TOWARDS THE CLASSIFICATION OF FINITE SIMPLE GROUPS WITH EXACTLY THREE OR FOUR SUPERCHARACTER THEORIES

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ABSTRACT. A supercharacter theory for a finite group G is a set of superclasses each of which is a union of conjugacy classes together with a set of sums of irreducible characters called supercharacters that together satisfy certain compatibility conditions. The aim of this paper is to give a description of some finite simple groups with exactly three or four supercharacter theories.

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1. Introduction

Throughout this paper G is a finite group, $d(n)$ denotes the number of divisors of positive integer n, $Irr(G)$ stands for the set of all ordinary irreducible character of G and $Con(G)$ is the set of all conjugacy classes of G. For other notations and terminology concerning character theory, we refer to the famous book of Isaacs [11]. Following Diaconis and Isaacs [7], a pair $(\mathcal{X}, \mathcal{K})$ together with the choices of characters χ_X is called a **supercharacter theory** of G if the following conditions are satisfied:

(1) $\mathcal{X}$ is a partition of $Irr(G)$ and $\mathcal{K}$ is a partition of $Con(G)$;
(2) $\{1\} \in \mathcal{K}$;
(3) the characters χ_X, $X \in \mathcal{X}$, are constant on the members of $\mathcal{K}$;
(4) $|\mathcal{X}| = |\mathcal{K}|$.

The elements of $\mathcal{X}$ and $\mathcal{K}$ are called **supercharacters** and **superclasses** of G, respectively. It is easy to see that $m(G) = (Irr(G), Con(G))$ and $M(G) = (\mathcal{X}, \mathcal{K})$, where $\mathcal{X} = \{\{1\}, Irr(G) \setminus \{1\}\}$ and $\mathcal{K} = \{\{1\}, Con(G) \setminus \{1\}\}$ are supercharacter theories of G which are called the trivial supercharacter theories of G. The set of all supercharacter theories of a finite group G is denoted by $Sup(G)$ and set $s(G) = |Sup(G)|$. These notions were first introduced by Andre for finite unitriangular groups using polynomial equations defining certain algebraic varieties [1, 2, 3].

Following Hendrickson [9], we assume that $Part(S)$ denotes the set of all partitions of a set S. If $\mathcal{X}$ and $\mathcal{Y}$ are two elements of $Part(S)$ then we say that $\mathcal{X}$ is a **refinement** of $\mathcal{Y}$ or $\mathcal{Y}$ is **coarser than** $\mathcal{X}$ and we write "$\mathcal{X} \preceq \mathcal{Y}$", if $[a]_{\mathcal{X}} \subseteq [a]_{\mathcal{Y}}$ for all $a \in S$. For two supercharacter theories $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$, we define $(\mathcal{X}, \mathcal{K}) \vee (\mathcal{Y}, \mathcal{L}) = (\mathcal{X} \vee \mathcal{Y}, \mathcal{K} \vee \mathcal{L})$. It is well-know that $(Part(S), \preceq)$ is a lattice which is called the **partition lattice** of S. By [10, Proposition 2.16], the join of two supercharacter theories of a group G is again a supercharacter theory for G, but it is possible to find a pair of supercharacter theories such that their meet is not a supercharacter theory. This shows that the set of all supercharacter theories of a finite group with usual join and meet don't constitute a lattice in general.

Burkett et al. [5] gave a classification of finite groups with exactly two supercharacter theories. They proved that a finite group G has exactly two supercharacter theories if and only if G is isomorphic to the cyclic group Z_3, the symmetric group S_3 or the simple group $Sp(6,2)$. The aim of this paper is to continue this work towards a classification of finite simple groups with exactly three or four supercharacter theories.

If p and $2p+1$ are primes then p is called a **Sophie Germain prime** and $2p+1$ is said to be a **safe prime**. The safe primes are recorded in on-line encyclopedia of integer sequences as A005385, see [16] for details. The first few members of this sequence is 5, 7, 11, 23, 47, 59, 83, 107, 167, 179, 227, 263, 347, 359, 383, 467, 479, 503, 563, 587, 719, 839, 863, 887, 983, 1019, 1187, 1283, 1307, 1319, 1367, 1439, 1487, 1523, 1619, 1823, 1907. These two sequences of prime numbers have several applications in public key cryptography and primality testing and it has been conjectured that there are infinitely many Sophie Germain primes, but this remains unproven [6].

2. Supercharacter Theory Construction for Sporadic Groups

The aim of this section is to compute some supercharacter theories for sporadic simple groups. We use the following simple lemma in our calculations:

Lemma 2.1. *Let G be a finite group, χ be a non-real valued irreducible character of G and $x \in G$ such that $\chi(x)$ is a non-real number. We also assume that each row and column of G has at most two non-real numbers. Define:*

$$\begin{aligned} \mathcal{X} &= \{\{\chi_1\}, \{\chi, \overline{\chi}\}, Irr(G) - \{\chi_1, \chi, \overline{\chi}\}\} \\ \mathcal{K} &= \{\{e\}, \{x^G, (x^{-1})^G\}, Con(G) - \{e, x^G, (x^{-1})^G\}\}. \end{aligned}$$

Then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory for G.

Proof. To prove, it is enough to investigate the main condition of supercharacter theory for the conjugacy classes x^G, $(x^{-1})^G$ and irreducible characters χ, $\overline{\chi}$. By definition of σ_X on $X = \{x^G, (x^{-1})^G\}$,

$$\begin{aligned} \sigma_X(x^{-1}) &= \chi(1)\chi(x^{-1}) + \overline{\chi}(1)\overline{\chi}(x^{-1}) \\ &= \chi(1)\overline{\chi(x)} + \chi(1)\chi(x) \\ &= \overline{\chi(1)}\ \overline{\chi(x)} + \chi(1)\chi(x) \\ &= \sigma_X(x). \end{aligned}$$

Therefore, σ_X is constant on the part $\{x^G, (x^{-1})^G\}$ of $\mathcal{K}$, as desired. □

The following lemma is important for constructing supercharacter theories on simple groups.

Lemma 2.2. *Suppose G is a finite group, $A = \{\chi(x) \mid \chi \in Irr(G)\}$ and $\mathbb{Q}(A)$ denotes the filed generated by $\mathbb{Q}$ and A. Then the following holds:*

1. *If $\mathcal{X}(G) = \{\{\chi, \overline{\chi}\} \mid \chi \in Irr(G)\}$ and $\mathcal{K}(G) = \{\{x^G, (x^{-1})^G\} \mid x \in G\}$ then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of G.*
2. *If $\Gamma = Gal(\frac{\mathbb{Q}(\mathbb{A})}{\mathbb{Q}})$, $\mathcal{X}(G)$ is the set of all orbits of Γ on $Irr(G)$ and $\mathcal{K}(G)$ is the set of all orbits of Γ on $Con(G)$ then $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of G.*

Proof. The proof follows from [7, p. 2360] and [15]. □

To calculate the supercharacter of a finite group G, we first sort the character table of G by the following GAP commands [19]:

```
u:=CharacterTable(G);
t:=CharacterTableWithSortedCharacters(u);
```

Then we prepare a GAP program to check whether or not a given pair $(\mathcal{X}, \mathcal{K})$ of a partition $\mathcal{K}$ for conjugacy classes and another partition $\mathcal{X}$ for irreducible characters constitutes a supercharacter theory. To find this pair of partitions, we usually apply Lemma 2.1.

