Extension of Summation Formulas involving Stirling series

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Abstract

This paper presents a family of rapidly convergent summation formulas for various finite sums of the form $\sum_{k=0}^{\lfloor x\rfloor} f(k)$, where x is a positive real number.

1 Introduction

In this paper we will use the Euler-Maclaurin summation formula [3, 5] to obtain rapidly convergent series expansions for finite sums involving Stirling series [1]. Our key tool will be the so called Weniger transformation [1].

For example, one of our summation formulas for the sum $\sum_{k=0}^{\lfloor x\rfloor}\sqrt{k}$, where $x\in\mathbb{R}^+$ is

$$\begin{aligned}\sum_{k=0}^{\lfloor x\rfloor}\sqrt{k} &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) - \sqrt{x}B_1(\{x\}) + \sqrt{x}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}(-1)^l\frac{(2l-3)!!}{2^l(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)}\\ &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \left(\frac{1}{2}-\{x\}\right)\sqrt{x} + \frac{\left(\frac{1}{4}\{x\}^2-\frac{1}{4}\{x\}+\frac{1}{24}\right)\sqrt{x}}{(x+1)}\\ &\quad + \frac{\left(\frac{1}{24}\{x\}^3+\frac{3}{16}\{x\}^2-\frac{11}{48}\{x\}+\frac{1}{24}\right)\sqrt{x}}{(x+1)(x+2)} + \frac{\left(\frac{1}{64}\{x\}^4+\frac{3}{32}\{x\}^3+\frac{21}{64}\{x\}^2-\frac{7}{16}\{x\}+\frac{53}{640}\right)\sqrt{x}}{(x+1)(x+2)(x+3)}\\ &\quad + \frac{\left(\frac{1}{128}\{x\}^5+\frac{19}{256}\{x\}^4+\frac{109}{384}\{x\}^3+\frac{29}{32}\{x\}^2-\frac{977}{768}\{x\}+\frac{79}{320}\right)\sqrt{x}}{(x+1)(x+2)(x+3)(x+4)} + \ldots,\end{aligned}$$

where the $B_l(x)$'s are the Bernoulli polynomials and $S_k^{(1)}(l)$ denotes the Stirling numbers of the first kind.

Most of the other formulas in this article have a similar shape.

Setting in the above formula for $\sum_{k=0}^{\lfloor x\rfloor}\sqrt{k}$ the variable $x := n\in\mathbb{N}$, we obtain

$$\begin{aligned}\sum_{k=0}^{n}\sqrt{k} &= \frac{2}{3}n^{\frac{3}{2}} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \sqrt{n}\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}(-1)^l\frac{(2l-3)!!}{2^l(l+1)!}B_{l+1}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}\\ &= \frac{2}{3}n^{\frac{3}{2}} + \frac{1}{2}\sqrt{n} - \frac{1}{4\pi}\zeta\left(\frac{3}{2}\right) + \frac{\sqrt{n}}{24(n+1)} + \frac{\sqrt{n}}{24(n+1)(n+2)} + \frac{53\sqrt{n}}{640(n+1)(n+2)(n+3)}\\ &\quad + \frac{79\sqrt{n}}{320(n+1)(n+2)(n+3)(n+4)} + \ldots,\end{aligned}$$

which is the corresponding formula given in our previous paper [8].

2 Definitions

As usual, we denote the floor of x by $\lfloor x \rfloor$ and the fractional part of x by $\{x\}$.

Definition 1. (Pochhammer symbol)[1]
We define the *Pochhammer symbol* (or rising factorial function) $(x)_k$ by

$$(x)_k := x(x+1)(x+2)(x+3)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt.$$

Definition 2. (Stirling numbers of the first kind)[1]
Let $k, l \in \mathbb{N}_0$ be two non-negative integers such that $k \geq l \geq 0$. We define the *Stirling numbers of the first kind* $S_k^{(1)}(l)$ as the connecting coefficients in the identity

$$(x)_k = (-1)^k \sum_{l=0}^{k} (-1)^l S_k^{(1)}(l) x^l,$$

where $(x)_k$ is the rising factorial function.

