralizer algebras (for example, in [HR] and [BH3]), and we use them here to recursively derive dimension formulas.

## 5 Dimensions formulas for symmetric group centralizer algebras

In the next two sections, we determine expressions for the dimensions of the irreducible modules for the centralizer algebras in Table 1. Our arguments will invoke standard combinatorial facts about representations of the symmetric group  $S_n$ . The dimensions will be expressed as integer combinations of Stirling numbers of the second kind. We begin by briefly reviewing some known

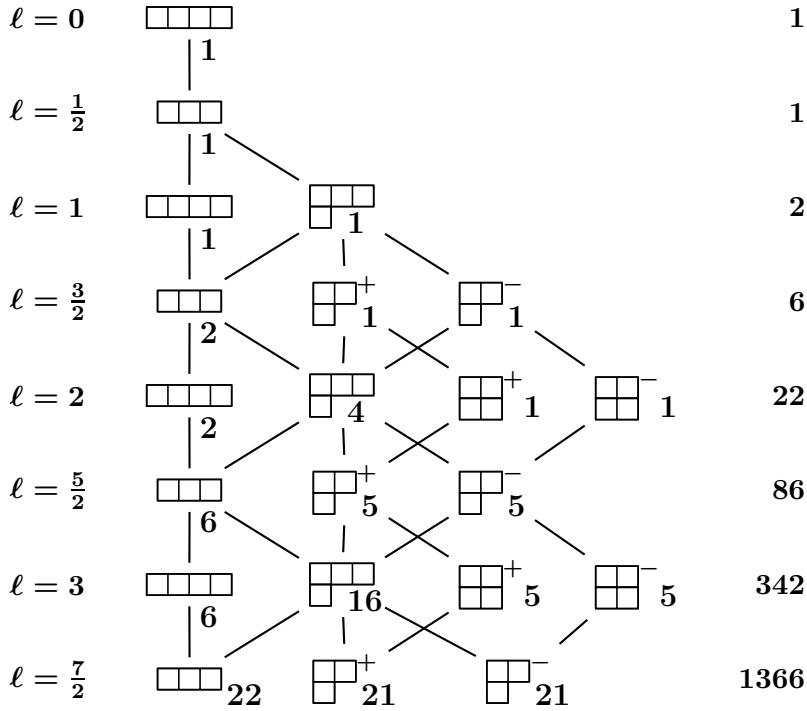


Figure 2: Levels  $\ell = 0, \frac{1}{2}, 1, \dots, 3, \frac{7}{2}$  of the Bratteli diagram for the pair  $(A_4, A_3)$ .

results about these numbers.

### 5.1 Stirling numbers of the second kind and Bell numbers

There are several commonly used notations for Stirling numbers of the second kind; for example,  $S(k, t)$  is used by Stanley [S1]. In [K], Knuth remarks “The lack of a widely accepted way to refer to these numbers has become almost scandalous,” and he goes on to make a convincing argument for adopting the notation  $\left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\}$ , which we will do here.

The *Stirling number*  $\left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\}$  of the *second kind* counts the number of ways to partition a set of  $k$  elements into  $t$  disjoint nonempty blocks. In particular,  $\left\{ \begin{smallmatrix} k \\ 0 \end{smallmatrix} \right\} = 0$  for all  $k \geq 1$ , and  $\left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\} = 0$  if  $t > k$ . By convention,  $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$ . These numbers satisfy the recurrence relations,

$$\left\{ \begin{smallmatrix} k+1 \\ t+1 \end{smallmatrix} \right\} = \sum_{r=t}^k \binom{k}{r} \left\{ \begin{smallmatrix} r \\ t \end{smallmatrix} \right\} \quad (5.1)$$

$$\left\{ \begin{smallmatrix} k+1 \\ t \end{smallmatrix} \right\} = t \left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} k \\ t-1 \end{smallmatrix} \right\}. \quad (5.2)$$

For  $k \geq 1$ ,

$$\sum_{t=0}^k \left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\} = \sum_{t=1}^k \left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\} = B(k), \quad (5.3)$$

where  $B(k)$  is the  $k$ th *Bell number*. More generally, for  $k \geq 1$ ,

$$\sum_{t=1}^n \left\{ \begin{matrix} k \\ t \end{matrix} \right\} =: B(k, n) \quad (5.4)$$

counts the number of ways to partition a set of  $k$  elements into at most  $n$  disjoint nonempty blocks, and  $B(k, n) = B(k)$  if  $n \geq k$ . Identifying  $P_0(\xi)$  with  $\mathbb{C}$ , we have  $\dim P_k(\xi) = B(2k)$  for all  $k \in \mathbb{Z}_{\geq 0}$ . In fact,  $\dim P_\ell(\xi) = B(2\ell)$  for all  $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ , which can be seen by taking  $\nu = \emptyset$  in Corollary 5.14 below.

## 5.2 Main result for symmetric group centralizer algebras

Our aim in this section is to establish Theorem 5.5, which gives the dimensions of the irreducible modules for the centralizer algebras  $Z_k(n) = \text{End}_{S_n}(M_n^{\otimes k})$  and  $Z_{k+\frac{1}{2}}(n) = \text{End}_{S_{n-1}}(M_n^{\otimes k})$ . Throughout, the notation  $\lambda^\# = [\lambda_2, \dots, \lambda_{\ell(\lambda)}]$  will designate a partition  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}]$  with its largest part  $\lambda_1$  removed, and  $f^\lambda$  will be the number of standard tableaux of shape  $\lambda$ , which is also the dimension of the irreducible symmetric group module labeled by  $\lambda$ . If  $\pi$  is a partition contained in  $\lambda$ , then  $f^{\lambda/\pi}$  denotes the number of standard tableaux with skew shape  $\lambda \setminus \pi$ . The Kostka number  $K_{\lambda, \gamma}$  counts the number of semistandard tableaux of shape  $\lambda$  and type  $\gamma$  (see Section 3).

**Theorem 5.5.** *Let  $k, n \in \mathbb{Z}_{\geq 0}$  and  $n \geq 1$ , and let the notation be as in Table 1.*

(a) *Assume  $\lambda = [\lambda_1, \dots, \lambda_n] \vdash n$ , and  $\lambda \in \Lambda_{k, S_n}$ . Then*

$$\begin{aligned} \dim Z_{k, n}^\lambda &= \sum_{t=0}^n \left\{ \begin{matrix} k \\ t \end{matrix} \right\} K_{\lambda, [n-t, 1^t]} = \sum_{t=|\lambda^\#|}^n \left\{ \begin{matrix} k \\ t \end{matrix} \right\} f^{\lambda/[n-t]} \\ &= f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-\lambda_2} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{|\lambda^\#|} + \sum_{t=n-\lambda_2+1}^n \left\{ \begin{matrix} k \\ t \end{matrix} \right\} f^{\lambda/[n-t]}. \end{aligned}$$

(b) *Assume  $\mu = [\mu_1, \dots, \mu_{n-1}] \vdash n-1$ , and  $\mu \in \Lambda_{k+\frac{1}{2}, S_{n-1}}$ . Then*

$$\begin{aligned} \dim Z_{k+\frac{1}{2}, n}^\mu &= \sum_{t=0}^{n-1} \left\{ \begin{matrix} k+1 \\ t+1 \end{matrix} \right\} K_{\mu, [n-1-t, 1^t]} = \sum_{t=|\mu^\#|}^{n-1} \left\{ \begin{matrix} k+1 \\ t+1 \end{matrix} \right\} f^{\mu/[n-1-t]} \\ &= f^{\mu^\#} \sum_{t=|\mu^\#|}^{n-1-\mu_2} \left\{ \begin{matrix} k+1 \\ t+t \end{matrix} \right\} \binom{t}{|\mu^\#|} + \sum_{t=n-\mu_2}^n \left\{ \begin{matrix} k+1 \\ t+1 \end{matrix} \right\} f^{\mu/[n-1-t]}. \end{aligned}$$

$$(c) \dim Z_k(n) = \dim Z_{2k, n}^{[n]} = \sum_{t=0}^n \left\{ \begin{matrix} 2k \\ t \end{matrix} \right\} = B(2k, n) \quad (= B(2k) \text{ if } n \geq 2k).$$

$$\begin{aligned} (d) \dim Z_{k+\frac{1}{2}}(n) &= \dim Z_{2k+\frac{1}{2}, n}^{[n-1]} = \sum_{t=0}^{n-3} \left\{ \begin{matrix} 2k+1 \\ t+1 \end{matrix} \right\} + \left( \left\{ \begin{matrix} 2k+1 \\ n-1 \end{matrix} \right\} + \left\{ \begin{matrix} 2k+1 \\ n \end{matrix} \right\} \right) \\ &= \sum_{t=1}^n \left\{ \begin{matrix} 2k+1 \\ t \end{matrix} \right\} = B(2k+1, n) \quad (= B(2k+1) \text{ if } n \geq 2k+1). \end{aligned}$$

**Remark 5.6.** When  $n > k$ , the top limit in the summation in part (a) can be taken to be  $k$  as the Stirling numbers  $\left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\}$  are 0 for  $t > k$ . When  $n \leq k$ , the term  $[n-t, 1^t]$  for  $t = n$  should be interpreted as the partition  $[1^n]$ . In that special case,  $K_{\lambda, [1^n]} = f^\lambda$ , the number standard tableaux of shape  $\lambda$ , as each entry in the tableau appears only once. The term  $t = n-1$  gives the same Kostka number  $K_{\lambda, [1^n]} = f^\lambda$ . The only time that the term  $t = 0$  contributes is when  $k = 0$ . The Stirling number  $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$ , and the Kostka number  $K_{\lambda, [n]} = 0$  if  $\lambda \neq [n]$  and  $K_{[n], [n]} = 1$ . Thus,  $\dim Z_0^\lambda(n) = \delta_{\lambda, [n]}$ , as expected, since  $M_n^{\otimes 0} = S_n^{[n]}$  by definition. In the proof to follow, we will assume  $k \geq 1$ .