Theorem 2.3. *The Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}$ and M_{24} have at least five supercharacter theories.*

Proof. We will present five supercharacter theories for each Mathieu group as follows:

- *The Mathieu group M_{11}.* Suppose the irreducible characters and conjugacy classes of the Mathieu group M_{11} are $Irr(M_{11}) = \{\chi_1, \chi_2, \ldots, \chi_{10}\}$ and $Con(M_{11}) = \{x_1^{M_{11}}, x_2^{M_{11}}, \ldots, x_{10}^{M_{11}}\}$, respectively. We now define:

$$\begin{aligned}
\mathcal{K}_1 &= \left\{\{1\}, \{x_i^{M_{11}}\}(2 \le i \le 8), \{x_9^{M_{11}}, x_{10}^{M_{11}}\}\right\}, \\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \le i \le 5), \{\chi_6, \chi_7\}, \{\chi_i\}(8 \le i \le 10)\}, \\
\mathcal{K}_2 &= \left\{\{1\}, \{x_i^{M_{11}}\}(2 \le i \le 6), \{x_7^{M_{11}}, x_8^{M_{11}}\}, \{x_i^{M_{11}}\}(9 \le i \le 10)\right\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_2\}, \{\chi_3, \chi_4\}, \{\chi_i\}(5 \le i \le 10)\}.
\end{aligned}$$

  By Lemma 2.1 and [10, Proposition 2.16], $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of M_{11}. Since $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, $m(M_{11})$ and $M(M_{11})$ are distinct, $s(M_{11}) \ge 5$.
- *The Mathieu group M_{12}.* We assume that the irreducible characters and conjugacy classes of the Mathieu group M_{12} are $Irr(M_{12}) = \{\chi_1, \chi_2, \ldots, \chi_{15}\}$ and $Con(M_{12}) = \{x_1^{M_{12}}, \ldots, x_{15}^{M_{12}}\}$, respectively. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \left\{\{1\}, \{x_i^{M_{12}}\}(2 \le i \le 13), \{x_{14}^{M_{12}}, x_{15}^{M_{12}}\}\right\}, \\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \le i \le 3), \{\chi_4, \chi_5\}, \{\chi_i\}(6 \le i \le 15)\}, \\
\mathcal{K}_2 &= \left\{\{1\}, \{x_i^{M_{12}}\}(2 \le i \le 5), \{x_6^{M_{12}}, x_7^{M_{12}}\}, \{x_i^{M_{12}}\}(8 \le i \le 10), \{x_{11}^{M_{12}}, x_{12}^{M_{12}}\}, \{x_i^{M_{12}}\}(13 \le i \le 15)\right\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_2, \chi_3\}, \{\chi_i\}(4 \le i \le 8), \{\chi_9, \chi_{10}\}, \{\chi_i\}(11 \le i \le 15)\}.
\end{aligned}$$

  Then by Lemma 2.1 and [10, Proposition 2.16], the pairs $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are three supercharacter theories of M_{12} different from $m(M_{12})$ and $M(M_{12})$. This proves that $s(M_{12}) \ge 5$, as desired.
- *The Mathieu group M_{22}.* Suppose $Irr(M_{22}) = \{\chi_1, \ldots, \chi_{12}\}$ and $Con(M_{22}) = \{x_1^{M_{22}}, \ldots, x_{12}^{M_{22}}\}$. We define:

$$\begin{aligned}
\mathcal{K}_1 &= \left\{\{1\}, \{x_i^{M_{22}}\}(2 \le i \le 10), \{x_{11}^{M_{22}}, x_{12}^{M_{22}}\}\right\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \le i \le 9), \{\chi_{10}, \chi_{11}\}, \{\chi_{12}\}\}, \\
\mathcal{K}_2 &= \left\{\{1\}, \{x_i^{M_{22}}\}(2 \le i \le 7), \{x_8^{M_{22}}, x_9^{M_{22}}\}, \{x_i^{M_{22}}\}(10 \le i \le 12)\right\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_2\}, \{\chi_3, \chi_4\}, \{\chi_i\}(5 \le i \le 12)\}.
\end{aligned}$$

Then by Lemma 2.1 the pairs $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ are supercharacter theories of M_{22}. We now apply [10, Proposition 2.16] to prove that $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ is another supercharacter theory for M_{22} which shows that $s(M_{22}) \geq 5$.

- *The Mathieu group* M_{23}. Suppose $Irr(M_{23}) = \{\chi_1, \ldots, \chi_{17}\}$ and $Con(M_{23}) = \{x_1^{M_{23}}, \ldots, x_{17}^{M_{23}}\}$ are the irreducible characters and conjugacy classes of the Mathieu group M_{23}, respectively. If we can present three supercharacter theories for M_{23} different from $m(M_{23})$ and $M(M_{23})$ then it can be easily proved that $s(M_{23}) \geq 5$, as desired. Define:

$$
\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{M_{23}}\}(2 \leq i \leq 15), \{x_{16}^{M_{23}}, x_{17}^{M_{23}}\}\Big\}, \\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 9), \{\chi_{10}, \chi_{11}\}, \{\chi_i\}(12 \leq i \leq 17)\}, \\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{M_{23}}\}(2 \leq i \leq 9), \{x_{10}^{M_{23}}, x_{11}^{M_{23}}\}, \{x_i^{M_{23}}\}(12 \leq i \leq 17)\Big\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 11), \{\chi_{12}, \chi_{13}\}, \{\chi_i\}(14 \leq i \leq 17)\}.
\end{aligned}
$$

  To complete the proof, it is enough to apply Lemma 2.1 and [10, Proposition 2.16] for proving that $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of M_{23}.

- *The Mathieu group* M_{24}. We now assume that $Irr(M_{24}) = \{\chi_1, \chi_2, \ldots, \chi_{26}\}$ and $Con(M_{24}) = \{x_1^{M_{24}}, x_2^{M_{24}}, \ldots, x_{26}^{M_{24}}\}$. If we define:

$$
\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{M_{24}}\}(2 \leq i \leq 20), \{x_{21}^{M_{24}}, x_{22}^{M_{24}}\}, \{x_i^{M_{24}}\}(23 \leq i \leq 26)\Big\}, \\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 4), \{\chi_5, \chi_6\}, \{\chi_i\}(7 \leq i \leq 26)\}, \\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{M_{24}}\}(2 \leq i \leq 24), \{x_{25}^{M_{24}}, x_{26}^{M_{24}}\}\Big\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 9), \{\chi_{10}, \chi_{11}\}, \{\chi_i\}(12 \leq i \leq 26)\}.
\end{aligned}
$$

  then by a similar calculation as other cases, we can prove that $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are three supercharacter theories different from $m(M_{24})$ and $M(M_{24})$, proving this case.