Definition 3. (Bernoulli numbers)[2]
We define the k-th *Bernoulli number* B_k as the k-th coefficient in the generating function relation

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \quad \forall x \in \mathbb{C} \text{ with } |x| < 2\pi.$$

Definition 4. (Euler numbers)[3]
We define the sequence of *Euler numbers* $\{E_k\}_{k=0}^{\infty}$ by the generating function identity

$$\frac{2e^x}{e^{2x}+1} = \sum_{k=0}^{\infty} \frac{E_k}{k!} x^k.$$

Definition 5. (Bernoulli polynomials)[2, 3]
We define for $n \in \mathbb{N}_0$ the n-th *Bernoulli polynomial* $B_n(x)$ via the following exponential generating function as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad \forall t \in \mathbb{C} \text{ with } |t| < 2\pi.$$

Using this expression, we see that the value of the n-th Bernoulli polynomial $B_n(x)$ at the point $x = 0$ is

$$B_n(0) = B_n,$$

which is the n-th Bernoulli number.

Definition 6. (Euler polynomials)[3]
We define for $n \in \mathbb{N}_0$ the n-th *Euler polynomial* $E_n(x)$ via the following exponential generating function as

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n.$$

Moreover, we have that [4]

$$E_n(0) = -2(2^{n+1}-1)\frac{B_{n+1}}{n+1} \quad \forall n \in \mathbb{N}_0.$$

3 Extended Summation Formulas involving Stirling Series

In this section we will prove our summation formulas for various finite sums of the form $\sum_{k=1}^{\lfloor x \rfloor} f(k)$. For this, we need the following

Lemma 7. *(Extended Euler-Maclaurin summation formula)[3, 5]*
Let f be an analytic function. Then for all $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} f(k) = \int_1^x f(t)dt + \sum_{k=1}^{m} (-1)^k \frac{B_k(\{x\})}{k!} f^{(k-1)}(x) - \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(1) + \frac{(-1)^{m+1}}{m!} \int_1^x B_m(\{t\}) f^{(m)}(t)dt,$$

where $B_m(x)$ is the m-th Bernoulli polynomial and $\{x\}$ denotes the fractional part of x. Therefore, for many functions f we have the asymptotic expansion

$$\sum_{k=1}^{\lfloor x \rfloor} f(k) \sim \int_1^x f(t)dt + C + \sum_{k=1}^{\infty} (-1)^k \frac{B_k(\{x\})}{k!} f^{(k-1)}(x) \quad \text{as } x \to \infty,$$

for some constant $C \in \mathbb{C}$

and

Lemma 8. *(Extended Boole summation formula)[3]*
Let f be an analytic function. Then for all $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} f(k) = \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{m} (-1)^{k+1} \frac{E_k(\{x\})}{k!} f^{(k)}(x) - \sum_{k=0}^{m} \frac{(2^{k+1}-1)B_{k+1}}{(k+1)!} f^{(k)}(1) + \frac{(-1)^m}{2m!} \int_1^x (-1)^{t-\{t\}} E_m(\{t\}) f^{(m+1)}(t)dt,$$

where $E_m(x)$ is the m-th Euler polynomial and $\{x\}$ denotes the fractional part of x. Therefore, for many functions f we have the asymptotic expansion

$$\sum_{k=1}^{\lfloor x\rfloor}(-1)^{k+1}f(k) \sim C + \frac{(-1)^{x-\{x\}}}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\frac{E_k(\{x\})}{k!}f^{(k)}(x) \quad \text{as } x \to \infty,$$

for some constant $C \in \mathbb{C}$.

From the above lemma we get by setting $x := n \in \mathbb{N}$ the following

Corollary 9. *(Boole Summation Formula)[3]*
Let f be an analytic function. Then for all $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n}(-1)^{k+1}f(k) = (-1)^n\sum_{k=0}^{m}(-1)^k\frac{(2^{k+1}-1)B_{k+1}}{(k+1)!}f^{(k)}(n) - \sum_{k=0}^{m}\frac{(2^{k+1}-1)B_{k+1}}{(k+1)!}f^{(k)}(1)$$
$$+\frac{(-1)^m}{2m!}\int_1^x(-1)^{t-\{t\}}E_m(\{t\})f^{(m+1)}(t)dt,$$

and the following asymptotic expansion

$$\sum_{k=1}^{n}(-1)^{k+1}f(k) \sim C + (-1)^n\sum_{k=0}^{\infty}(-1)^k\frac{(2^{k+1}-1)B_{k+1}}{(k+1)!}f^{(k)}(n) \quad \text{as } n \to \infty,$$

for some constant $C \in \mathbb{C}$.