*Proof.* (a) For  $1 \leq t \leq n$ , the linear span of  $t$ -element ordered subsets of  $\{1, 2, \dots, n\}$  forms an  $S_n$ -module isomorphic to the permutation module  $\mathcal{M}^{[n-t, 1^t]}$ . For  $t = n-1$  and  $t = n$ , both modules are isomorphic to  $\mathcal{M}^{[1^n]}$ . We claim that

$$M_n^{\otimes k} = \sum_{t=1}^n \left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\} \mathcal{M}^{[n-t, 1^t]}. \quad (5.7)$$

This can be seen as follows: Let  $u_1, \dots, u_n$  be the basis for  $M_n$  that  $S_n$  permutes. For each set partition of  $\{1, \dots, k\}$  into  $t$  blocks, we get a copy of  $\mathcal{M}^{[n-t, 1^t]}$  spanned by the vectors  $u_{j_1} \otimes u_{j_2} \otimes \dots \otimes u_{j_k}$ , where  $j_a = j_b$  if and only if  $a, b$  are in the same part of the set partition. There are  $\left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\}$  such set partitions. The multiplicity of  $S_n^\lambda$  in  $M_n^{\otimes k}$  is obtained from (5.7) by observing that  $S_n^\lambda$  has multiplicity  $K_{\lambda, [n-t, 1^t]}$  in  $\mathcal{M}^{[n-t, 1^t]}$ . By Schur-Weyl duality, the multiplicity of  $S_n^\lambda$  in  $M_n^{\otimes k}$  equals  $\dim Z_k^\lambda(n)$ , and therefore,  $\dim Z_{k,n}^\lambda = \sum_{t=1}^n \left\{ \begin{smallmatrix} k \\ t \end{smallmatrix} \right\} K_{\lambda, [n-t, 1^t]}$ .

The second equality in part (a) follows from the fact that  $K_{\lambda, [n-t, 1^t]} = 0$  unless  $\lambda_1 \geq n-t$ , i.e. unless  $t \geq n - \lambda_1 = |\lambda^\#|$ , and from the fact that a semistandard tableau, whose entries are  $n-t$  zeros and the numbers  $1, 2, \dots, t$ , must have the  $n-t$  zeros in the first row and have a standard filling of the skew shape  $\lambda/[n-t]$ . To see that the last line of part (a) holds, observe that when  $n-t \geq \lambda_2$ , any standard tableau of shape  $\lambda/[n-t]$ , has  $\lambda_1 - (n-t) = t - (n - \lambda_1)$  entries chosen from  $\{1, 2, \dots, t\}$  in its first row. There are

$$\binom{t}{t - (n - \lambda_1)} = \binom{t}{n - \lambda_1} = \binom{t}{|\lambda^\#|}$$

ways to select those entries. The remaining integers from  $\{1, 2, \dots, t\}$  fill the shape  $\lambda^\#$  to give a standard tableau. Therefore,  $f^{\lambda/[n-t]} = \binom{t}{|\lambda^\#|} f^{\lambda^\#}$  if  $n - \lambda_2 \geq t$ .

For part (b), identifying  $S_{n-1}$  with the permutations of  $S_n$  that fix  $n$ , we see that restriction from  $S_n$  to  $S_{n-1}$  gives  $M_n = M_{n-1} \oplus \mathbb{C}u_n$ , where  $M_{n-1}$  is the permutation module of  $S_{n-1}$  spanned by the vectors  $u_1, \dots, u_{n-1}$ . Hence,  $M_n^{\otimes k} \cong \bigoplus_{s=0}^k \binom{k}{s} M_{n-1}^{\otimes s}$  as an  $S_{n-1}$ -module, which together with (a) implies

$$\begin{aligned}
\dim Z_{k+\frac{1}{2},n}^\mu &= \sum_{s=0}^k \binom{k}{s} \dim Z_{s,n-1}^\mu \\
&= \sum_{s=0}^k \binom{k}{s} \left( \sum_{t=1}^{n-1} \left\{ \begin{matrix} s \\ t \end{matrix} \right\} K_{\mu,[n-1-t,1^t]} \right) \\
&= \sum_{t=0}^{n-1} \left( \sum_{s=0}^k \binom{k}{s} \left\{ \begin{matrix} s \\ t \end{matrix} \right\} \right) K_{\mu,[n-1-t,1^t]} \\
&= \sum_{t=0}^{n-1} \left( \sum_{s=t}^k \binom{k}{s} \left\{ \begin{matrix} s \\ t \end{matrix} \right\} \right) K_{\mu,[n-1-t,1^t]} \\
&= \sum_{t=0}^{n-1} \left\{ \begin{matrix} k+1 \\ t+1 \end{matrix} \right\} K_{\mu,[n-1-t,1^t]} \quad \text{using (5.1)}.
\end{aligned}$$

This establishes the first equality in (b) for  $k$  and all  $n \geq 1$ . The remainder of (b) can be shown by arguments similar to the ones used for part (a).

Part (c) is an immediate consequence of (a), since  $S_n^{[n]}$  is the trivial  $S_n$ -module, and  $M_n^{\otimes k}$  is isomorphic, as an  $S_n$ -module, to its dual module, so that

$$\begin{aligned}
\dim Z_k(n) &= \dim Z_{2k,n}^{[n]} = \sum_{t=0}^n \left\{ \begin{matrix} 2k \\ t \end{matrix} \right\} K_{[n],[n-t,1^t]} \\
&= \sum_{t=0}^n \left\{ \begin{matrix} 2k \\ t \end{matrix} \right\} = B(2k, n) \quad (= B(2k) \text{ if } n \geq 2k).
\end{aligned}$$

Part (d) follows readily from (b) for similar reasons.  $\square$

**Remark 5.8.** In [D, Prop. 2.1], it was shown using the exponential generating functions of Goupil and Chauve [GC] that for the partition  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}]$ , the multiplicity of  $S_n^\lambda$  in  $M_n^{\otimes k}$  (i.e.  $\dim Z_{k,n}^\lambda$ ) equals  $f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-2} \binom{t}{|\lambda^\#|} \left\{ \begin{matrix} k \\ t \end{matrix} \right\}$  whenever  $1 \leq k \leq n - \lambda_2$ . This is a special case of part (a) of Proposition 5.10. As mentioned earlier, this result was used in [D] to bound the mixing time of a Markov chain on  $S_n$ .

**Remark 5.9.** Suppose  $\nu = [\nu_1, \dots, \nu_{\ell(\nu)}]$  is a partition with  $0 \leq |\nu| \leq k$ , and for  $n \geq 2k$ , let  $[n - |\nu|, \nu]$  be the partition of  $n$  given by  $[n - |\nu|, \nu] := [n - |\nu|, \nu_1, \dots, \nu_{\ell(\nu)}]$ . In the next proposition, we obtain an expression for the dimension of the irreducible  $Z_k(n)$ -module  $Z_{k,n}^{[n-|\nu|,\nu]}$  (and for the  $Z_{k+\frac{1}{2}}(n)$ -module  $Z_{k+\frac{1}{2},n}^{[n-1-|\nu|,\nu]}$  when  $n-1 \geq 2k$ ). We prove that both dimensions equal  $f^\nu = \dim S_k^\nu$  when  $|\nu| = k$ . When  $\nu = [k]$ ,  $\dim Z_{k,n}^{[n-k,k]} = f^{[k]} = 1$  for all  $n \geq 2k$ . When  $n = 2k - 1$ , the kernel of the map  $P_k(n) \rightarrow Z_k(n)$  is one-dimensional, since  $[n - k, k]$  is not a partition in that case. In [BH2], we describe the kernel of the map  $P_k(n) \rightarrow Z_k(n)$  for all  $n < 2k$ .