This completes the proof. □

Theorem 2.4. *The Leech lattice groups have at least five supercharacter theories.*

Proof. There are seven Leech lattice simple groups. These are HS, J_2, Co_1, Co_2, Co_3, McL and Suz. Our main proof will consider seven cases as follows:

- *The Higman-Sims group HS.* To establish five supercharacter theories for HS, we assume that $Irr(HS) = \{\chi_i\}_{1 \leq i \leq 24}$ and $Con(HS) = \{x_i^{HS}\}_{1 \leq i \leq 24}$. Define:

$$
\begin{aligned}
\mathcal{K}_1 &= \{\{1\}, \{x_i^{HS}\}(2 \leq i \leq 22), \{x_{23}^{HS}, x_{24}^{HS}\}\}, \\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 10), \{\chi_{11}, \chi_{12}\}, \{\chi_i\}(13 \leq i \leq 24)\}, \\
\mathcal{K}_2 &= \{\{1\}, \{x_i^{HS}\}(2 \leq i \leq 18), \{x_{19}^{HS}, x_{20}^{HS}\}, \{x_i^{HS}\}(21 \leq i \leq 24)\}, \\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 13), \{\chi_{14}, \chi_{15}\}, \{\chi_i\}(16 \leq i \leq 24)\}.
\end{aligned}
$$

  Then by Lemma 2.1, one can see that $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ are supercharacter theories of HS. We now apply [10, Proposition 2.16], to prove that $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ is another supercharacter theory of HS. These supercharacter theories are different from $m(HS)$ and $M(HS)$ which concludes that $s(HS) \geq 5$.

- *The Conway group Co_1.* Suppose $Irr(Co_1) = \{\chi_i\}_{1\leq i\leq 101}$ and $Con(Co_1) = \{x_i^{Co_1}\}_{1\leq i\leq 101}$. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Co_1}\}(2 \leq i \leq 96), \{x_{97}^{Co_1}, x_{98}^{Co_1}\}, \{x_i^{Co_1}\}(99 \leq i \leq 101)\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 26), \{\chi_{27}, \chi_{28}\}, \{\chi_i\}(29 \leq i \leq 101)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Co_1}\}(2 \leq i \leq 77), \{x_{78}^{Co_1}, x_{79}^{Co_1}\}, \{x_i^{Co_1}\}(80 \leq i \leq 101)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 16), \{\chi_{17}, \chi_{18}\}, \{\chi_i\}(19 \leq i \leq 101)\}.
\end{aligned}$$

By Lemma 2.1, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ are supercharacter theories of Co_1 and by [10, Proposition 2.16], $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$. Since these supercharacter theories are different from $m(Co_1)$ and $M(Co_1)$, $s(Co_1) \geq 5$. Hence the result follows.

- *The second Conway group Co_2.* Let $Irr(Co_2) = \{\chi_i\}_{1\leq i\leq 60}$ and $Con(Co_2) = \{x_i^{Co_2}\}_{1\leq i\leq 60}$. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Co_2}\}(2 \leq i \leq 45), \{x_{46}^{Co_2}, x_{47}^{Co_2}\}, \{x_i^{Co_2}\}(48 \leq i \leq 58), \{x_{59}^{Co_2}, x_{60}^{Co_2}\}\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 11), \{\chi_{12}, \chi_{13}\}, \{\chi_i\}(14 \leq i \leq 30), \{\chi_{31}, \chi_{32}\}, \{\chi_i\}(33 \leq i \leq 60)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Co_2}\}(2 \leq i \leq 52), \{x_{53}^{Co_2}, x_{54}^{Co_2}\}, \{x_i^{Co_2}\}(55 \leq i \leq 60)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 9), \{\chi_{10}, \chi_{11}\}, \{\chi_i\}(12 \leq i \leq 60)\}.
\end{aligned}$$

By Lemma 2.1 and [10, Proposition 2.16], $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of the Conway group Co_2. Since these supercharacter theories are different from $m(Co_2)$ and $M(Co_2)$, $s(Co_2) \geq 5$.

- *The third Conway group Co_3.* We assume that $Irr(Co_3) = \{\chi_i\}_{1\leq i\leq 42}$ and $Con(Co_3) = \{x_i^{Co_3}\}_{1\leq i\leq 42}$. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Co_3}\}(2 \leq i \leq 23), \{x_{24}^{Co_3}, x_{25}^{Co_3}\}, \{x_i^{Co_3}\}(26 \leq i \leq 35), \{x_{36}^{Co_3}, x_{37}^{Co_3}\}, \{x_i^{Co_3}\}(38 \leq i \leq 42)\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 5), \{\chi_6, \chi_7\}, \{\chi_i\}(8 \leq i \leq 17), \{\chi_{18}, \chi_{19}\}, \{\chi_i\}(20 \leq i \leq 42)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Co_3}\}(2 \leq i \leq 37), \{x_{38}^{Co_3}, x_{39}^{Co_3}\}, \{x_i^{Co_3}\}(40 \leq i \leq 42)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 15), \{\chi_{16}, \chi_{17}\}, \{\chi_i\}(18 \leq i \leq 42)\}.
\end{aligned}$$

Then $m(Co_3)$, $M(Co_3)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are five supercharacter theories for Co_3. Thus $s(Co_3) \geq 5$, as required.

- *The McLaughlin group McL.* Suppose $Irr(McL) = \{\chi_i\}_{1\leq i\leq 24}$ and $Con(McL) = \{x_i^{McL}\}_{1\leq i\leq 24}$. We now define:

$$\begin{aligned}
\mathcal{K}_1 &= \{\{1\}, \{x_i^{McL}\}(2 \leq i \leq 15), \{x_{16}^{McL}, x_{17}^{McL}\}, \{x_i^{McL}\}(18 \leq i \leq 24)\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 6), \{\chi_7, \chi_8\}, \{\chi_i\}(9 \leq i \leq 24)\},\\
\mathcal{K}_2 &= \{\{1\}, \{x_i^{McL}\}(2 \leq i \leq 12), \{x_{13}^{McL}, x_{14}^{McL}\}, \{x_i^{McL}\}(15 \leq i \leq 24)\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 20), \{\chi_{21}, \chi_{22}\}, \{\chi_i\}(23 \leq i \leq 24)\}.
\end{aligned}$$

Since $m(McL)$, $M(McL)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are five supercharacter theories of McL, $s(McL) \geq 5$.

- *The Suzuki group Suz.* The irreducible characters and conjugacy classes for the Suzuki group Suz are $Irr(Suz) = \{\chi_i\}_{1\leq i\leq 43}$ and $Con(Suz) = \{x_i^{Suz}\}_{1\leq i\leq 43}$, respectively. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \big\{\{1\}, \{x_i^{Suz}\}(2\leq i\leq 40), \{x_{41}^{Suz}, x_{42}^{Suz}\}, \{x_{43}^{Suz}\}\big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 24), \{\chi_{25}, \chi_{26}\}, \{\chi_i\}(27\leq i\leq 43)\},\\
\mathcal{K}_2 &= \big\{\{1\}, \{x_i^{Suz}\}(2\leq i\leq 34), \{x_{35}^{Suz}, x_{36}^{Suz}\}, \{x_i^{Suz}\}(37\leq i\leq 43)\big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 12), \{\chi_{13}, \chi_{14}\}, \{\chi_i\}(15\leq i\leq 43)\}.
\end{aligned}$$

  Since this group has five supercharacter theories $m(Suz)$, $M(Suz)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$, we conclude that $s(Suz) \geq 5$.