As well as the next key result found by J. Weniger:

Lemma 10. *(Generalized Weniger transformation)[1]*
For every inverse power series $\sum_{k=1}^{\infty}\frac{a_k(x)}{x^{k+1}}$, where $a_k(x)$ is any function in x, the following transformation formula holds

$$\sum_{k=1}^{\infty}\frac{a_k(x)}{x^{k+1}} = \sum_{k=1}^{\infty}\frac{(-1)^k}{(x)_{k+1}}\sum_{l=1}^{k}(-1)^l S_k^{(1)}(l)a_l(x)$$
$$= \sum_{k=1}^{\infty}\frac{(-1)^k\sum_{l=1}^{k}(-1)^l S_k^{(1)}(l)a_l(x)}{x(x+1)(x+2)\cdots(x+k)}.$$

Now, we prove as an example the following

Theorem 11. *(Extended summation formulas for the harmonic series)*
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x\rfloor}\frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l}{l+1}S_k^{(1)}(l)B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

We also have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

Proof. Applying the extended Euler-Maclaurin summation formula to the function $f(x) := \frac{1}{x}$, we get that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} \sim \log(x) + \gamma - \sum_{k=1}^{\infty} \frac{B_k(\{x\})}{kx^k}.$$

Applying now the generalized Weniger transformation to the equivalent series

$$\begin{aligned} \sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} &\sim \log(x) + \gamma - \frac{B_1(\{x\})}{x} - \sum_{k=2}^{\infty} \frac{B_k(\{x\})}{kx^k} \\ &\sim \log(x) + \gamma - \frac{B_1(\{x\})}{x} - \sum_{k=1}^{\infty} \frac{B_{k+1}(\{x\})}{(k+1)x^{k+1}}, \end{aligned}$$

we get the first claimed formula. To obtain the second expression, we apply the generalized Weniger transformation to the identity

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} \sim \log(x) + \gamma - x \sum_{k=1}^{\infty} \frac{B_k(\{x\})}{kx^{k+1}}.$$

□

At this point, we want to give an overview on summation formulas obtained with this method:

1.) **Extended summation formulas for the harmonic series:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l+1} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

2.) **Extended summation formulas for the partial sums of $\zeta(2)$:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2} = \zeta(2) - \frac{1}{x} - \frac{B_1(\{x\})}{x^2} + \frac{1}{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2} = \zeta(2) - \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

3.) **Extended summation formulas for the partial sums of $\zeta(3)$:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} - \frac{B_1(\{x\})}{x^3} + \frac{1}{2x^2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (l+2) S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (l+1) S_k^{(1)}(l) B_l(\{x\})}{x^2(x+1)(x+2)\cdots(x+k)}.$$

4.) **Extended summation formulas for the sum of the square roots:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) - \sqrt{x} B_1(\{x\}) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-3)!!}{2^l (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)},$$

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) - \sqrt{x} B_1(\{x\}) + \frac{B_2(\{x\})}{4\sqrt{x}} + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-1)!!}{2^{l+1} (l+2)!} S_k^{(1)}(l) B_{l+2}(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta\left(\frac{3}{2}\right) + x\sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-5)!!}{2^{l-1} l!} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

5.) **Extended summation formulas for the partial sums of $\zeta(-3/2)$:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} k\sqrt{k} = \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) - x^{\frac{3}{2}}B_1(\{x\}) + \frac{3}{2}x^{\frac{3}{2}}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l-5)!!}{2^{l-1}(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)},$$

$$\sum_{k=0}^{\lfloor x \rfloor} k\sqrt{k} = \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) - x^{\frac{3}{2}}B_1(\{x\}) + \frac{3}{4}\sqrt{x}B_2(\{x\}) - \frac{B_3(\{x\})}{4\sqrt{x}} + \frac{3}{2}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l-1)!!}{2^{l+1}(l+3)!}S_k^{(1)}(l)B_{l+3}(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=0}^{\lfloor x \rfloor} k\sqrt{k} = \frac{2}{5}x^{\frac{5}{2}} - \frac{3}{16\pi^2}\zeta\left(\frac{5}{2}\right) + \frac{3}{2}x^{\frac{5}{2}}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l-7)!!}{2^{l-2}l!}S_k^{(1)}(l)B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

