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LINK HOMOLOGY AND THE NABLA OPERATOR

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Abstract. In recent work, Elias and Hogancamp develop a recurrence for the Poincaré series of the triply graded Hochschild homology of certain links, one of which is the (n, n) torus link. In this case, Elias and Hogancamp give a combinatorial formula for this homology that is reminiscent of the combinatorics of the modified Macdonald polynomial eigenoperator ∇. We give a combinatorial formula for the homologies of all links considered by Elias and Hogancamp. Our first formula is not easily computable, so we show how to transform it into a computable version. Finally, we conjecture a direct relationship between the (n, n) torus link case of our formula and the symmetric function ∇p_{1^n}.

1. Introduction

We begin by establishing some notation from knot theory, following [EH16]. The remaining sections of the paper will take a more combinatorial perspective.

The *braid group on n strands*, denoted Br_n, can be defined by the presentation

$$\mathrm{Br}_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,|\, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\, \sigma_i\sigma_j = \sigma_j\sigma_i \rangle \tag{1}$$

for all $1 \leq i \leq n-2$ and $|i - j| \geq 2$. This group can be pictured as all ways to "braid" together n strands, where σ_i corresponds to crossing string $i+1$ over string i and the group operation is concatenation. One particularly notable braid is the *full twist braid* on n strands, denoted FT_n, which can be written

$$\mathrm{FT}_n = \left((\sigma_1)(\sigma_2\sigma_1)\ldots(\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1)\right)^2. \tag{2}$$

where multiplication is left to right. We will also need an operation ω on braids which corresponds to rotation around the horizontal axis. We define ω on Br_n by $\omega(\sigma_i) = \sigma_i$ and $\omega(\alpha\beta) = \omega(\beta)\omega(\alpha)$. Then ω is an anti-involution on Br_n. All of our braids will have the property that the string that begins in column i also ends in column i for all i; these are sometimes called *perfect braids*.

Given a braid with n strands, one can form a *link* (i.e. nonintersecting collection of knots) by identifying the top of the strand that begins in position i with the bottom of the strand that ends in position i for $1 \leq i \leq n$. The result is called a *closed braid*. Alexander proved that every link can be represented by a closed braid (although this representation is not unique) [Ale23]. The closure of a perfect braid is a link that consists of n separate unknots linked together.

In [EH16], Elias and Hogancamp assign a complex C_v to every binary word v. We describe this assignment here – see Figure 1 for an example. Say $v \in \{0,1\}^n$ with $|v| = m$. We begin with two braids, the full twist braid FT_{n-m} and a certain recursively defined complex K_m [EH16], which sits to the right of FT_{n-m}. For $i = 1$ to n, we feed string i into the leftmost available position in K_m if $v_i = 1$; otherwise, we feed string i into the leftmost available position in FT_{n-m}. All crossings that occur are forced to be "positive," i.e. the right strand crosses over the left strand.

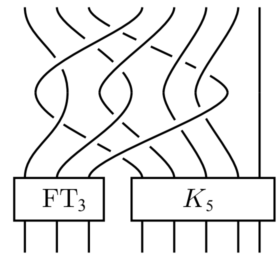


FIGURE 1. We have drawn the complex $C_{10101101}$, where FT_3 is the full twist braid and K_5 is a certain complex defined recursively in [EH16]. This figure is used courtesy of [EH16].

This induces a braid $\beta_v \in \mathrm{Br}_n$ that occurs before the adjacent FT_{n-m} and K_m. The final complex C_v is obtained by performing $\omega(\beta_v)$, followed by β_v, followed by the adjacent FT_{n-m} and K_m. We note that C_{0^n} is the full twist braid FT_n and that the closure of this braid is the (n, n) torus link. The combinatorics of other links, in particular the (m, n) torus link for m and n coprime, has been studied by a variety of authors in recent years [GORS14, GN15]. Haglund gives an overview of this work from a combinatorial perspective in [Hag16].

Elias and Hogancamp map each complex C_v to a graded Soergel bimodule and then consider the *Hochschild homology* of this bimodule; this is sometimes called Khovanov-Rozansky homology [Kho07, KR08]. This homology has three gradings: the bimodule degree (using the variable Q), the homological degree (T), and the Hochschild degree (A). After the grading shifts $q = Q^2$, $t = T^2Q^{-2}$, and $a = AQ^{-2}$, Elias and Hogancamp give a recurrence for the Poincaré series of this triply graded homology, which they denote $f_v(q, a, t)$. They also give a combinatorial formula for the special case $f_{0^n}(q, a, t)$. We will give two combinatorial formulas for $f_v(q, a, t)$ for every $v \in \{0, 1\}^n$.