This completes the proof. □

Theorem 2.5. *The Monster sections have at least five supercharacter theories.*

Proof. The Monster sections are eight simple groups He, HN, Th, Fi_{22}, Fi_{23}, Fi_{24}', B and M. We will present five supercharacter theories in each case as follows:

- *The Held group He.* Suppose $Irr(He) = \{\chi_i\}_{1\leq i\leq 33}$ and $Con(He) = \{x_i^{He}\}_{1\leq i\leq 33}$ are irreducible characters and conjugacy classes of the group He, respectively. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \big\{\{1\}, \{x_i^{He}\}(2\leq i\leq 27), \{x_{28}^{He}, x_{29}^{He}\}, \{x_i^{He}\}(30\leq i\leq 33)\big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 29), \{\chi_{30}, \chi_{31}\}, \{\chi_i\}(32\leq i\leq 33)\},\\
\mathcal{K}_2 &= \big\{\{1\}, \{x_i^{He}\}(2\leq i\leq 25), \{x_{26}^{He}, x_{27}^{He}\}, \{x_i^{He}\}(28\leq i\leq 33)\big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 6), \{\chi_7, \chi_8\}, \{\chi_i\}(9\leq i\leq 33)\}.
\end{aligned}$$

  Since $m(He)$, $M(He)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are five supercharacter theories for He, $s(He) \geq 5$.
- *The Harada-Norton group HN.* This group has exactly 54 conjugacy classes and irreducible characters. Suppose $Irr(HN) = \{\chi_i\}_{1\leq i\leq 54}$ and $Con(HN) = \{x_i^{HN}\}_{1\leq i\leq 54}$. We also define:

$$\begin{aligned}
\mathcal{K}_1 &= \big\{\{1\}, \{x_i^{HN}\}(2\leq i\leq 38), \{x_{39}^{HN}, x_{40}^{HN}\}, \{x_i^{HN}\}(41\leq i\leq 54)\big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 50), \{\chi_{51}, \chi_{52}\}, \{\chi_i\}(53\leq i\leq 54)\},\\
\mathcal{K}_2 &= \big\{\{1\}, \{x_i^{HN}\}(2\leq i\leq 52), \{x_{53}^{HN}, x_{54}^{HN}\}\big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 34), \{\chi_{35}, \chi_{36}\}, \{\chi_i\}(37\leq i\leq 54)\}.
\end{aligned}$$

  By Lemma 2.1 and [10, Proposition 2.16], $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of HN and apart from supercharacter theories $m(HN)$ and $M(HN)$, it concludes that $m(HN) \geq 5$.
- *The Thompson group Th.* The Thompson group Th has exactly 48 conjugacy classes and irreducible characters. We assume that $Irr(Th) = \{\chi_i\}_{1\leq i\leq 48}$ and $Con(Th) = \{x_i^{Th}\}_{1\leq i\leq 48}$ and define:

$$\begin{aligned}
\mathcal{K}_1 &= \big\{\{1\}, \{x_i^{Th}\}(2\leq i\leq 24), \{x_{25}^{Th}, x_{26}^{Th}\}, \{x_i^{Th}\}(27\leq i\leq 39), \{x_{40}^{Th}, x_{41}^{Th}\}, \{x_i^{Th}\}(42\leq i\leq 48)\big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 8), \{\chi_9, \chi_{10}\}, \{\chi_i\}(11\leq i\leq 34), \{\chi_{35}, \chi_{36}\}, \{\chi_i\}(37\leq i\leq 48)\},\\
\mathcal{K}_2 &= \big\{\{1\}, \{x_i^{Th}\}(2\leq i\leq 36), \{x_{37}^{Th}, x_{38}^{Th}\}, \{x_i^{Th}\}(39\leq i\leq 48)\big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 21), \{\chi_{22}, \chi_{23}\}, \{\chi_i\}(24\leq i\leq 48)\}.
\end{aligned}$$

Again apply Lemma 2.1 and [10, Proposition 2.16] to deduce that $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and finally $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories for Th. Thus $s(Th) \geq 5$.

- *The Fischer group* Fi_{22}. The Fischer group Fi_{22} has exactly 65 conjugacy classes and irreducible characters. Set $Irr(Fi_{22}) = \{\chi_i\}_{1\leq i\leq 65}$ and $Con(Fi_{22}) = \{x_i^{Fi_{22}}\}_{1\leq i\leq 65}$. We also define:

$$
\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Fi_{22}}\}(2 \leq i \leq 35), \{x_{36}^{Fi_{22}}, x_{37}^{Fi_{22}}\}, \{x_i^{Fi_{22}}\}(38 \leq i \leq 60), \{x_{61}^{Fi_{22}}, x_{62}^{Fi_{22}}\}, \{x_i^{Fi_{22}}\}(63 \leq i \leq 65)\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 39), \{\chi_{40}, \chi_{41}\}, \{\chi_i\}(42 \leq i \leq 50), \{\chi_{51}, \chi_{52}\}, \{\chi_i\}(53 \leq i \leq 65)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Fi_{22}}\}(2 \leq i \leq 54), \{x_{55}^{Fi_{22}}, x_{56}^{Fi_{22}}\}, \{x_i^{Fi_{22}}\}(57 \leq i \leq 65)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 42), \{\chi_{43}, \chi_{44}\}, \{\chi_i\}(45 \leq i \leq 65)\}.
\end{aligned}
$$

Since $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and finally $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$, are three supercharacter theories of Fi_{22} different from $m(Fi_{22})$ and $M(Fi_{22})$, $s(Fi_{22}) \geq 5$.

- *The Fischer group* Fi_{23}. This group has exactly 98 conjugacy classes and irreducible characters. Set $Irr(Fi_{23}) = \{\chi_i\}_{1\leq i\leq 98}$ and $Con(Fi_{23}) = \{x_i^{Fi_{23}}\}_{1\leq i\leq 98}$. Define:

$$
\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Fi_{23}}\}(2 \leq i \leq 79), \{x_{80}^{Fi_{23}}, x_{81}^{Fi_{23}}\}, \{x_i^{Fi_{23}}\}(82 \leq i \leq 98)\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 16), \{\chi_{17}, \chi_{18}\}, \{\chi_i\}(19 \leq i \leq 98)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Fi_{23}}\}(2 \leq i \leq 62), \{x_{63}^{Fi_{23}}, x_{64}^{Fi_{23}}\}, \{x_i^{Fi_{23}}\}(65 \leq i \leq 79), \{x_{80}^{Fi_{23}}, x_{81}^{Fi_{23}}\}, \{x_i^{Fi_{23}}\}(82 \leq i \leq 98)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 14), \{\chi_{15}, \chi_{16}\}, \{\chi_{17}, \chi_{18}\}, \{\chi_i\}(19 \leq i \leq 98)\},\\
\mathcal{C}_1 &= (\mathcal{X}_1, \mathcal{K}_1), \quad \mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2), \quad \mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2.
\end{aligned}
$$

Since $\{m(Fi_{23}), M(Fi_{23}), \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\} \subseteq Sup(Fi_{23})$, $s(Fi_{23}) \geq 5$.

- *The Fischer group* Fi_{24}'. The largest Fischer group has exactly 108 conjugacy classes and irreducible characters. Set $Irr(Fi_{24}') = \{\chi_i\}_{1\leq i\leq 108}$ and $Con(Fi_{24}') = \{x_i^{Fi_{24}'}\}_{1\leq i\leq 108}$. Define:

$$
\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Fi_{24}'}\}(2 \leq i \leq 105), \{x_{106}^{Fi_{24}'}, x_{107}^{Fi_{24}'}\}, \{x_{108}^{Fi_{24}'}\}\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 98), \{\chi_{99}, \chi_{100}\}, \{\chi_i\}(101 \leq i \leq 108)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Fi_{24}'}\}(2 \leq i \leq 80), \{x_{81}^{Fi_{24}'}, x_{82}^{Fi_{24}'}\}, \{x_i^{Fi_{24}'}\}(83 \leq i \leq 108)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 100), \{\chi_{101}, \chi_{102}\}, \{\chi_i\}(103 \leq i \leq 108)\}.
\end{aligned}
$$

Since $m(Fi_{24}')$, $M(Fi_{24}')$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of Fi_{24}, $s(Fi_{24}) \geq 5$.