6.) **Extended summation formulas for the partial sums of $\zeta(-5/2)$:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} k^2\sqrt{k} = \frac{2}{7}x^{\frac{7}{2}} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) - x^{\frac{5}{2}}B_1(\{x\}) + \frac{15}{4}x^{\frac{5}{2}}\sum_{k=1}^{\infty}(-1)^{k}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l-7)!!}{2^{l-2}(l+1)!}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)},$$

$$\sum_{k=0}^{\lfloor x \rfloor} k^2\sqrt{k} = \frac{2}{7}x^{\frac{7}{2}} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) - x^{\frac{5}{2}}B_1(\{x\}) + \frac{5}{4}x^{\frac{3}{2}}B_2(\{x\}) - \frac{5}{8}\sqrt{x}B_3(\{x\}) + \frac{5B_4(\{x\})}{64\sqrt{x}} + \frac{15}{4}\sum_{k=1}^{\infty}(-1)^{k}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l-1)!!}{2^{l+1}(l+4)!}S_k^{(1)}(l)B_{l+4}(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=0}^{\lfloor x \rfloor} k^2\sqrt{k} = \frac{2}{7}x^{\frac{7}{2}} + \frac{15}{64\pi^3}\zeta\left(\frac{7}{2}\right) + \frac{15}{4}x^{\frac{7}{2}}\sum_{k=1}^{\infty}(-1)^{k}\frac{\sum_{l=1}^{k}\frac{(-1)^l(2l-9)!!}{2^{l-3}l!}S_k^{(1)}(l)B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

7.) **Extended summation formulas for the sum of the inverse square roots:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{\sqrt{k}} = 2\sqrt{x} + \zeta\left(\frac{1}{2}\right) - \frac{B_1(\{x\})}{\sqrt{x}} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-1)!!}{2^l (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{\sqrt{k}} = 2\sqrt{x} + \zeta\left(\frac{1}{2}\right) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-3)!!}{2^{l-1} l!} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

8.) **Extended summation formulas for the partial sums of $\zeta(3/2)$:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k\sqrt{k}} = \zeta\left(\frac{3}{2}\right) - \frac{2}{\sqrt{x}} - \frac{B_1(\{x\})}{x\sqrt{x}} + \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l+1)!!}{2^{l+1} (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k\sqrt{k}} = \zeta\left(\frac{3}{2}\right) - \frac{2}{\sqrt{x}} + 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l-1)!!}{2^l l!} S_k^{(1)}(l) B_l(\{x\})}{\sqrt{x}(x+1)(x+2)\cdots(x+k)}.$$

9.) **Extended summation formulas for the partial sums of $\zeta(5/2)$:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2\sqrt{k}} = \zeta\left(\frac{5}{2}\right) - \frac{2}{3x^{\frac{3}{2}}} - \frac{B_1(\{x\})}{x^2\sqrt{x}} + \frac{4}{3x\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l+3)!!}{2^{l+2} (l+1)!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2\sqrt{k}} = \zeta\left(\frac{5}{2}\right) - \frac{2}{3x^{\frac{3}{2}}} + \frac{4}{3\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(-1)^l (2l+1)!!}{2^{l+1} l!} S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

10.) **Extended Generalized Faulhaber Formulas:**
For every positive real number $x \in \mathbb{R}^+$ and for every complex number $m \in \mathbb{C} \setminus \{-1\}$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{x^{m+1}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l} S_k^{(1)}(l) B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x\rfloor} k^m = \frac{1}{m+1}x^{m+1} + \zeta\left(-m\right) - x^m B_1(\{x\}) + \frac{x^m}{m+1}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\binom{m+1}{l+1}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=1}^{\lfloor x\rfloor} k^m = \frac{1}{m+1}x^{m+1} + \zeta\left(-m\right) + \frac{1}{m+1}\sum_{k=1}^{\lfloor m+1\rfloor}(-1)^k\binom{m+1}{k}B_k(\{x\})x^{m-k+1}$$
$$+ (-1)^{\lceil m+1\rceil}\frac{x^{m-\lceil m+1\rceil+2}}{m+1}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\binom{m+1}{l+\lceil m+1\rceil-1}S_k^{(1)}(l)B_{l+\lceil m+1\rceil-1}(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