In Section 2, we define a symmetric function $L_v(x; q, t)$ which we call the *link symmetric function*. Its definition is reminiscent of the combinatorics of the Macdonald eigenoperator ∇, introduced in [BGHT99]. We prove that $f_v(q, a, t)$ is equal to a certain inner product with $L_v(x; q, t)$.

The main weakness of our first formula is that it is a sum over infinitely many objects, so it is not clear how to compute using this formula. We address this issue in Section 3, obtaining a finite formula for $L_v(x; q, t)$ using a collection of combinatorial objects we call *barred Fubini words*.

We close by presenting some conjectures in Section 4. In particular, we conjecture that

$$L_{0^n}(x; q, t) = (1 - q)^{-n} \nabla p_{1^n}. \tag{3}$$

where the terminology is defined in Section 4. A proof of this conjecture would provide the first combinatorial interpretation for ∇p_{1^n}. There has been much recent work establishing combinatorial interpretations for ∇e_n [CM15] and ∇p_n [Ser16]. We believe that $L_v(x; q, t)$ is also related to Macdonald polynomials for general v, although we do not have an explicit conjecture in this direction.

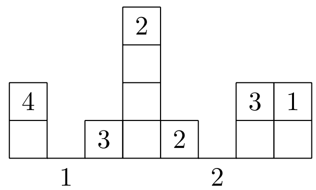


FIGURE 2. We have depicted the example $\gamma = 20141022$ and $\pi = 41322231$ by drawing bottom-justified columns with heights γ_1, γ_2, ..., γ_8 and the labels π_i are placed as high as possible in each column. In this example, we compute $\text{area}(\gamma) = 6$, $\text{dinv}(\gamma, \pi) = 7$, where the contributing pairs are in columns $(1, 7)$, $(1, 8)$, $(2, 3)$, $(2, 5)$, $(3, 5)$, $(5, 7)$, $(7, 8)$, and $x^\pi = x_1^2 x_2^3 x_3^2 x_4$.

2. An infinite formula

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{P} = \{1, 2, 3, \ldots\}$. We begin by defining two statistics.

Definition 2.1. *Given words $\gamma \in \mathbb{N}^n$ and $\pi \in \mathbb{P}^n$, we define*

$$\text{area}(\gamma) = |\gamma| - \#\{1 \le i \le n : \gamma_i > 0\} \tag{4}$$

$$\begin{aligned} \text{dinv}(\gamma, \pi) = \#\{1 \le i < j \le n : \gamma_i = \gamma_j, \pi_i > \pi_j\} \\ + \#\{1 \le i < j \le n : \gamma_i + 1 = \gamma_j, \pi_i < \pi_j\} \end{aligned} \tag{5}$$

$$x^\pi = \prod_{i=1}^{n} x_{\pi_i}. \tag{6}$$

In Figure 2, we draw a diagram for $\gamma = 20141022$ and $\pi = 41322231$. Area counts the empty boxes in such a diagram, dinv counts certain pairs of labels, and x^π records all labels that appear in the diagram.

Definition 2.2. *Given $n \in \mathbb{P}$ and $v \in \{0, 1\}^n$, define*

$$L_v = L_v(x; q, t) = \sum_{\substack{\gamma \in \mathbb{N}^n,\, \pi \in \mathbb{P}^n \\ \gamma_i = 0 \Leftrightarrow v_i = 1}} q^{\text{area}(\gamma)} t^{\text{dinv}(\gamma, \pi)} x^\pi. \tag{7}$$

Perhaps the first thing to note about L_v is that it can be expressed as a sum of LLT polynomials [LLT97]; as a result, it is symmetric in the x_i variables. More precisely, each $\gamma \in \mathbb{N}^n$ can be associated with an n-tuple $\lambda(\gamma)$ of single cell partitions in the plane, where the ith cell is placed on diagonal γ_i and the order is not changed. Using the notation of [HHL05], the unicellular LLT polynomial $G_{\lambda(\gamma)}(x; t)$ can be used to write

$$L_v = \sum_{\substack{\gamma \in \mathbb{N}^n \\ \gamma_i = 0 \Leftrightarrow v_i = 1}} q^{\text{area}(\gamma)} G_{\lambda(\gamma)}(x; t). \tag{8}$$

Since LLT polynomials are symmetric, every L_v is also symmetric.