- *The Baby Monster group* B. This group has exactly 184 conjugacy classes and irreducible characters. Suppose $Irr(B) = \{\chi_i\}_{1\leq i\leq 184}$, $Con(B) = \{x_i^B\}_{1\leq i\leq 184}$ and define:

$$
\begin{aligned}
\mathcal{K}_1 &= \{\{1\}, \{x_i^B\}(2 \leq i \leq 177), \{x_{178}^B, x_{179}^B\}, \{x_i^B\}(180 \leq i \leq 184)\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 177), \{\chi_{178}, \chi_{179}\}, \{\chi_i\}(180 \leq i \leq 184)\},\\
\mathcal{K}_2 &= \{\{1\}, \{x_i^B\}(2 \leq i \leq 171), \{x_{172}^B, x_{173}^B\}, \{x_i^B\}(174 \leq i \leq 184)\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 20), \{\chi_{21}, \chi_{22}\}, \{\chi_i\}(23 \leq i \leq 184)\}.
\end{aligned}
$$

Since $m(B)$, $M(B)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of B, $s(B) \geq 5$, as desired.

- *The Monster group M.* The largest sporadic group M has exactly 198 conjugacy classes and irreducible characters. Set $Irr(M) = \{\chi_i\}_{1\leq i\leq 194}$ and $Con(M) = \{x_i^M\}_{1\leq i\leq 194}$.

$$\begin{aligned}
\mathcal{K}_1 &= \{\{1\}, \{x_i^M\}(2 \leq i \leq 192), \{x_{193}^M, x_{194}^M\}\}\,,\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 46), \{\chi_{47}, \chi_{48}\}, \{\chi_i\}(49 \leq i \leq 194)\}\,,\\
\mathcal{K}_2 &= \{\{1\}, \{x_i^M\}(2 \leq i \leq 188), \{x_{189}^M, x_{190}^M\}, \{x_i^M\}(191 \leq i \leq 194)\}\,,\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 123), \{\chi_{124}, \chi_{125}\}, \{\chi_i\}(126 \leq i \leq 194)\}\,.
\end{aligned}$$

  We can see that the group M has at least 5 supercharacter theories as $m(M)$, $M(M)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$. This shows that $s(M) \geq 5$.

This completes the proof. □

Theorem 2.6. *The Pariahs have at least five supercharacter theories.*

Proof. The Pariahs are six sporadic groups J_1, $O'N$, J_3, Ru, J_4 and Ly. Our main proof will consider six separate cases as follows:

- *Janko group J_1.* The first Janko group J_1 has exactly 15 conjugacy classes and irreducible characters. Set $Irr(J_1) = \{\chi_i\}_{1\leq i\leq 15}$ and $Con(J_1) = \{x_i^{J_1}\}_{1\leq i\leq 15}$.

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{J_1}\}(2 \leq i \leq 3), \{x_4^{J_1}, x_5^{J_1}\}, \{x_i^{J_1}\}(6 \leq i \leq 7), \{x_8^{J_1}, x_9^{J_1}\}, \{x_{10}^{J_1}\}, \{x_{11}^{J_1}, x_{12}^{J_1}\}, \{x_i^{J_1}\}(13 \leq i \leq 15)\Big\}\,,\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_2, \chi_3\}, \{\chi_i\}(4 \leq i \leq 6), \{\chi_7, \chi_8\}, \{\chi_i\}(9 \leq i \leq 12), \{\chi_{13}, \chi_{14}\}, \{\chi_{15}\}\}\,,\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{J_1}\}(2 \leq i \leq 12), \{x_{13}^{J_1}, x_{14}^{J_1}, x_{15}^{J_1}\}\Big\}\,,\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 8), \{\chi_9, \chi_{10}, \chi_{11}\}, \{\chi_i\}(12 \leq i \leq 15)\}\,.
\end{aligned}$$

  Since $m(J_1)$, $M(J_1)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of J_1, $s(J_1) \geq 5$.
- *The O'Nan group $O'N$.* The O'Nan group $O'N$ has exactly 30 conjugacy classes and irreducible characters. Set $Irr(O'N) = \{\chi_i\}_{1\leq i\leq 30}$ and $Con(O'N) = \{x_i^{O'N}\}_{1\leq i\leq 30}$.

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{O'N}\}(2 \leq i \leq 21), \{x_{22}^{O'N}, x_{23}^{O'N}, x_{24}^{O'N}\}, \{x_i^{O'N}\}(25 \leq i \leq 30)\Big\}\,,\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 25), \{\chi_{26}, \chi_{27}, \chi_{28}\}, \{\chi_i\}(29 \leq i \leq 30)\}\,,\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{O'N}\}(2 \leq i \leq 26), \{x_{27}^{O'N}, x_{28}^{O'N}\}, \{x_i^{O'N}\}(29 \leq i \leq 30)\Big\}\,,\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 28), \{\chi_{29}, \chi_{30}\}\}\,.
\end{aligned}$$

  Apply Lemma 2.1 and [10, Proposition 2.16] to deduce that $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of $O'N$. Hence $s(O'N) \geq 5$, as desired.
- *The Janko group J_3.* We assume that $Irr(J_3) = \{\chi_i\}_{1\leq i\leq 21}$ and $Con(J_3) = \{x_i^{J_3}\}_{1\leq i\leq 21}$. Define:

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{J_3}\}(2 \leq i \leq 9), \{x_{10}^{J_3}, x_{11}^{J_3}, x_{12}^{J_3}\}, \{x_i^{J_3}\}(13 \leq i \leq 21)\Big\}\,,\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 13), \{\chi_{14}, \chi_{15}, \chi_{16}\}, \{\chi_i\}(17 \leq i \leq 21)\}\,,\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{J_3}\}(2 \leq i \leq 19), \{x_{20}^{J_3}, x_{21}^{J_3}\}\Big\}\,,\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_2, \chi_3\}, \{\chi_i\}(4 \leq i \leq 21)\}\,.
\end{aligned}$$

  Since $m(J_3)$, $M(J_3)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ are supercharacter theories of J_3, $s(J_3) \geq 5$.

- *The Rudvalis group Ru.* Set $Irr(Ru) = \{\chi_i\}_{1\leq i\leq 36}$ and $Con(Ru) = \{x_i^{Ru}\}_{1\leq i\leq 36}$ and define:

$$\begin{aligned}\mathcal{K}_1 &= \{\{1\}, \{x_i^{Ru}\}(2\leq i\leq 34), \{x_{35}^{Ru}, x_{36}^{Ru}\}\},\\ \mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 33), \{\chi_{34}, \chi_{35}\}, \{\chi_{36}\}\},\\ \mathcal{K}_2 &= \{\{1\}, \{x_i^{Ru}\}(2\leq i\leq 31), \{x_{32}^{Ru}, x_{33}^{Ru}, x_{34}^{Ru}\}, \{x_i^{Ru}\}(35\leq i\leq 36)\},\\ \mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 16), \{\chi_{17}, \chi_{18}, \chi_{19}\}, \{\chi_i\}(20\leq i\leq 36)\}.\end{aligned}$$