11.) **Generalized Faulhaber Formulas:**
For every complex number $m \in \mathbb{C}\setminus\{-1\}$ and every natural number $n \in \mathbb{N}$, we have that

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}n^{m+1} + \zeta\left(-m\right) + \frac{n^{m+1}}{m+1}\sum_{k=1}^{\infty}\frac{(-1)^k\sum_{l=1}^{k}\binom{m+1}{l}B_l S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}$$

and

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}n^{m+1} + \zeta\left(-m\right) + \frac{1}{2}n^m + \frac{n^m}{m+1}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\binom{m+1}{l+1}B_{l+1}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}$$

and

$$\sum_{k=1}^{n} k^m = \frac{1}{m+1}n^{m+1} + \zeta\left(-m\right) + \frac{1}{m+1}\sum_{k=1}^{\lfloor m+1\rfloor}(-1)^k\binom{m+1}{k}B_k n^{m-k+1}$$
$$+ (-1)^{\lceil m+1\rceil}\frac{n^{m-\lceil m+1\rceil+2}}{m+1}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\binom{m+1}{l+\lceil m+1\rceil-1}B_{l+\lceil m+1\rceil-1}S_k^{(1)}(l)}{(n+1)(n+2)\cdots(n+k)}.$$

12.) **Extended convergent versions of Stirling's formula:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x\rfloor}\log(k) = x\log(x) - x + \frac{1}{2}\log\left(2\pi\right) - \log(x)B_1(\{x\}) + \sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{(-1)^l}{l(l+1)}S_k^{(1)}(l)B_{l+1}(\{x\})}{(x+1)(x+2)\cdots(x+k)},$$

$$\sum_{k=1}^{\lfloor x\rfloor}\log(k)=x\log(x)-x+\frac{1}{2}\log(2\pi)-\log(x)B_1(\{x\})+\frac{B_2(\{x\})}{2x}$$
$$+\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{(-1)^l}{(l+1)(l+2)}S_k^{(1)}(l)B_{l+2}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}$$

and that

$$\sum_{k=1}^{\lfloor x\rfloor}\log(k)=x\log(x)-x+\frac{1}{2}\log(2\pi)-\log(x)B_1(\{x\})+x\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=2}^{k}\frac{(-1)^l}{l(l-1)}S_k^{(1)}(l)B_l(\{x\})}{(x+1)(x+2)\cdots(x+k)}.$$

13.) **Extended first logarithmic summation formulas:**
For every positive real number $x\in\mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x\rfloor}k\log(k)=\frac{1}{2}x^2\log(x)-\frac{1}{4}x^2+\frac{1}{12}-\zeta^{'}(-1)-x\log(x)B_1(\{x\})+\frac{1}{2}\log(x)B_2(\{x\})$$
$$+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l}{l(l+1)(l+2)}S_k^{(1)}(l)B_{l+2}(\{x\})}{(x+1)(x+2)\cdots(x+k)}$$

and

$$\sum_{k=0}^{\lfloor x\rfloor}k\log(k)=\frac{1}{2}x^2\log(x)-\frac{1}{4}x^2+\frac{1}{12}-\zeta^{'}(-1)-x\log(x)B_1(\{x\})+\frac{1}{2}\log(x)B_2(\{x\})-\frac{B_3(\{x\})}{6x}$$
$$+\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{(-1)^l}{(l+1)(l+2)(l+3)}S_k^{(1)}(l)B_{l+3}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

14.) **Extended second logarithmic summation formula:**
For every positive real number $x\in\mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x\rfloor}\frac{\log(k)}{k}=\frac{1}{2}\log(x)^2+\gamma_1-\frac{\log(x)}{x}B_1(\{x\})+\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{(-1)^l}{(l+1)!}S_{l+1}^{(1)}(2)S_k^{(1)}(l)B_{l+1}(\{x\})}{x(x+1)(x+1)\cdots(x+k)}$$
$$+\log(x)\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{1}{l+1}S_k^{(1)}(l)B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