We also remark that L_{1^n} is equal to the modified Macdonald polynomial $\widetilde{H}_{1^n}(x; q, t)$, which is also equal to the graded Frobenius series of the coinvariants of $\mathfrak{S}_n$ with grading in t.

Next, we note that the Poincaré series $f_v(q, a, t)$ can be recovered as a certain inner product of L_v. We follow the standard notation for symmetric functions and their usual inner product, as described in Chapter 7 of [Sta99]. Before we can prove Theorem 2.1, we need the following lemma.

Lemma 2.1.

$$L_{0^n} = \frac{1}{1-q} L_{10^{n-1}}. \tag{9}$$

Proof. By definition,

$$L_{0^n} = \sum_{\gamma,\pi \in \mathbb{P}^n} q^{\operatorname{area}(\gamma)} t^{\operatorname{dinv}(\gamma,\pi)} x^{\pi}. \tag{10}$$

Our aim is to show that

$$L_{0^n} = q^n L_{0^n} + \left(1 + q + \ldots + q^{n-1}\right) L_{10^{n-1}} \tag{11}$$

which clearly implies the lemma.

If $\gamma_i > 1$ for all i, then let γ' be the word obtained by decrementing each entry in γ by 1. Set $\pi' = \pi$. Note that the pair (γ', π') has

$$\operatorname{area}(\gamma') = \operatorname{area}(\gamma) - n \tag{12}$$

$$\operatorname{dinv}(\gamma', \pi') = \operatorname{dinv}(\gamma, \pi) \tag{13}$$

$$x^{\pi'} = x^{\pi}. \tag{14}$$

Furthermore, every pair of words of positive integers can be obtained as (γ', π') in this fashion. This case corresponds to the first term on the right-hand side of (11).

The other case we must consider is if $\gamma_i = 1$ for some i. Let k be the rightmost position such that $\gamma_k = 1$. Then we define

$$\gamma'' = (\gamma_k - 1)(\gamma_{k+1} - 1) \ldots (\gamma_n - 1)\gamma_1\gamma_2 \ldots \gamma_{k-1} \tag{15}$$

$$\pi'' = \pi_k \pi_{k+1} \ldots \pi_n \pi_1 \pi_2 \ldots \pi_{k-1}. \tag{16}$$

It is straightforward to check that

$$\operatorname{area}(\gamma'') = \operatorname{area}(\gamma) - (n - k) \tag{17}$$

$$\operatorname{dinv}(\gamma'', \pi'') = \operatorname{dinv}(\gamma, \pi) \tag{18}$$

$$x^{\pi''} = x^{\pi}. \tag{19}$$

Furthermore, by construction we have $\gamma''_1 = 0$ and the other entries of γ'' are greater than 0. Summing over all values of k and pairs (γ'', π'') obtained in this way, we get the remaining terms in the right-hand side of (11). □

Theorem 2.1. *For any* $v \in \{0,1\}^n$,

$$f_v(q,a,t) = \sum_{d=0}^{n} \langle L_v, e_{n-d} h_d \rangle a^d. \tag{20}$$

Proof. Let us denote the right-hand side of the statement in the theorem by $L_v(q,a,t)$. In [EH16], the authors prove that $f_v(q,a,t)$ satisfies a certain recurrence. We will use their recurrence as our definition of $f_v(q,a,t)$.

Given $v \in \{0,1\}^n$ and $w \in \{0,1\}^{n-|v|}$, we form a word $u \in \{0,1,2\}^n$ that depends on v and w. We set $u_i = 1$ if $v_i = 1$. If $v_i = 0$, say that we are at the

jth zero in v, counting from left to right. Then we set $u_i = 2w_j$. For example, if $v = 10110100$ and $w = 0110$ then $u = 10112120$. We form a product

$$P_{v,w}(a,t) = \prod_{i\,:\,v_i=1} \left(t^{\#\{j<i\,:\,u_j=1\}+\#\{j>i\,:\,u_j=2\}} + a \right). \tag{21}$$

Then the recurrence in [EH16] is

$$f_v(q,a,t) = \sum_{w\in\{0,1\}^{n-|v|}} q^{n-|v|-|w|} P_{v,w}(a,t) f_w(q,a,t) \tag{22}$$

with base cases $f_\emptyset(q,a,t) = 1$ and $f_{0^n}(q,a,t) = (1-q)^{-1} f_{10^{n-1}}(q,a,t)$. We use this as the definition of $f_v(q,a,t)$.