**Proposition 5.10.** *Assume  $\nu = [\nu_1, \dots, \nu_{\ell(\nu)}]$  is a partition with  $0 \leq |\nu| \leq k$ .*

(a) *If  $0 \leq 2k \leq n$ , then*

$$\dim Z_{k,n}^{[n-|\nu|,\nu]} = f^\nu \sum_{t=|\nu|}^k \binom{t}{|\nu|} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \quad (= f^\nu \text{ when } |\nu| = k). \quad (5.11)$$

(b) *If  $0 \leq 2k \leq n - 1$ , then*

$$\dim Z_{k+\frac{1}{2},n}^{[n-1-|\nu|,\nu]} = f^\nu \sum_{t=|\nu|}^k \binom{t}{|\nu|} \left\{ \begin{matrix} k+1 \\ t+1 \end{matrix} \right\} \quad (= f^\nu \text{ when } |\nu| = k). \quad (5.12)$$

*Proof.* (a) From Theorem 5.5 (a), we know for the partition  $[n - |\nu|, \nu] = [n - |\nu|, \nu_1, \dots, \nu_{\ell(\nu)}]$  that

$$\dim Z_{k,n}^{[n-|\nu|,\nu]} = f^\nu \sum_{t=|\nu|}^{n-\nu_1} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{|\nu|} + \sum_{t=n-\nu_1+1}^n \left\{ \begin{matrix} k \\ t \end{matrix} \right\} f^{[n-|\nu|,\nu]/[n-t]}. \quad (5.13)$$

Since  $\nu_1 \leq |\nu| \leq k$ , and we are assuming  $n \geq 2k$ , it follows that  $n - \nu_1 \geq n - k \geq k$ . Thus, the first summation equals  $f^\nu \sum_{t=|\nu|}^k \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \binom{t}{|\nu|}$ , and the second is 0, which is the assertion in (a). The argument for part (b) is completely analogous.  $\square$

The partition algebras  $P_k(\xi)$  and  $P_{k+\frac{1}{2}}(\xi)$  are generically semisimple for all  $\xi \in \mathbb{C}$  with  $\xi \notin \{0, 1, \dots, 2k - 1\}$  (see [MS] or [HR, Thm. 3.7]). Assume  $\nu = [\nu_1, \dots, \nu_{\ell(\nu)}]$  is a partition with  $0 \leq |\nu| \leq k$ , and let  $P_{k,\xi}^\nu$  denote the irreducible  $P_k(\xi)$ -module and  $P_{k+\frac{1}{2},\xi}^\nu$  denote the irreducible  $P_{k+\frac{1}{2}}(\xi)$ -module indexed by  $\nu$ . The dimension of  $P_{k,\xi}^\nu$  (resp.  $P_{k+\frac{1}{2},\xi}^\nu$ ) is the same for all generic values of  $\xi$ . Therefore, we can apply Proposition 5.10 with  $n = 2k$  and  $\lambda = [n - |\nu|, \nu_1, \dots, \nu_{\ell(\nu)}] \vdash n$  to conclude the following:

**Corollary 5.14.** *Let  $\nu$  be a partition with  $0 \leq |\nu| \leq k$ . For  $\xi \notin \{0, 1, \dots, 2k - 1\}$ , let  $P_{k,\xi}^\nu$  denote the irreducible  $P_k(\xi)$ -module and  $P_{k+\frac{1}{2},\xi}^\nu$  denote the irreducible  $P_{k+\frac{1}{2}}(\xi)$ -module indexed by  $\nu$ . Then*

$$\begin{aligned} \dim P_{k,\xi}^\nu &= f^\nu \sum_{t=|\nu|}^k \binom{t}{|\nu|} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} && (= f^\nu \text{ when } |\nu| = k) \\ \dim P_{k+\frac{1}{2},\xi}^\nu &= f^\nu \sum_{t=|\nu|}^k \binom{t}{|\nu|} \left\{ \begin{matrix} k+1 \\ t+1 \end{matrix} \right\} && (= f^\nu \text{ when } |\nu| = k). \end{aligned}$$

### 5.3 Bijective proof of Theorem 5.5 (a)

By Theorem 5.5(a), we know that  $\dim Z_{k,n}^\lambda$  equals the number of pairs  $(P, T)$  where  $P$  is a set partition of  $\{1, 2, \dots, k\}$  into  $t$  blocks for some  $t \in \{1, \dots, n\}$ , and  $T$  is a semistandard tableau of shape  $\lambda$  filled with  $n - t$  zeros and  $t$  distinct numbers from  $\mathbb{Z}_{>0}$ , so that  $T$  has type  $[n - t, 1^t]$ . By Section 4, we know that this dimension is also equal to the number of paths from the root of the

Bratteli diagram  $\mathcal{B}(S_n, S_{n-1})$  to  $\lambda \in \Lambda_{k, S_n}$ . Such paths (also referred to as vacillating tableaux) are given by a sequence of partitions

$$\left( \lambda^{(0)} = [n], \lambda^{(\frac{1}{2})} = [n-1], \lambda^{(1)}, \lambda^{(1+\frac{1}{2})}, \dots, \lambda^{(k-1)}, \lambda^{(k-\frac{1}{2})}, \lambda^{(k)} = \lambda \right)$$

such that  $\lambda^{(i)} \in \Lambda_{i, S_n}$ ,  $\lambda^{(i-\frac{1}{2})} \in \Lambda_{i-\frac{1}{2}, S_{n-1}}$  for each  $i$ , and

- (a)  $\lambda^{(i-\frac{1}{2})} = \lambda^{(i-1)} - \square$ ,
- (b)  $\lambda^{(i)} = \lambda^{(i-\frac{1}{2})} + \square$ ,

for each integer  $1 \leq i \leq k$ . In this section, we demonstrate a bijection between paths and pairs  $(P, T)$ , thereby giving a combinatorial proof of Theorem 5.5(a). The corresponding bijection for Theorem 5.5(b) is gotten by applying this same bijection on paths to  $\mu \in \Lambda_{k-\frac{1}{2}, S_{n-1}}$ . We assume familiarity with the RSK row-insertion algorithm (see for example [S2, Sec. 7.1.1]), and let  $T \leftarrow b$  denote row insertion of the integer  $b$  into the semistandard tableau  $T$ . The bijection here, which works for all  $n \geq 1$  and  $k \geq 0$ , extends that of [CDDSY, Thm. 2.4], which holds for  $n \geq 2k$ . It is easily adaptable to give a combinatorial proof of Theorem 5.5 (b).

**Bijection from paths to pairs  $(P, T)$ :** Given a path  $(\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \dots, \lambda^{(k)} = \lambda)$  to  $\lambda \in \Lambda_{k, S_n}$ , we recursively construct a sequence  $(P_0, T_0), (P_{\frac{1}{2}}, T_{\frac{1}{2}}), (P_1, T_1), \dots, (P_k, T_k)$  such that, for each  $i$ ,  $P_i$  is a set partition of  $\{1, \dots, [i]\}$  into  $t$  blocks, and  $T_i$  is a semistandard tableau of shape  $\lambda^{(i)}$  with  $n-t$  zeros and nonzero entries from the set  $\max(P_i)$  whose elements are the maximal entries in the  $t$  blocks of  $P_i$ . Then  $(P_k, T_k)$  is the pair associated with the path  $(\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \dots, \lambda^{(k)} = \lambda)$ .

Let  $P_0 = \emptyset$  and let  $T_0$  be the semistandard tableau of shape  $[n]$  and type  $[n]$ , i.e., with each entry equal to 0. Then for each integer  $i = 1, 2, \dots, k$ , perform these steps.

- (1) Construct  $(P_{i-\frac{1}{2}}, T_{i-\frac{1}{2}})$  from  $(P_{i-1}, T_{i-1})$  as follows: Let  $b$  be the unique nonnegative integer such that  $T_{i-1} = (T_{i-\frac{1}{2}} \leftarrow b)$ . Since  $b \in T_{i-1}$ , we know that  $0 \leq b < i$ . If  $b = 0$ , then  $P_{i-\frac{1}{2}}$  is obtained by adding the block  $\{i\}$  to  $P_{i-1}$ . If  $b > 0$ , then  $P_{i-\frac{1}{2}}$  is obtained by adding  $i$  to the block that contains  $b$  in  $P_{i-1}$ .
- (2) Construct  $(P_i, T_i)$  from  $(P_{i-\frac{1}{2}}, T_{i-\frac{1}{2}})$  by letting  $P_i$  equal  $P_{i-\frac{1}{2}}$  and  $T_i$  be the column strict tableau obtained from  $T_{i-\frac{1}{2}}$  by adding the entry  $i$  in the box  $\lambda^i \setminus \lambda^{i-\frac{1}{2}}$ .

By the above construction,  $P_i$  is a set partition of  $\{1, \dots, [i]\}$  for each  $i$ , and if  $P_i$  has  $t$  parts, then  $T_i$  is a semistandard tableau with  $n-t$  zeros and with the elements of  $\max(P_i)$  as its nonzero entries.