15.) **Extended third logarithmic summation formula:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{\log(k)}{k^2} = -\zeta'(2) - \frac{\log(x)}{x} - \frac{1}{x} - \frac{\log(x)}{x^2} B_1(\{x\}) + \frac{1}{x}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sum_{l=1}^{k}\frac{1}{l+1}\left(\sum_{m=0}^{l-1}\frac{m+1}{l-m}\right)S_k^{(1)}(l)B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)} + \frac{\log(x)}{x}\sum_{k=1}^{\infty}\frac{(-1)^k\sum_{l=1}^{k}S_k^{(1)}(l)B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

16.) **Extended fourth logarithmic summation formula:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \log(k)^2 = x\log(x)^2 - 2x\log(x) + 2x + \frac{\gamma^2}{2} - \frac{\pi^2}{24} - \frac{\log(2)^2}{2} - \log(2)\log(\pi) - \frac{\log(\pi)^2}{2} + \gamma_1 - \log(x)^2 B_1(\{x\}) + \frac{\log(x)}{x}B_2(\{x\}) + 2\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{(-1)^l}{(l+2)!}S_{l+1}^{(1)}(2)S_k^{(1)}(l)B_{l+2}(\{x\})}{x(x+1)(x+2)\cdots(x+k)} + 2\log(x)\sum_{k=1}^{\infty}(-1)^k\frac{\sum_{l=1}^{k}\frac{1}{(l+1)(l+2)}S_k^{(1)}(l)B_{l+2}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

17.) **Extended summation formula for partial sums of the Gregory-Leibniz series:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=0}^{\lfloor x \rfloor} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} + \frac{(-1)^{x-\{x\}}}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\sum_{l=0}^{k}(-1)^l 2^l S_k^{(1)}(l)E_l(\{x\})}{(2x+1)(2x+2)(2x+3)\cdots(2x+k+1)}.$$

Setting $x := n \in \mathbb{N}$, we get that

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} + (-1)^{n+1}\sum_{k=0}^{\infty}\frac{(-1)^k\sum_{l=0}^{k}\frac{(-1)^l}{l+1}2^l(2^{l+1}-1)S_k^{(1)}(l)B_{l+1}}{(2n+1)(2n+2)(2n+3)\cdots(2n+k+1)}.$$

18.) **Extended formula for the partial sums of the alternating harmonic series:**
For every positive real number $x \in \mathbb{R}^+$, we have that

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{(-1)^{k+1}}{k} = \log(2) + \frac{(-1)^{x-\{x\}}}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\frac{\sum_{l=0}^{k}(-1)^l S_k^{(1)}(l)E_l(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = \log(2) + (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^{k} \frac{(-1)^l}{l+1} (2^{l+1}-1) S_k^{(1)}(l) B_{l+1}}{n(n+1)(n+2)\cdots(n+k)}.$$

4 Other Extended Summation Formulas for Finite Sums

In this section, we denote by $\eta(s) := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}$ the Dirichlet eta function.

1.) **The extended alternating Faulhaber formula:**

$$\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^{x-\{x\}}}{2(m+1)} \sum_{k=0}^{m} (-1)^{k+1} (k+1) \binom{m+1}{k+1} E_k(\{x\}) x^{m-k} \quad \forall m \in \mathbb{N}_0.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=1}^{n} (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^n}{m+1} \sum_{k=0}^{m} (-1)^k (2^{k+1}-1) \binom{m+1}{k+1} B_{k+1} n^{m-k} \quad \forall m \in \mathbb{N}_0.$$

2.) **The extended generalized alternating Faulhaber formula:**

$$\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^{x-\{x\}} x^m}{2(m+1)} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{\sum_{l=0}^{k} (l+1) \binom{m+1}{l+1} S_k^{(1)}(l) E_l(\{x\})}{(x+1)(x+2)\cdots(x+k)} \quad \forall m \in \mathbb{C}.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=1}^{n} (-1)^{k+1} k^m = \eta(-m) + \frac{(-1)^n n^m}{(m+1)} \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^{k} \binom{m+1}{l+1} (2^{l+1}-1) S_k^{(1)}(l) B_{l+1}}{(n+1)(n+2)\cdots(n+k)} \quad \forall m \in \mathbb{C}.$$