  Apply again Lemma 2.1 and [10, Proposition 2.16] to deduce that the Rudvalis group Ru has at least five supercharacter theories as $m(Ru)$, $M(Ru)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$. Hence $s(Ru) \geq 5$, as required.
- *The Janko group J_4.* This group has exactly 62 conjugacy classes and irreducible characters. Define $Irr(J_4) = \{\chi_i\}_{1\leq i\leq 62}$, $Con(J_4) = \{x_i^{J_4}\}_{1\leq i\leq 62}$ and define:

$$\begin{aligned}\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{J_4}\}(2\leq i\leq 49), \{x_{50}^{J_4}, x_{51}^{J_4}, x_{52}^{J_4}\}, \{x_i^{J_4}\}(53\leq i\leq 62)\Big\},\\ \mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 52), \{\chi_{53}, \chi_{54}, \chi_{55}\}, \{\chi_i\}(56\leq i\leq 62)\},\\ \mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{J_4}\}(2\leq i\leq 42), \{x_{43}^{J_4}, x_{44}^{J_4}, x_{45}^{J_4}\}, \{x_i^{J_4}\}(46\leq i\leq 49), \{x_{50}^{J_4}, x_{51}^{J_4}, x_{52}^{J_4}\}, \{x_i^{J_4}\}(53\leq i\leq 62)\Big\},\\ \mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 52), \{\chi_{53}, \chi_{54}, \chi_{55}\}, \{\chi_{56}, \chi_{57}, \chi_{58}\}, \{\chi_i\}(59\leq i\leq 62)\}.\end{aligned}$$

  The Janko group J_4 has at least five supercharacter theories as $m(J_4)$, $M(J_4)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$. Therefore, $s(J_4) \geq 5$.
- *The Lyons group Ly.* The Lyons group Ly has exactly 53 irreducible characters and conjugacy classes. Set $Irr(Ly) = \{\chi_i\}_{1\leq i\leq 53}$ and $Con(Ly) = \{x_i^{Ly}\}_{1\leq i\leq 53}$ and define:

$$\begin{aligned}\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{Ly}\}(2\leq i\leq 37), \{x_{38}^{Ly}, x_{39}^{Ly}, x_{40}^{Ly}, x_{41}^{Ly}, x_{42}^{Ly}\}, \{x_i^{Ly}\}(43\leq i\leq 53)\Big\},\\ \mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 38), \{\chi_{39}, \chi_{40}, \chi_{41}, \chi_{42}, \chi_{43}\}, \{\chi_i\}(44\leq i\leq 53)\},\\ \mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{Ly}\}(2\leq i\leq 37), \{x_{38}^{Ly}, x_{39}^{Ly}, x_{40}^{Ly}, x_{41}^{Ly}, x_{42}^{Ly}\}, \{x_i^{Ly}\}(43\leq i\leq 50), \{x_{51}^{Ly}, x_{52}^{Ly}, x_{53}^{Ly}\}\Big\},\\ \mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2\leq i\leq 25), \{\chi_{26}, \chi_{27}, \chi_{28}\}, \{\chi_i\}(29\leq i\leq 38), \{\chi_{39}, \chi_{40}, \chi_{41}, \chi_{42}, \chi_{43}\}, \{\chi_i\}(44\leq i\leq 53)\}.\end{aligned}$$

  Since the group Ly has at least five supercharacter theories as $m(Ly)$, $M(Ly)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$, $s(Ly) \geq 5$.

This completes the proof. □

Lemma 2.7. *The Janko group J_2 has at least three supercharacter theories.*

Proof. A simple investigation shows that the supercharacter theories of the Janko group J_2 are $m(J_2)$, $M(J_2)$, $\mathcal{C} = (\mathcal{X}, \mathcal{K})$ such that

$$\begin{aligned}\mathcal{K} &= \Big\{\{1\}, \{x_i^{J_2}\}(2\leq i\leq 6), \{x_7^{J_2}, x_8^{J_2}\}, \{x_9^{J_2}, x_{10}^{J_2}\}, \{x_i^{J_2}\}(11\leq i\leq 14), \{x_{15}^{J_2}, x_{16}^{J_2}\}, \{x_{17}^{J_2}, x_{18}^{J_2}\}, \{x_{19}^{J_2}\}, \{x_{20}^{J_2}, x_{21}^{J_2}\}\Big\},\\ \mathcal{X} &= \{\{1\}, \{\chi_2, \chi_3\}, \{\chi_4, \chi_5\}, \{\chi_i\}(6\leq i\leq 7), \{\chi_8, \chi_9\}, \{\chi_i\}(10\leq i\leq 13), \{\chi_{14}, \chi_{15}\}, \{\chi_{16}, \chi_{17}\}, \{\chi_i\}(18\leq i\leq 20)\},\end{aligned}$$

proving the lemma. □

We run a GAP program to be sure that $s(J_2) = 3$, but our program after some days stopped, because it needs a huge amount of RAM.

Conjecture 2.8. $s(J_2) = 3$.

3. Supercharacter Theory Construction for Alternating and Suzuki Groups

The intention of this section is to move a step towards a classification of the finite simple groups with exactly three and four supercharacter theories. We start with the cyclic group of order p.

Lemma 3.1. *Suppose p is prime. Then,*

1. $s(Z_p) = 3$ *if and only if* $p = 5$,
2. $s(Z_p) = 4$ *if and only if p is a Sophie Germain prime.*

Proof. It is clear that p is an odd prime. By [9, Theorem 6.32 and Table 1], $s(Z_p) = d(p-1)$.

1. $s(Z_p) = 3$. In this case, $s(Z_p) = d(p-1)$ and so $p \geq 5$. Since $p-1$ is an even integer, the case of $p > 5$ cannot be occurred and so $p = 5$, as desired.
2. $s(Z_p) = 4$. Since $d(p-1) = 4$, $p-1 = q^3$, q is prime, or $p-1 = 2r$, where r is prime. If $p-1 = q^3$ then $q = 2$ and $p = 9$, a contradiction. So, $p-1 = 2r$, where r is prime. This shows that p is a Sophie Germain prime.

This completes the proof. □

The following well-known results are crucial in the classification of alternating simple groups with exactly three or four supercharacter theories.

Theorem 3.2. *The following are hold:*

1. (Berggren [4]) *Every irreducible characters of the alternating group A_n are real valued if and only if* $n \in \{1, 2, 5, 6, 10, 14\}$.
2. (Grove [8, Proposition 8.2.1]) *If K is a conjugacy class in S_n, $K \subset A_n$, and $\sigma \in K$, then K is a conjugacy class in A_n if and only if some odd elements of S_n commutes with σ; if that is not the case, then the conjugacy class K splits as the union of two A_n-classes, each of size $|K|/2$. If λ is the (partition) type of σ then K splits if and only if the parts of λ are all odd and all different from each other.*
3. *Suppose $x \in A_n$ is a product of r pair-wise disjoint cycles including all fixed points as singleton cycles. Then x^{A_n} is non-real if and only if $\sum_{j=1}^{m} \frac{r_j - 1}{2}$ is odd.*

Theorem 3.3. *The simple alternating group A_n has exactly three supercharacter theories if and only if $n = 5$ or 7. There is no simple alternating groups with exactly four supercharacter theories.*

Proof. Suppose $n \geq 5$. It is easy to see that the alternating groups A_5 and A_7 have exactly three supercharacter theories. Our main proof will consider three separate cases as follows:

1. *All character values of A_n are real.* Since $n \geq 5$, Theorem 3.2(1) implies that $n = 5, 6, 10$ or 14. In what follows three non-trivial supercharacter theories for the alternating groups A_6, A_{10} and A_{14} are presented.
   (a) By a GAP program, one can see that the group A_5 has exactly 3 supercharacter theories $m(A_5)$, $M(A_5)$ and $\mathcal{C} = (\mathcal{X}, \mathcal{K})$ such that

$$\begin{aligned}
\mathcal{K} &= \left\{\{1\}, \{x_2^{A_5}\}, \{x_3^{A_5}\}, \{x_4^{A_5}, x_5^{A_5}\}\right\}, \\
\mathcal{X} &= \{\{1\}, \{\chi_2, \chi_3\}, \{\chi_i\}(4 \leq i \leq 5)\}.
\end{aligned}$$

Here, $Irr(A_5) = \{\chi_1, \chi_2, \chi_3, \chi_4, \chi_5\}$ and $Con(A_5) = \{x_1^{A_5}, x_2^{A_5}, x_3^{A_5}, x_4^{A_5}, x_5^{A_5}\}$.