The map is bijective since the above construction can be reversed: Given a pair  $(P, T)$  consisting of a set partition  $P$  of  $\{1, \dots, k\}$  into  $t$  blocks and a semistandard tableau  $T$  of shape  $\lambda \in \Lambda_{k, S_n}$  filled with  $n-t$  zeros and the elements of  $\max(P)$ , we produce a path  $(\lambda^0 = [n], \lambda^{\frac{1}{2}}, \dots, \lambda^k = \lambda)$  by performing these steps: Start with  $P_k = P$ ,  $T_k = T$ , and work backwards to produce the sequence  $(P_k, T_k), (P_{k-\frac{1}{2}}, T_{k-\frac{1}{2}}), (P_{k-1}, T_{k-1}), \dots, (P_0, T_0)$  as follows:

- (1)' Construct  $(P_{i-\frac{1}{2}}, T_{i-\frac{1}{2}})$  from  $(P_i, T_i)$  by letting  $P_{i-\frac{1}{2}} = P_i$  and deleting  $i$  from  $T_i$ .

(2)' Construct  $(P_{i-1}, T_{i-1})$  from  $(P_{i-\frac{1}{2}}, T_{i-\frac{1}{2}})$  by the following procedure: If  $i$  is a singleton block in  $P_{i-\frac{1}{2}}$ , then  $T_{i-1} = (T_{i-\frac{1}{2}} \leftarrow 0)$ . If  $i$  is not a singleton block in  $P_{i-\frac{1}{2}}$ , then  $T_{i-1} = (T_{i-\frac{1}{2}} \leftarrow b)$ , where  $b$  is the second largest element of the block containing  $i$ . Let  $P_{i-1}$  be obtained by deleting  $\{i\}$  from  $P_{i-\frac{1}{2}}$ .

If  $\lambda^{(i)}$  is the partition shape of  $T_i$  for  $i = 0, \frac{1}{2}, 1, \dots, k$ , then  $(\lambda^{(0)}, \lambda^{(\frac{1}{2})}, \dots, \lambda^{(k)})$  to  $\lambda$  is the corresponding path in the Bratteli diagram. This bijection is illustrated in the next example.

**Example 5.15.** If  $n = 4, k = 3$ , and  $\lambda = [2, 2]$ , then

$$\dim Z_{3,4}^{[2,2]} = \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} K_{[2,2],[3,1]} + \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} K_{[2,2],[2,1,1]} + \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} K_{[2,2],[1,1,1,1]} = 1 \cdot 0 + 3 \cdot 1 + 1 \cdot 2 = 5.$$

This is the subscript 5 on  $[2, 2]$  at level 3 in the Bratteli diagram in Figure 1. The five corresponding pairs  $(P, T)$  of set partitions and semistandard tableaux are:

$$(\{1, 2 \mid 3\}, \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 2 & 3 \\ \hline \end{array}), \quad (\{1, 3 \mid 2\}, \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 2 & 3 \\ \hline \end{array}), \quad (\{1 \mid 2, 3\}, \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 1 & 3 \\ \hline \end{array}), \quad (\{1 \mid 2 \mid 3\}, \begin{array}{|c|c|} \hline 0 & 2 \\ \hline 1 & 3 \\ \hline \end{array}), \quad (\{1 \mid 2 \mid 3\}, \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 3 \\ \hline \end{array}).$$

The five paths to  $[2, 2] \in \Lambda_{3, S_4}$  and the corresponding bijections with these pairs are illustrated in Figure 3.

## 5.4 Dimension Examples

**Example 5.16.** Assume  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}] \vdash n$  and  $1 \leq \lambda_2 \leq 2$ . We claim that under these assumptions, Theorem 5.5(a) simplifies to

$$\dim Z_{k,n}^\lambda = \begin{cases} f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-2} \binom{t}{|\lambda^\#|} \begin{Bmatrix} k \\ t \end{Bmatrix} + f^\lambda \left( \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + \begin{Bmatrix} k \\ n \end{Bmatrix} \right) & \text{if } \lambda_1 > 1, \\ \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + \begin{Bmatrix} k \\ n \end{Bmatrix} & \text{if } \lambda_1 = 1, \end{cases} \quad (5.17)$$

where  $\lambda^\# = [\lambda_2, \dots, \lambda_{\ell(\lambda)}]$ . In particular, this formula holds for all  $\lambda$  when  $n \leq 5$ . When  $\lambda_1 = 1$ , then  $\lambda = [1^n]$ , and this says  $\dim Z_{k,n}^{[1^n]} = \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + \begin{Bmatrix} k \\ n \end{Bmatrix}$  for all  $n \geq 1$ .

To verify this assertion, we start from the last line of Theorem 5.5 (a),

$$\dim Z_{k,n}^\lambda = f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-\lambda_2} \begin{Bmatrix} k \\ t \end{Bmatrix} \binom{t}{|\lambda^\#|} + \sum_{t=n-\lambda_2+1}^n \begin{Bmatrix} k \\ t \end{Bmatrix} f^{\lambda/[n-t]}. \quad (5.18)$$

When  $\lambda_1 = 1$ , then  $\lambda = [1^n]$  and  $\lambda^\# = [1^{n-1}]$ , and this reduces to

$$\dim Z_{k,n}^{[1^n]} = f^{[1^{n-1}]} \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + f^{[1^n]} \begin{Bmatrix} k \\ n \end{Bmatrix} = \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + \begin{Bmatrix} k \\ n \end{Bmatrix}.$$

level	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$
$\ell = 0$	$\boxed{00000}$ $\emptyset$	$\boxed{00000}$ $\emptyset$	$\boxed{00000}$ $\emptyset$	$\boxed{00000}$ $\emptyset$	$\boxed{00000}$ $\emptyset$
$\ell = \frac{1}{2}$	$\boxed{0000} \leftarrow 0$ $\{1\}$	$\boxed{0000} \leftarrow 0$ $\{1\}$	$\boxed{0000} \leftarrow 0$ $\{1\}$	$\boxed{0000} \leftarrow 0$ $\{1\}$	$\boxed{0000} \leftarrow 0$ $\{1\}$
$\ell = 1$	$\boxed{00001}$ $\{1\}$	$\boxed{0000}$ $\{1\}$	$\boxed{0000}$ $\{1\}$	$\boxed{0000}$ $\{1\}$	$\boxed{0000}$ $\{1\}$
$\ell = 1\frac{1}{2}$	$\boxed{0000} \leftarrow 1$ $\{1, 2\}$	$\boxed{0001} \leftarrow 0$ $\{1   2\}$	$\boxed{000} \leftarrow 0$ $\{1   2\}$	$\boxed{000} \leftarrow 0$ $\{1   2\}$	$\boxed{000} \leftarrow 0$ $\{1   2\}$
$\ell = 2$	$\boxed{0000}$ $\{1, 2\}$	$\boxed{0001}$ $\{1   2\}$	$\boxed{0002}$ $\{1   2\}$	$\boxed{000}$ $\{1   2\}$	$\boxed{000}$ $\{1   2\}$
$\ell = 2\frac{1}{2}$	$\boxed{000} \leftarrow 0$ $\{1, 2   3\}$	$\boxed{000} \leftarrow 1$ $\{1, 3   2\}$	$\boxed{000} \leftarrow 2$ $\{1   2, 3\}$	$\boxed{002} \leftarrow 0$ $\{1   2   3\}$	$\boxed{001} \leftarrow 0$ $\{1   2   3\}$
$\ell = 3$	$\boxed{000}$ $\{1, 2   3\}$	$\boxed{000}$ $\{1, 3   2\}$	$\boxed{000}$ $\{1   2, 3\}$	$\boxed{002}$ $\{1   2   3\}$	$\boxed{001}$ $\{1   2   3\}$

Figure 3: The bijection between the five paths to  $\lambda = [2, 2] \in \Lambda_{3, S_4}$  in the Bratteli diagram  $\mathcal{B}(S_4, S_3)$  and pairs  $(P, T)$  of Example 5.15 consisting of a set partition  $P$  of  $\{1, 2, 3\}$  into  $t$  blocks and a semistandard tableau  $T$  filled with  $4 - t$  zeroes and the maximum entries of the blocks of  $P$ .

When  $\lambda_1 > 1$ , and  $\lambda_2 = 1$ , then  $\lambda^\# = [1^{n-\lambda_1}]$ , and

$$\begin{aligned}
\dim Z_{k,n}^\lambda &= f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-1} \begin{Bmatrix} k \\ t \end{Bmatrix} \binom{t}{|\lambda^\#|} + f^\lambda \begin{Bmatrix} k \\ n \end{Bmatrix} \\
&= f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-2} \begin{Bmatrix} k \\ t \end{Bmatrix} \binom{t}{|\lambda^\#|} + f^{\lambda^\#} \begin{Bmatrix} k \\ n-1 \end{Bmatrix} \binom{n-1}{|\lambda^\#|} + f^\lambda \begin{Bmatrix} k \\ n \end{Bmatrix} \\
&= f^{\lambda^\#} \sum_{t=|\lambda^\#|}^{n-2} \binom{t}{|\lambda^\#|} \begin{Bmatrix} k \\ t \end{Bmatrix} + f^\lambda \left( \begin{Bmatrix} k \\ n-1 \end{Bmatrix} + \begin{Bmatrix} k \\ n \end{Bmatrix} \right),
\end{aligned}$$

since  $f^\lambda = f^{\lambda^\#} \binom{n-1}{n-\lambda_1}$ . Finally, when  $\lambda_2 = 2$ , the assertion is exactly (5.18), as  $f^{\lambda/[1]} = f^\lambda$ .