3.) **The Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x \rfloor} a^k = \frac{a^x}{\log(a)} + \frac{1}{1-a} + a^x \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{\log(a)^{l-1}}{l!} S_k^{(1)}(l) B_l(\{x\}) x^l}{(x+1)(x+2)\cdots(x+k)} \quad \forall a \neq 1.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=0}^{n} a^k = \frac{a^n}{\log(a)} + \frac{1}{1-a} + a^n \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{\log(a)^{l-1}}{l!} S_k^{(1)}(l) B_l n^l}{(n+1)(n+2)\cdots(n+k)} \quad \forall a \neq 1.$$

4.) **The alternating Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x\rfloor}(-1)^k a^k = \frac{1}{1+a} + \frac{(-1)^{x-\{x\}}}{2}a^x \sum_{k=0}^{\infty}(-1)^k \frac{\sum_{l=0}^{k}\frac{\log(a)^l}{l!}S_k^{(1)}(l)E_l(\{x\})x^l}{(x+1)(x+2)\cdots(x+k)} \ \ \forall a\neq -1.$$

Setting $x := n \in \mathbb{N}$, we get

$$\sum_{k=0}^{n}(-1)^k a^k = \frac{1}{1+a} + (-1)^{n+1}a^n \sum_{k=0}^{\infty}(-1)^k \frac{\sum_{l=0}^{k}\frac{\log(a)^l}{(l+1)!}(2^{l+1}-1)S_k^{(1)}(l)B_{l+1}n^l}{(n+1)(n+2)\cdots(n+k)} \ \ \forall a\neq -1.$$

5.) **The Euler-Maclaurin Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x\rfloor} a^k = \frac{a^x}{\log(a)} + \frac{1}{1-a} + a^x\sum_{k=1}^{\infty}(-1)^k\frac{\log(a)^{k-1}}{k!}B_k(\{x\}) \ \text{ for } \ \frac{1}{e^{2\pi}} < a \neq 1 < e^{2\pi}.$$

6.) **The Euler-Maclaurin alternating Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x\rfloor}(-1)^k a^k = \frac{1}{1+a} + (-1)^{x-\{x\}}\frac{a^x(a-1)}{a+1}\sum_{k=0}^{\infty}(-1)^k\frac{\log(a)^{k-1}}{k!}B_k(\{x\}) \ \text{ for } \ \frac{1}{e^{2\pi}} < a < e^{2\pi}.$$

7.) **The Exponential Geometric Summation Formula:**

$$\sum_{k=0}^{\lfloor x\rfloor} e^k = e^x + \frac{1}{1-e} + e^x\sum_{k=1}^{\infty}(-1)^k\frac{B_k(\{x\})}{k!} \ \ \forall x\in\mathbb{R}^+.$$

8.) **The Self-Counting Summation Formula:**
Let $\{a_k\}_{k=1}^{\infty} := \{1,2,2,3,3,3,4,4,4,4,...\}$ be the self-counting sequence [6, 7] defined by

$$a_k := \left\lfloor \frac{1}{2} + \sqrt{2k}\right\rfloor.$$

Then, we have that

$$\begin{aligned}\sum_{k=1}^{\lfloor x\rfloor} a_k = {} & \frac{x\sqrt{8x+1}}{3} - \frac{5\sqrt{8x+1}}{24} - \frac{\sqrt{8x+1}}{2}B_1(\{x\}) + B_1\left(\left\{\frac{\sqrt{8x+1}}{2}-\frac{1}{2}\right\}\right)B_1(\{x\}) \\ & + \frac{1}{2}B_1\left(\left\{\frac{\sqrt{8x+1}}{2}-\frac{1}{2}\right\}\right) - \frac{\sqrt{8x+1}}{4}B_2\left(\left\{\frac{\sqrt{8x+1}}{2}-\frac{1}{2}\right\}\right) \\ & + \frac{1}{6}B_3\left(\left\{\frac{\sqrt{8x+1}}{2}-\frac{1}{2}\right\}\right).\end{aligned}$$

9.) **Slowly convergent summation formula for the sum of the square roots:**

$$\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \sqrt{x} B_1(\{x\}) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\text{FresnelS}\left(2\sqrt{k}\sqrt{x}\right)}{k^{\frac{3}{2}}}.$$

10.) **Slowly convergent summation formula for the harmonic series:**

$$\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + 2 \sum_{k=1}^{\infty} \text{CosIntegral}(2\pi k x).$$

5 Conclusion

This paper presents an overview on some rapidly convergent summation formulas obtained by applying the Weniger transformation [1].

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