(b) Suppose $Irr(A_6) = \{\chi_1, \chi_2, \ldots, \chi_7\}$ and $Con(A_6) = \{x_1^{A_6}, x_2^{A_6}, \ldots, x_7^{A_6}\}$. Define $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$ and $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ as follows:

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_2^{A_6}\}, \{x_3^{A_6}, x_4^{A_6}\}, \{x_5^{A_6}\}, \{x_6^{A_6}, x_7^{A_6}\}\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_2, \chi_3\}, \{\chi_4, \chi_5\}, \{\chi_6\}, \{\chi_7\}\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_2^{A_6}, x_5^{A_6}\}, \{x_3^{A_6}, x_4^{A_6}\}, \{x_6^{A_6}\}, \{x_7^{A_6}\}\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_2, \chi_3, \chi_7\}, \{\chi_4\}, \{\chi_5\}, \{\chi_6\}\},
\end{aligned}$$

If $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ then $\mathcal{C}_3$ is a supercharacter theory for A_6 different from $\mathcal{C}_1$, $\mathcal{C}_2$, $m(A_6)$ and $M(A_6)$. Therefore, A_6 has at least five supercharacter theories.

(c) Suppose $Irr(A_{10}) = \{\chi_1, \chi_2, \ldots, \chi_{24}\}$, $Con(A_{10}) = \{x_1^{A_{10}}, x_2^{A_{10}}, \ldots, x_{24}^{A_{10}}\}$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$, where

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{A_{10}}\}(2 \le i \le 19), \{x_{20}^{A_{10}}, x_{21}^{A_{10}}\}, \{x_i^{A_{10}}\}(22 \le i \le 24)\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \le i \le 19), \{\chi_{20}, \chi_{21}\}, \{\chi_i\}(22 \le i \le 24)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{A_{10}}\}(2 \le i \le 22), \{x_{23}^{A_{10}}, x_{24}^{A_{10}}\}\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \le i \le 11), \{\chi_{12}, \chi_{13}\}, \{\chi_i\}(14 \le i \le 24)\}.
\end{aligned}$$

Then $\mathcal{C}_i = (\mathcal{X}_i, \mathcal{K}_i)$, $1 \le i \le 3$, are three supercharacter theories of A_{10} different from $m(A_{10})$ and $M(A_{10})$.

(d) We claim that the alternating group A_{14} has at least 5 supercharacter theories. These are $m(A_{14})$, $M(A_{14})$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$ such that

$$\begin{aligned}
\mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{A_{14}}\}(2 \le i \le 70), \{x_{71}^{A_{14}}, x_{72}^{A_{14}}\}\Big\},\\
\mathcal{X}_1 &= \{\{1\}, \{\chi_i\}(2 \le i \le 19), \{\chi_{20}, \chi_{21}\}, \{\chi_i\}(22 \le i \le 72)\},\\
\mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{A_{14}}\}(2 \le i \le 67), \{x_{68}^{A_{14}}, x_{69}^{A_{14}}\}, \{x_i^{A_{14}}\}(70 \le i \le 72)\Big\},\\
\mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \le i \le 56), \{\chi_{57}, \chi_{58}\}, \{\chi_i\}(59 \le i \le 72)\}.
\end{aligned}$$

(2) *The alternating group A_n has exactly one pair of non-real valued irreducible character.* In this case, we will prove $n \in \{7, 9, 11, 15, 18, 19, 23\}$. Since the number of non-real irreducible characters is equal to the number of conjugacy classes that are not preserved by the inversion mapping, we need those n for which A_n has exactly two such conjugacy classes. Suppose $x \in A_n$. By [12, Proposition 12.17(2)], $x^{S_n} = x^{A_n} \cup (12)x(12)^{A_n}$ if and only if $C_{S_n}(x) = C_{A_n}(x)$. By Theorem 3.2(2), the last one is satisfied if and only if $\dfrac{n-r}{2}$ is odd, where r is the number of cycles in decomposition of x. Our aim is to find all natural numbers n such that A_n has exactly one non-real class. Our main proof will consider the following separate cases:

(a) $n \equiv 0\ (mod\ 4)$. If $n \ge 8$ then $[1, n-1]$ and $[3, n-3]$ are two partitions with given properties. So, $n = 4$ which contradicts by our main assumption that $n \ge 5$.

(b) $n \equiv 1\ (mod\ 4)$. If $n \ge 13$ then $[1, 3, n-4]$ and $[1, 5, n-6]$ are two partitions with this property that all parts are odd and $\frac{n-3}{2}$ $(r = 3)$ is odd. Thus $n = 5$ or 9. By Theorem 3.2(1), the alternating group A_5 does not have non-real class and the alternating group A_9 has exactly a unique pair of non-real class.

(c) $n \equiv 2\ (mod\ 4)$. If $n \ge 22$ then $[1, 3, 5, n-9]$ and $[1, 3, 7, n-11]$ are two partitions with given properties and so $n = 6, 10, 14$ or 18. By Theorem 3.2(1), the alternating groups

A_6, A_{10} and A_{14} don't have non-real conjugacy class, but the alternating group A_{18} has a unique pair of conjugate non-real characters.

(d) $n \equiv 3 \ (mod\ 4)$. If $n \geq 31$ then $[1, 3, 5, 7, n-16]$ and $[1, 3, 5, 9, n-18]$ are two partitions with this property that all parts are odd and $\frac{n-5}{2}$ $(r = 5)$ is odd. Thus $n = 7, 11, 15, 19, 23$ or 27. Since [27] and $[1, 5, 7, 9, 11]$ are two partitions with mentioned properties, the case of $n = 27$ cannot be happened. Other cases are solution of our problem.