**Example 5.19.** Since  $f^{[n]} = 1$ , Corollary 5.14 implies for all  $k \geq n$  and all generic values of  $\xi$  that

$$\dim P_{k,\xi}^{[n]} = \sum_{t=n}^k \binom{t}{n} \begin{Bmatrix} k \\ t \end{Bmatrix} \quad \text{and} \quad \dim P_{k+\frac{1}{2},\xi}^{[n]} = \sum_{t=n}^k \binom{t}{n} \begin{Bmatrix} k+1 \\ t+1 \end{Bmatrix}. \quad (5.20)$$

The last line (5.18) of part (a) of Theorem 5.5 gives

$$\dim Z_{k,n}^{[n]} = \sum_{t=0}^n \left\{ \begin{matrix} k \\ t \end{matrix} \right\} = B(k, n) \quad (= B(k) \text{ when } n \geq k), \quad (5.21)$$

while the last line of part (b) of Theorem 5.5 says

$$\dim Z_{k+\frac{1}{2},n}^{[n-1]} = \sum_{t=1}^n \left\{ \begin{matrix} k+1 \\ j \end{matrix} \right\} = B(k+1, n) \quad (= B(k+1) \text{ when } n \geq k+1). \quad (5.22)$$

**Remark 5.23.** A (*Gelfand*) *model* for an algebra is a module in which each irreducible module appears as a direct summand with multiplicity one. In [HRe], Halverson and Reeks construct models for certain diagram algebras, including the partition algebras  $P_k(\xi)$  for generic  $\xi$ , using basis diagrams invariant under reflection about the horizontal axis (the symmetric diagrams) and the diagram conjugation action of  $P_k(\xi)$  on them. The model  $\mathfrak{M}_{P_k}$  for  $P_k(\xi)$  decomposes into submodules  $\mathfrak{M}_{P_k} = \bigoplus_{r,p} \mathfrak{M}_{P_k}^{r,p}$ , where  $\mathfrak{M}_{P_k}^{r,p} = \bigoplus_{\substack{\nu \vdash r \\ \text{odd}(\nu)=p}} P_{k,\xi}^\nu$  according to the size  $r = |\nu|$  of the partition  $\nu$  and its number  $p = \text{odd}(\nu)$  of odd parts. By enumerating symmetric diagrams, they determine that

$$\dim \mathfrak{M}_{P_k}^{r,p} = \sum_{t=r}^k \binom{r}{p} (r-p-1)!! \binom{t}{r} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} = \binom{r}{p} (r-p-1)!! \sum_{t=r}^k \binom{t}{r} \left\{ \begin{matrix} k \\ t \end{matrix} \right\}, \quad (5.24)$$

where  $r-p$  is even and  $(r-p-1)!! = (r-p-1)(r-p-3)\cdots 3 \cdot 1$ . The factor  $\binom{r}{p} (r-p-1)!!$  comes from the fact (see [HRe]) that

$$\sum_{\substack{\nu \vdash r \\ \text{odd}(\nu)=p}} f^\nu = |\Gamma^{r,p}| = \binom{r}{p} (r-p-1)!!, \quad (5.25)$$

where  $\Gamma^{r,p}$  is the set of involutions (elements of order 2) with  $p$  fixed points in the symmetric group  $S_r$ . Corollary 5.14 and (5.25) give an alternate proof of (5.24):

$$\dim \mathfrak{M}_{P_k}^{r,p} = \sum_{\substack{\nu \vdash r \\ \text{odd}(\nu)=p}} \dim P_{k,\xi}^\nu = \sum_{\substack{\nu \vdash r \\ \text{odd}(\nu)=p}} f^\nu \left( \sum_{t=|\nu|}^k \binom{t}{|\nu|} \left\{ \begin{matrix} k \\ t \end{matrix} \right\} \right) = \binom{r}{p} (r-p-1)!! \sum_{t=r}^k \binom{t}{r} \left\{ \begin{matrix} k \\ t \end{matrix} \right\}.$$

## 6 Dimension formulas for alternating group centralizer algebras

The restriction rules in (3.6) combined with Theorem 2.7 (b) can be used to derive expressions for the dimensions of the irreducible modules for the alternating group centralizer algebras  $\widehat{Z}_k(n)$  and  $\widehat{Z}_{k+\frac{1}{2}}(n)$  from the dimension formulas for irreducible modules for  $Z_k(n)$  and  $Z_{k+\frac{1}{2}}(n)$  in Theorem 5.5.

**Theorem 6.1.** *Assume  $k \in \mathbb{Z}_{\geq 0}$ . The dimensions of the irreducible modules for  $\widehat{Z}_k(n)$  and  $\widehat{Z}_{k+\frac{1}{2}}(n)$  are as follows (using notation from Table 1).*

(a) For  $\lambda \vdash n$  and  $\lambda \in \Lambda_{k, A_n}$ ,

$$\begin{aligned} \dim \widehat{Z}_{k,n}^\lambda &= \dim Z_{k,n}^\lambda + \dim Z_{k,n}^{\lambda^*}, & \text{if } \lambda > \lambda^*, \\ \dim \widehat{Z}_{k,n}^{\lambda^+} &= \dim \widehat{Z}_{k,n}^{\lambda^-} = \dim Z_{k,n}^\lambda, & \text{if } \lambda = \lambda^*, \end{aligned}$$

where  $\dim Z_{k,n}^\lambda$  and  $\dim Z_{k,n}^{\lambda^*}$  are given by the formula in Theorem 5.5 (a).

(b) For  $\mu \vdash n-1$  and  $\mu \in \Lambda_{k, A_{n-1}}$ ,

$$\begin{aligned} \dim \widehat{Z}_{k+\frac{1}{2},n}^\mu &= \dim Z_{k+\frac{1}{2},n}^\mu + \dim Z_{k+\frac{1}{2},n}^{\mu^*}, & \text{if } \mu > \mu^*, \\ \dim \widehat{Z}_{k+\frac{1}{2},n}^{\mu^+} &= \dim \widehat{Z}_{k+\frac{1}{2},n}^{\mu^-} = \dim Z_{k+\frac{1}{2},n}^\mu, & \text{if } \mu = \mu^*, \end{aligned}$$

where  $\dim Z_{k+\frac{1}{2},n}^\mu$  and  $\dim Z_{k+\frac{1}{2},n}^{\mu^*}$  are given by the formula in Theorem 5.5 (b).

The next corollary gives some particular instances of Theorem 6.1 of special interest.

**Corollary 6.2.** Assume  $k \in \mathbb{Z}_{\geq 0}$  and  $r \geq 2$ . Recall the definitions of the Bell numbers  $B(k, n)$  and  $B(k)$  from (5.3) and (5.4).

$$\begin{aligned} \text{(a)} \quad \dim \widehat{Z}_{k,n}^{[n]} &= \dim Z_{k,n}^{[n]} + \dim Z_{k,n}^{[1^n]} = \sum_{t=0}^n \binom{k}{t} + \binom{k}{n-1} + \binom{k}{n} \\ &= B(k, n) + \binom{k}{n-1} + \binom{k}{n}. \end{aligned}$$

$$\text{(b)} \quad \dim \widehat{Z}_k(n) = \dim \widehat{Z}_{2k,n}^{[n]} = B(2k, n) + \binom{2k}{n-1} + \binom{2k}{n}.$$

In particular,  $\dim \widehat{Z}_k(n) = B(2k) + 1$  if  $n = 2k + 1$ , and  $\dim \widehat{Z}_k(n) = B(2k)$  if  $n > 2k + 1$ .

$$\begin{aligned} \text{(c)} \quad \dim \widehat{Z}_{k+\frac{1}{2},n}^{[n-1]} &= \dim Z_{k+\frac{1}{2},n}^{[n-1]} + \dim \widehat{Z}_{k+\frac{1}{2},n}^{[1^{n-1}]} = \sum_{j=1}^n \binom{k+1}{j} + \binom{k+1}{n-1} + \binom{k+1}{n} \\ &= B(k+1, n) + \binom{k+1}{n-1} + \binom{k+1}{n} = \dim \widehat{Z}_{k+1,n}^{[n]}. \end{aligned}$$

$$\text{(d)} \quad \dim \widehat{Z}_{k+\frac{1}{2}}(n) = \dim \widehat{Z}_{2k+\frac{1}{2},n}^{[n-1]} = B(2k+1, n) + \binom{2k+1}{n-1} + \binom{2k+1}{n}.$$

In particular,  $\dim \widehat{Z}_{k+\frac{1}{2}}(n) = B(2k+1) + 1$  if  $n = 2k + 2$ , and  $\dim \widehat{Z}_{k+\frac{1}{2}}(n) = B(2k+1)$  if  $n > k + 2$ .