The goal of this proof is to show that $L_v(q,a,t)$ satisfies (21). As discussed in [Hag08], taking the inner product with $e_{n-d}h_d$ can be thought of as replacing π with a word containing $n-d$ $\underline{0}$'s and d 1's. For the purposes of computing $\operatorname{dinv}(\gamma,\pi)$ we consider $\underline{0}$ to be less than itself, but we do not make this convention for 1. For example, if $\gamma = 1111$ and $\pi = \underline{0}1\underline{0}1$, we have $\operatorname{dinv}(\gamma,\pi) = 2$, where the two pairs we count are $(1,3)$ and $(1,2)$. With these definitions, we can write

$$L_v(q,a,t) = \sum_{\substack{\gamma\in\mathbb{N}^n,\, \pi\in\{\underline{0},1\}^n \\ \gamma_i=0 \Leftrightarrow v_i=1}} q^{\operatorname{area}(\gamma)} t^{\operatorname{dinv}(\gamma,\pi)} a^{\#1\text{'s in }\pi}. \tag{23}$$

Given such a word γ, we form a word u by setting $u_i = 1$ if $\gamma_i = 0$, $u_i = 2$ if $\gamma_i = 1$, and $u_i = 0$ otherwise. From this word u we construct another word $w \in \{0,1\}^{n-|v|}$ by scanning u from left to right and appending a 1 to w whenever we see a 2 in u and appending a 0 to w whenever we see a 0 in u. For example, if $\gamma = 013021$ we have $u = 120102$ and $w = 1001$.

Now we can explain why $L_v(q,a,t)$ satisfies (22). First, we note that the $q^{n-|v|-|w|}$ term counts the contribution of empty boxes in row 1 to area. We also claim that $P_{v,w}(a,t)$ uniquely counts the contributions from dinv pairs (i,j) with either $\gamma_i = \gamma_j = 0$ or $\gamma_i = 0$ and $\gamma_j = 1$. For each such pair, say that the pair *projects onto* j if $\gamma_i = \gamma_j = 0$ or i if $\gamma_i = 0$ and $\gamma_j = 1$. Then every such pair projects onto a unique i such that $\gamma_i = 0$, which is equivalent to $v_i = 1$. Furthermore, the number of pairs projecting onto a particular i is 0 if $\pi_i = 1$ and

$$\#\{j < i \,:\, \gamma_j = 0\} + \#\{j > i \,:\, \gamma_j = 1\} = \#\{j < i \,:\, u_j = 1\} + \#\{j > i \,:\, u_j = 2\} \tag{24}$$

if $\pi_i = \underline{0}$. Hence, $P_{v,w}(a,t)$ accounts for the contribution all such dinv pairs. By induction, $L_w(q,a,t)$ accounts for all other area and all other dinv pairs. The $v = 0^n$ case follows from Lemma 2.1. □

For the sake of comparison with [EH16], we give a simplified formula that directly computes $f_v(q,a,t)$ from Theorem 2.1. Given $\gamma \in \mathbb{N}^n$ and $1 \le i \le n$, let

$$\operatorname{dinv}_i(\gamma) = \#\{j < i : \gamma_j = \gamma_i\} + \#\{j > i : \gamma_j = \gamma_i + 1\}. \tag{25}$$

Corollary 2.1.

$$f_v(q,a,t) = \sum_{\substack{\gamma\in\mathbb{N}^n \\ \gamma_i=0\Leftrightarrow v_i=1}} q^{\operatorname{area}(\gamma)} \prod_{i=1}^{n} \left(a + t^{\operatorname{dinv}_i(\gamma)} \right) \tag{26}$$

where, as before, $\operatorname{area}(\gamma) = |\gamma| - \#\{1 \le i \le n : \gamma_i > 0\}$.

If $v = 0^n$ and $a = 0$, this is exactly Theorem 1.9 in [EH16].

3. A finite formula

Although the combinatorial definition of L_v is straightforward, it is not computationally effective[1] since it is a sum over infinitely many words $\gamma \in \mathbb{N}^n$. We rectify this issue in Theorem 3.1 below. The idea is to compress the vectors γ while altering the statistics so that the link polynomial L_v is not changed.