Hence, the alternating group A_n, $n \geq 5$, has a unique non-real conjugacy class if and only if $n \in \{7, 9, 11, 15, 18, 19, 23\}$. A simple calculations by GAP shows that the alternating group A_7 has exactly three supercharacter theories $m(A_7)$, $M(A_7)$ and $\mathcal{C} = (\mathcal{X}, \mathcal{K})$, where

$$\begin{aligned} \mathcal{K} &= \Big\{\{1\}, \{x_i^{A_7}\}(2 \leq i \leq 7), \{x_8^{A_7}, x_9^{A_7}\}\Big\}, \\ \mathcal{X} &= \{\{1\}, \chi_2, \{\chi_3, \chi_4\}, \{\chi_i\}(5 \leq i \leq 9)\}. \end{aligned}$$

The alternating group A_9 has at least 5 supercharacter theories $m(A_9)$, $M(A_9)$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$. The partitions of conjugacy classes and irreducible characters of $\mathcal{C}_1$ and $\mathcal{C}_2$ are defined as follows:

$$\begin{aligned} \mathcal{K}_1 &= \Big\{\{1\}, \{x_i^{A_9}\}(2 \leq i \leq 12), \{x_{13}^{A_9}, x_{14}^{A_9}\}, \{x_i^{A_9}\}(15 \leq i \leq 18)\Big\}, \\ \mathcal{X}_1 &= \{\{1\}, \{\chi_2\}, \{\chi_3, \chi_4\}, \{\chi_i\}(5 \leq i \leq 18)\}, \\ \mathcal{K}_2 &= \Big\{\{1\}, \{x_i^{A_9}\}(2 \leq i \leq 16), \{x_{17}^{A_9}, x_{18}^{A_9}\}\Big\}, \\ \mathcal{X}_2 &= \{\{1\}, \{\chi_i\}(2 \leq i \leq 6), \{\chi_7, \chi_8\}, \{\chi_i\}(9 \leq i \leq 18)\}. \end{aligned}$$

The alternating groups A_{11}, A_{15}, A_{18}, A_{19} and A_{23} has at least one non-real irreducible character and one non-rational irreducible real character. So, by Lemma 2.1 it has at least five supercharacter theories m, M, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3 = \mathcal{C}_1 \vee \mathcal{C}_2$, proving this case.

(3) *The alternating group A_n has at least two pairs of non-real valued irreducible characters.* We first assume that $n > 24$. If $A = \{\chi(x) \mid \chi \in Irr(G)\}$ and $\mathbb{Q}(A)$ denotes the filed generated by $\mathbb{Q}$ and A then by [14, Theorem], the character table of A_n has both irrational and non-real character values. On the other hand, by [13, Theorem 2.5.13], each row or column of the character table of A_n contains at most one pair of irrational numbers. Now by Lemma 2.1, we have at least five supercharacter theories, as required.

Next we assume that $n \leq 24$. Set $\Gamma_1 = \{5, 6, 7, 9, 10, 11, 14, 15, 18, 19, 23\}$ and $\Gamma_2 = \{8, 12, 13, 16, 17, 20, 21, 22, 24\}$. If $n \in \Gamma_1$ then the number of supercharacter theories of A_n are investigated in Cases (1) and (2). So, we have to prove that $s(A_n) \geq 5$, when $n \in \Gamma_2$. By an easy calculation with GAP, one can see that if $n \in \Gamma_2$ then A_n has at least two pairs of non-real valued irreducible characters and by Lemma 2.1, $s(A_n) \geq 5$.

Hence the result. □

In the end of this paper we prove that the simple Suzuki group $Sz(q), q = 2^{2n+1}$ has at least six super character theories.

Theorem 3.4. *The Suzuki group $Sz(q)$ has at least 6 supercharacter theories as: $m(Sz(q))$, $M(Sz(q))$, $\mathcal{C}_1 = (\mathcal{X}_1, \mathcal{K}_1)$, $\mathcal{C}_2 = (\mathcal{X}_2, \mathcal{K}_2)$, $\mathcal{C}_3 = (\mathcal{X}_3, \mathcal{K}_3)$ and $\mathcal{C}_4 = (\mathcal{X}_4, \mathcal{K}_4)$, where*

$$\begin{aligned}
\mathcal{K}_1 &= \{\{1\}, \{\rho, \rho^{-1}\}, Con(G(q)) - \{\rho, \rho^{-1}\}\},\\
\mathcal{X}_1 &= \{\{1\}, \{W_1, W_2\}, Irr(G(q)) - \{W_1, W_2\}\},\\
\mathcal{K}_2 &= \{\{1\}, \{\pi_0\}, Con(G(q)) - \{\pi_0\}\},\\
\mathcal{X}_2 &= \{\{1\}, \{X_i\}, Irr(G(q)) - \{X_i\}\},\\
\mathcal{K}_3 &= \{\{1\}, \{\pi_1\}, Con(G(q)) - \{\pi_1\}\},\\
\mathcal{X}_3 &= \{\{1\}, \{Y_j\}, Irr(G(q)) - \{Y_j\}\},\\
\mathcal{K}_4 &= \{\{1\}, \{\pi_2\}, Con(G(q)) - \{\pi_2\}\},\\
\mathcal{X}_4 &= \{\{1\}, \{Z_k\}, Irr(G(q)) - \{Z_k\}\}.
\end{aligned}$$

Proof. The schematic form of the character table of $Sz(q), q = 2^{2n+1}$, is shown in Table 1, see [17, 18] for details.

Table 1. The Schematic Form of the Character Table of $Sz(q)$.

Irreducible Characters	Degrees	#Irreducible Characters
X	q^2	1
X_i	$q^2 + 1$	$q/2 - 1$
Y_j	$(q - r + 1)(q - 1)$	$(q + r)/4$
Z_k	$(q + r + 1)(q - 1)$	$(q - r)/4$
W_l	$r(q - 1)/2$	2

In this table, the first column designates the characters, the second column indicates the degrees, and the last one is the number of characters of each degree.

Suppose $2q = r^2$. The Suzuki group $Sz(q)$ contains cyclic groups of order $q - 1$, $q + r + 1$ and $q - r + 1$. These subgroups are denoted by A_0, A_1 and A_2, respectively. We also assume that π_i is a typical non-identity element of A_i, $i = 0, 1, 2$, $\sigma = (0, 1)$ and $\rho = (1, 0)$.

Let ε_0, ε_1 and ε_2 be a $(q - 1)^{th}$, a primitive $(q + r + 1)^{th}$ and a $(q - r + 1)^{th}$ root of unity. We also assume that ξ_0, ξ_1 and ξ_2 are generators of A_0, A_1 and A_2, respectively. Define ε_0^i, ε_1^i and ε_2^i as follows:

$$\begin{aligned}
\varepsilon_0^i(\xi_0^j) &= \varepsilon_0^{ij} + \varepsilon_0^{-ij};\ (\ i = 1, \ldots, q/2 - 1),\\
\varepsilon_1^i(\xi_1^k) &= \varepsilon_1^{ik} + \varepsilon_1^{ikq} + \varepsilon_1^{-ik} + \varepsilon_1^{-ikq};\ (\ i = 1, \ldots, q + r),\\
\varepsilon_2^i(\xi_2^k) &= \varepsilon_2^{ik} + \varepsilon_2^{ikq} + \varepsilon_2^{-ik} + \varepsilon_2^{-ikq}.
\end{aligned}$$

The functions ε_0^i, ε_1^i and ε_2^i are characters of A_0, A_1 and A_2, respectively. Following Suzuki [17], the character table of $Sz(q)$, is computed in Table 2.

We now apply Lemma 2.1 to construct four supercharacter theories given the statement of this theorem. By considering $m(Sz(q))$ and $M(Sz(q))$, it can be proved that $s(Sz(q)) \geq 6$. □

Table 2. The character table of $Sz(q)$.

	1	σ	ρ, ρ^{-1}	π_0	π_1	π_2
X	q^2	0	0	1	-1	-1
X_i	q^2+1	1	1	$\varepsilon_0^i(\pi_0)$	0	0
Y_j	$(q-r+1)(q-1)$	$r-1$	-1	0	$-\varepsilon_1^j(\pi_0)$	0
Z_k	$(q+r+1)(q-1)$	$-r-1$	-1	0	0	$-\varepsilon_2^k(\pi_0)$
W_l	$r(q-1)/2$	$-r/2$	$\pm r\sqrt{-1}/2$	0	1	-1

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