**Remark 6.3.** Part (b) of Corollary 6.2 was shown by Bloss [B11, B12] by different methods. Part (d) extends that result to the centralizer algebras  $\widehat{Z}_{k+\frac{1}{2}}(n)$  and gives some indication of how the algebras  $\widehat{Z}_{k+\frac{1}{2}}(n)$  “fill the gap” between the integer levels.

**Example 6.4.** Corollary 6.2 (c) says that for  $k = 3$  and  $n = 4$ ,

$$\dim \widehat{Z}_{3+\frac{1}{2},4}^{[3]} = \sum_{j=1}^4 \binom{4}{j} + \binom{4}{3} + \binom{4}{4} = 1 + 7 + 2(6 + 1) = 22.$$

This is the subscript on the partition [3] in the last row of the Bratteli diagram in Figure 2.

## 7 The centralizer algebra $\text{QZ}_k(n) := \text{End}_{S_n}(\mathbb{R}_n^{\otimes k})$ for $\mathbb{R}_n = S_n^{[n-1,1]}$ and its relatives

In [DO], Daugherty and Orellana investigated the centralizer algebra  $\text{QZ}_k(n) := \text{End}_{S_n}(\mathbb{R}_n^{\otimes k})$ , where  $\mathbb{R}_n = S_n^{[n-1,1]}$ , and proved that there is a variant of the partition algebra, that they termed the *quasi-partition algebra* and denoted  $\text{QP}_k(n)$ . They exhibited an algebra homomorphism  $\text{QP}_k(n) \rightarrow \text{End}_{S_n}(\mathbb{R}_n^{\otimes k})$  and showed that this mapping is always a surjection and is an isomorphism when  $n \geq 2k$ . The irreducible modules  $\text{QZ}_{k,n}^\lambda$  for  $\text{QZ}_k(n)$  are indexed by partitions  $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n] \vdash n$ .

In this last section, we determine a formula for the dimensions of these irreducible modules. The dimension expression we obtain holds for arbitrary values of  $k$  and  $n$  and differs from that in [DO, Thm. 4.6], which is valid for  $n > k + \lambda_2$ , and is more closely related to the one in [D, Cor. 2.2], which holds for  $n \geq k + \lambda_2$ . We also extend these results to the case of the corresponding centralizer algebra of the alternating group:  $\widehat{\text{QZ}}_k(n) := \text{End}_{A_n}(\mathbb{R}_n^{\otimes k})$ .

We adopt the notation in Table 2 for various centralizer algebras and their irreducible modules associated with  $\mathbb{R}_n^{\otimes k}$ . In this table, for all  $k \in \mathbb{Z}_{\geq 0}$ ,  $\text{q}\Lambda_{k,S_n}$  (resp.  $\text{q}\Lambda_{k,A_n}$ ) is the set of indices for the irreducible  $\text{QZ}_k(n)$ -summands (resp.  $\widehat{\text{QZ}}_k(n)$ -summands) in  $\mathbb{R}_n^{\otimes k}$  with multiplicity at least one; similarly  $\text{q}\Lambda_{k,S_{n-1}}$  (resp.  $\text{q}\Lambda_{k,A_{n-1}}$ ) is the set of indices for the irreducible  $\text{QZ}_{k+\frac{1}{2}}(n)$ -summands (resp.  $\widehat{\text{QZ}}_{k+\frac{1}{2}}(n)$ -summands) in  $\mathbb{R}_n^{\otimes k}$  with multiplicity at least one.

centralizer algebra	irreducible modules
$\text{QZ}_k(n) := \text{End}_{S_n}(\mathbb{R}_n^{\otimes k})$	$\text{QZ}_{k,n}^\lambda, \lambda \vdash n, \lambda \in \text{q}\Lambda_{k,S_n} \subseteq \Lambda_{S_n}$
$\text{QZ}_{k+\frac{1}{2}}(n) := \text{End}_{S_{n-1}}(\mathbb{R}_n^{\otimes k})$	$\text{QZ}_{k+\frac{1}{2},n}^\mu, \mu \vdash n-1, \mu \in \text{q}\Lambda_{k+\frac{1}{2},S_{n-1}} \subseteq \Lambda_{S_{n-1}}$
$\widehat{\text{QZ}}_k(n) := \text{End}_{A_n}(\mathbb{R}_n^{\otimes k})$	$\widehat{\text{QZ}}_{k,n}^\lambda, \lambda \vdash n, \lambda > \lambda^*, \lambda \in \text{q}\Lambda_{k,A_n} \subseteq \Lambda_{A_n}$ $\widehat{\text{QZ}}_{k,n}^{\lambda^\pm}, \lambda \vdash n, \lambda = \lambda^*, \lambda \in \text{q}\Lambda_{k,A_n} \subseteq \Lambda_{A_n}$
$\widehat{\text{QZ}}_{k+\frac{1}{2}}(n) := \text{End}_{A_{n-1}}(\mathbb{R}_n^{\otimes k})$	$\widehat{\text{QZ}}_{k+\frac{1}{2},n}^\mu, \mu \vdash n-1, \mu > \mu^*, \mu \in \text{q}\Lambda_{k+\frac{1}{2},A_{n-1}} \subseteq \Lambda_{A_{n-1}}$ $\widehat{\text{QZ}}_{k+\frac{1}{2},n}^{\mu^\pm}, \mu \vdash n, \mu = \mu^*, \mu \in \text{q}\Lambda_{k+\frac{1}{2},A_{n-1}} \subseteq \Lambda_{A_{n-1}}$

Table 2: Notation for the centralizer algebras and modules associated with the tensor product  $\mathbb{R}_n^{\otimes k}$  of the reflection module  $\mathbb{R}_n = S_n^{[n-1,1]}$  of  $S_n$  and its restriction to  $S_{n-1}$ ,  $A_n$ , and  $A_{n-1}$ .

The permutation module  $M_n$  of the symmetric group satisfies  $M_n \cong \mathbb{R}_n \oplus S_n^{[n]}$ , where  $\mathbb{R}_n = S_n^{[n-1,1]}$  is the  $(n-1)$ -dimensional reflection representation of  $S_n$  and  $S_n^{[n]}$  is the trivial module. Applying Proposition 2.11 (a) and Theorem 6.1 gives the following:

**Theorem 7.1.** Let  $k, n \in \mathbb{Z}_{\geq 0}$  with  $n \geq 1$ . The dimensions of the irreducible modules for  $\mathbb{Q}Z_k(n)$ ,  $\mathbb{Q}Z_{k+\frac{1}{2}}(n)$ ,  $\widehat{\mathbb{Q}Z}_k(n)$  and  $\widehat{\mathbb{Q}Z}_{k+\frac{1}{2}}(n)$  are as follows (using notation from Tables 1 and 2).

(a) For  $\lambda \vdash n$ ,  $\lambda \in \mathfrak{q}\Lambda_{k, S_n}$ ,

$$\dim \mathbb{Q}Z_{k,n}^\lambda = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell,n}^\lambda.$$

(b) For  $\mu \vdash n-1$ ,  $\mu \in \mathfrak{q}\Lambda_{k, S_{n-1}}$ ,

$$\dim \mathbb{Q}Z_{k+\frac{1}{2},n}^\mu = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell+\frac{1}{2},n}^\mu.$$

(c) For  $\lambda \vdash n$  with  $\lambda \in \mathfrak{q}\Lambda_{k, A_n}$ ,

$$\dim \widehat{\mathbb{Q}Z}_{k,n}^\lambda = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim \widehat{Z}_{\ell,n}^\lambda = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} (\dim Z_{\ell,n}^\lambda + \dim Z_{\ell,n}^{\lambda^*}), \quad \text{if } \lambda > \lambda^*,$$

$$\dim \widehat{\mathbb{Q}Z}_{k,n}^{\lambda^\pm} = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim \widehat{Z}_{\ell,n}^{\lambda^\pm} = \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell,n}^\lambda = \dim \mathbb{Q}Z_{k,n}^\lambda, \quad \text{if } \lambda = \lambda^*.$$

(d) For  $\mu \vdash n-1$  with  $\mu \in \mathfrak{q}\Lambda_{k+\frac{1}{2}, A_{n-1}}$ ,

$$\begin{aligned} \dim \widehat{\mathbb{Q}Z}_{k+\frac{1}{2},n}^\mu &= \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim \widehat{Z}_{\ell+\frac{1}{2},n}^\mu \\ &= \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} (\dim Z_{\ell+\frac{1}{2},n}^\mu + \dim Z_{\ell+\frac{1}{2},n}^{\mu^*}), \quad \text{if } \mu > \mu^*, \end{aligned}$$

$$\begin{aligned} \dim \widehat{\mathbb{Q}Z}_{k+\frac{1}{2},n}^{\mu^\pm} &= \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim \widehat{Z}_{\ell+\frac{1}{2},n}^{\mu^\pm} \\ &= \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \dim Z_{\ell+\frac{1}{2},n}^\mu = \dim \mathbb{Q}Z_{k+\frac{1}{2},n}^\mu, \quad \text{if } \mu = \mu^*. \end{aligned}$$

Applying Proposition 2.11(b) and Theorem 7.1 gives the following:

**Corollary 7.2.** Let  $k, n \in \mathbb{Z}_{\geq 0}$  with  $n > 0$ , and let the notation be as in Table 2.

(a)  $\dim \mathbb{Q}Z_k(n) = \dim \mathbb{Q}Z_{2k,n}^{[n]} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell, n)$   
 $\left( = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell) = 1 + \sum_{\ell=1}^{2k} (-1)^{\ell-1} \mathbb{B}(2k-\ell) \quad \text{if } n \geq 2k+2 \right).$

(b)  $\dim \mathbb{Q}Z_{k+\frac{1}{2}}(n) = \dim \mathbb{Q}Z_{2k+\frac{1}{2},n}^{[n-1]} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell+1, n)$   
 $\left( = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell+1) = \mathbb{B}(2k) \quad \text{if } n \geq 2k+1 \right).$



$$(c) \dim \widehat{\mathbb{QZ}}_k(n) = \dim \widehat{\mathbb{QZ}}_{2k,n}^{[n]} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \left( \mathbb{B}(\ell, n) + \left\{ \begin{matrix} \ell \\ n-1 \end{matrix} \right\} + \left\{ \begin{matrix} \ell \\ n \end{matrix} \right\} \right) \\ \left( = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell) = 1 + \sum_{\ell=1}^{2k} (-1)^{\ell-1} \mathbb{B}(2k-\ell) \quad \text{if } n \geq 2k+2 \right).$$

$$(d) \dim \widehat{\mathbb{QZ}}_{k+\frac{1}{2}}(n) = \dim \widehat{\mathbb{QZ}}_{2k+\frac{1}{2},n}^{[n-1]} \\ = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \left( \mathbb{B}(\ell+1, n) + \left\{ \begin{matrix} \ell+1 \\ n-1 \end{matrix} \right\} + \left\{ \begin{matrix} \ell+1 \\ n \end{matrix} \right\} \right) \\ \left( = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell+1) = \mathbb{B}(2k) \quad \text{if } n \geq 2k+2 \right).$$

*Proof.* The first two equalities in parts (a)-(d) of the corollary follow from Proposition 2.11 (b), (5.21), (5.22), and Corollary 6.2 (b) and (d). The final equality in (a) and (c) can be seen as follows: Let  $v_\ell$  be the number of set partitions of  $\{1, \dots, \ell\}$ , with no blocks of size 1. Then, as shown in [Be, Sec. 3.5],  $v_\ell + v_{\ell+1} = \mathbb{B}(\ell)$ , (the  $\ell$ th Bell number). Now [SW, Sec. 1] implies that  $v_{2k} = \sum_{\ell=0}^{2k} (-1)^{2k-\ell} \binom{2k}{\ell} \mathbb{B}(\ell)$ . However, substituting the expression  $v_\ell + v_{\ell+1}$  for  $\mathbb{B}(\ell)$  shows that the telescoping sum  $1 + \sum_{\ell=1}^{2k} (-1)^{\ell-1} \mathbb{B}(2k-\ell) = v_{2k}$  also. Hence, the two expressions for  $v_{2k}$  equal. The final equality in parts (b) and (d) is a well-known property of Bell numbers (see for example [SW, (1.2)]).  $\square$

**Remark 7.3.** The result from Corollary 7.2 (a) that  $\dim \mathbb{QZ}_k(n) = 1 + \sum_{\ell=1}^{2k} (-1)^{\ell-1} \mathbb{B}(2k-\ell)$  when  $n \geq 2k+2$  was shown in [DO, Cor. 2.6]. As noted there, the sequence  $\{v_\ell\}$  is #A000296 in [OEIS].

The results in Theorem 7.1 enable us to conclude the following for generic quasi-partition algebras.

**Corollary 7.4.** *Let  $\nu$  be a partition with  $0 \leq |\nu| \leq k$ . For  $\xi \notin \{0, 1, \dots, 2k-1\}$ , let  $\mathbb{QP}_{k,\xi}^\nu$  denote the irreducible  $\mathbb{QP}_k(\xi)$ -module. Then*

$$\dim \mathbb{QP}_{k,\xi}^\nu = f^\nu \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left( \sum_{t=|\nu|}^{\ell} \binom{\ell}{t} \left\{ \begin{matrix} \ell \\ t \end{matrix} \right\} \right) \quad (= f^\nu \text{ when } |\nu| = k).$$

The Bratteli diagrams constructed using the reflection module  $R_n$  for the pairs  $(S_n, S_{n-1})$ ,  $(A_n, A_{n-1})$  for  $n = 6$  are displayed in A.2 and A.4 of the Appendix. The subscript on a partition at level  $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$  is the dimension of the irreducible module for the centralizer algebra  $\mathbb{QZ}_\ell(6)$ . For  $k \in \mathbb{Z}_{\geq 0}$ ,  $\text{Ind}_{S_{n-1}}^{S_n} \text{Res}_{S_{n-1}}^{S_n} (R_n^{\otimes k})$  is isomorphic as an  $S_n$ -module to  $R_n^{\otimes k} \oplus R_n^{\otimes(k+1)}$ . This implies that the subscripts on level  $k + \frac{1}{2}$  are gotten from level  $k$  by Pascal addition; however, the subscripts on level  $k+1$  are obtained by first performing Pascal addition from level  $k + \frac{1}{2}$  and then subtracting the corresponding subscript from level  $k$ .

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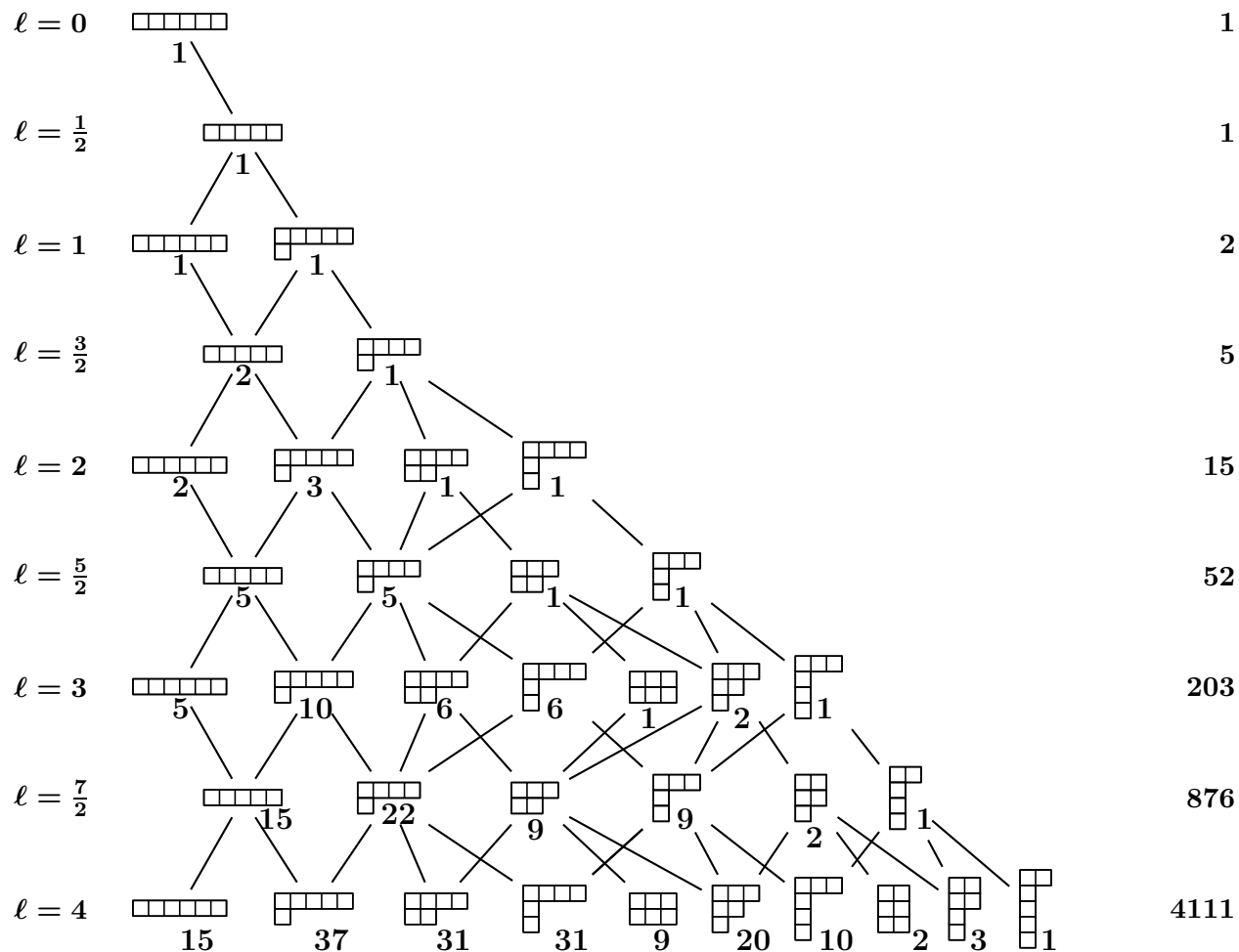
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## A Appendix: Bratteli Diagrams

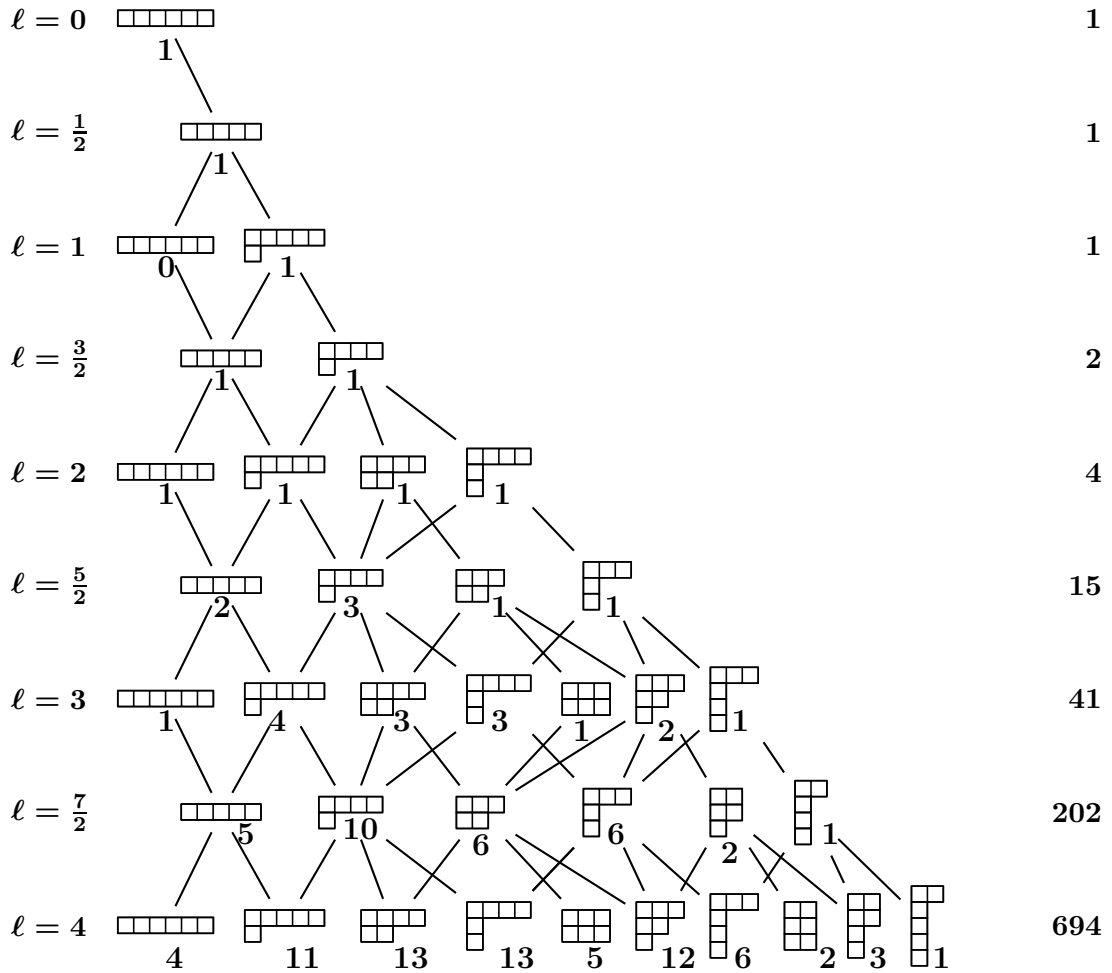
### A.1 Levels $\ell = 0, \frac{1}{2}, 1, \dots, \frac{7}{2}, 4$ of the Bratteli diagram $\mathcal{B}(S_6, S_5)$

Level  $\ell = \frac{7}{2}$  is the first time the centralizer algebra loses a dimension from the generic dimension, which is the 7th Bell number  $B(7) = 877$ .

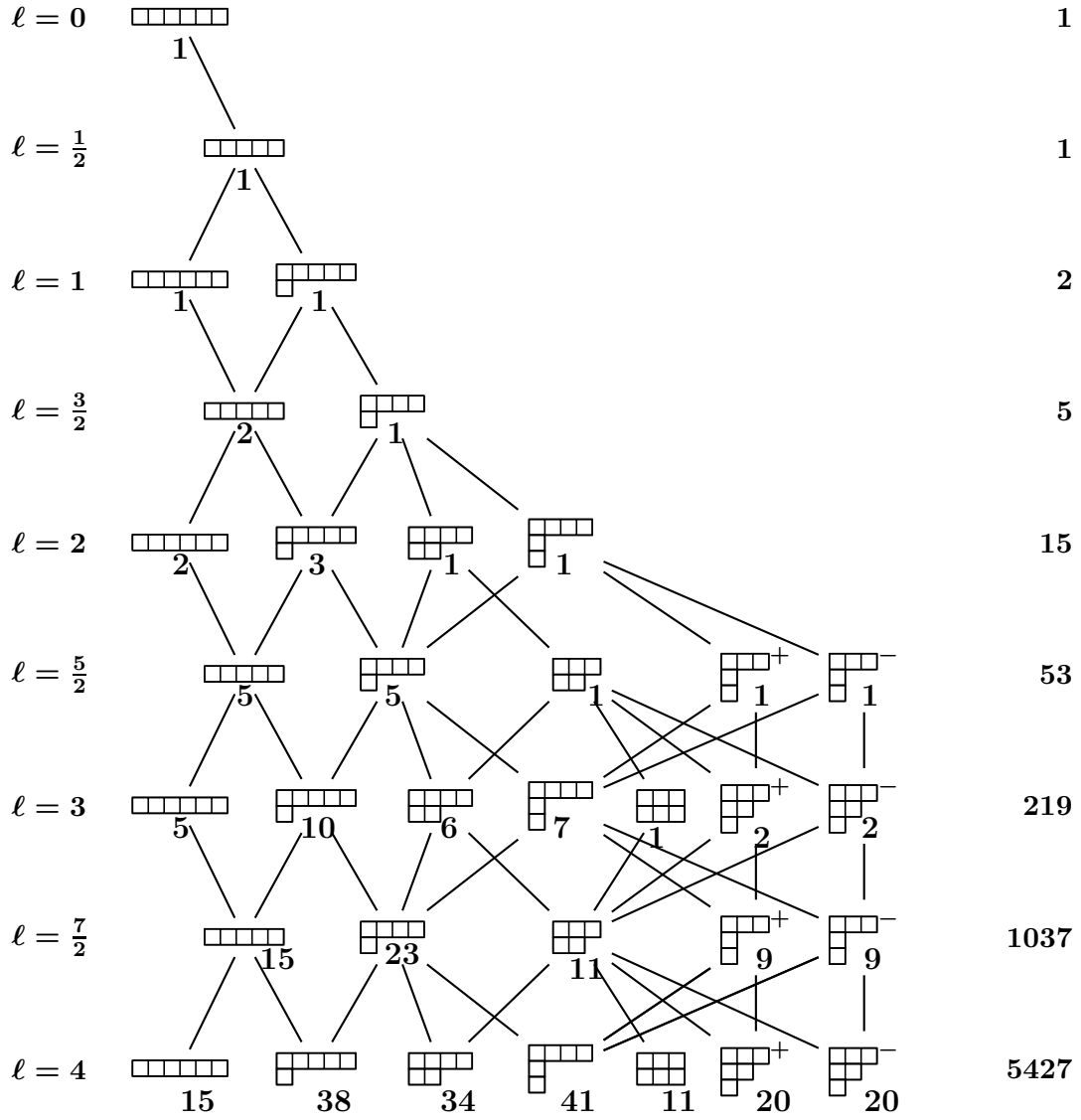


## A.2 Levels $\ell = 0, \frac{1}{2}, 1, \dots, \frac{7}{2}, 4$ of the quasi-Bratteli diagram $\mathcal{QB}(S_6, S_5)$

To calculate the subscripts on the half-integer rows, use Pascal addition of the subscripts from the row above. To calculate the subscripts on integer level rows, first use Pascal addition from the row above, and then subtract the subscript on the same partition from two rows above.



**A.3 Levels  $\ell = 0, \frac{1}{2}, 1, \dots, \frac{7}{2}, 4$  of the Bratteli diagram  $\mathcal{B}(A_6, A_5)$**



**A.4 Levels  $\ell = 0, \frac{1}{2}, 1, \dots, \frac{7}{2}, 4$  of the quasi-Bratteli diagram  $\mathcal{QB}(A_6, A_5)$**

To calculate the subscripts on the half-integer rows, use Pascal addition of the subscripts from the row above. To calculate the subscripts on integer level rows, first use Pascal addition from the row above, and then subtract the subscript on the same partition from two rows above.