Definition 3.1. *A word $\gamma \in \mathbb{N}^n$ is a* Fubini word *if every integer $0 \leq k \leq \max(\gamma)$ appears in γ.*

For example, 41255103 is a Fubini word but 20141022 is not a Fubini word, since it contains a 4 but not a 3. We call these Fubini words because they are counted by the Fubini numbers ([Slo], A000670), which also count ordered partitions of the set $\{1, 2, \ldots, n\}$. We will actually be interested in certain decorated Fubini words.

Definition 3.2. *Given $v \in \{0,1\}^n$, we say that a Fubini word γ is* associated *with v if either*

- *$v = 0^n$ and the only zero in γ occurs at γ_1, or*
- *$v \neq 0^n$ and $\gamma_i = 0$ if and only if $v_i = 1$.*

Definition 3.3. *A* barred Fubini word *associated with v is a Fubini word γ associated with v where we may place bars over certain entries. Specifically, the entry γ_j may be barred if*

1. *$\gamma_j > 0$,*
2. *γ_j is unique in γ, and*
3. *for each $i < j$ we have $\gamma_i < \gamma_j$, i.e. γ_j is a left-to-right maximum in γ.*

We denote the collection of barred Fubini words associated with v by $\overline{\mathcal{F}}_v$.

For example,

$$\overline{\mathcal{F}}_0 = \{0\} \tag{27}$$

$$\overline{\mathcal{F}}_{00} = \{01, 0\overline{1}\} \tag{28}$$

$$\overline{\mathcal{F}}_{000} = \{011, 012, 0\overline{1}2, 01\overline{2}, 0\overline{1}\overline{2}, 021, 0\overline{2}1\}. \tag{29}$$

The sequence $|\overline{\mathcal{F}}_{0^n}|$ for $n \in \mathbb{N}$ begins $1, 1, 2, 7, 35, 226, \ldots$ and seems to appear in the OEIS as A014307 [Slo]. One way to define sequence A014307 is that it has exponential generating function

$$\sqrt{\frac{e^z}{2 - e^z}}. \tag{30}$$

This sequence is given several combinatorial interpretations in [Ren15]. It would be interesting to obtain a bijection between $\overline{\mathcal{F}}_{0^n}$ and one of the collections of objects in [Ren15]. See Figure 3 for more examples of barred Fubini words.

Given a barred Fubini word γ and a word $\pi \in \mathbb{P}^n$, we modify the dinv statistic slightly:

$$\begin{aligned} \operatorname{dinv}(\gamma, \pi) = \#\{1 \leq i < j \leq n : \gamma_i = \gamma_j, \pi_i > \pi_j\} \\ + \#\{1 \leq i < j \leq n : \gamma_i + 1 = \gamma_j, \pi_i < \pi_j, \gamma_j \text{ is not barred}\} \end{aligned} \tag{31}$$

[1] There are also infinitely many $\pi \in \mathbb{P}^n$, but this problem can be rectified with standardization [Hag08].

v	$\overline{\mathcal{F}}_v$
111	000
011	$100, \bar{1}00$
101	$010, 0\bar{1}0$
110	$001, 00\bar{1}$
001	$110, 120, 1\bar{2}0, \bar{1}20, \bar{1}\bar{2}0, 210, \bar{2}10$
010	$101, 102, 10\bar{2}, \bar{1}02, \bar{1}0\bar{2}, 201, \bar{2}01$
100	$011, 012, 0\bar{1}2, 01\bar{2}, 0\bar{1}\bar{2}, 021, 0\bar{2}1$
000	$011, 012, 0\bar{1}2, 01\bar{2}, 0\bar{1}\bar{2}, 021, 0\bar{2}1$

FIGURE 3. We have listed the barred Fubini words $\overline{\mathcal{F}}_v$ for each $v \in \{0,1\}^3$.

We also let $\mathrm{bar}(\gamma)$ be the number of barred entries in γ. We have the following result.

Theorem 3.1. *For $v \in \{0,1\}^n$,*

$$L_v = \sum_{\substack{\gamma \in \overline{\mathcal{F}}_v \\ \pi \in \mathbb{P}^n}} q^{\mathrm{area}(\gamma)+\mathrm{bar}(\gamma)} t^{\mathrm{dinv}(\gamma,\pi)} (1-q)^{-\mathrm{bar}(\gamma)-\chi(v=0^n)} x^\pi \tag{32}$$

where χ of a statement is 1 if the statement is true and 0 if it is false.

Proof. Assume, for now, that $v \neq 0^n$. Let $\overline{\mathcal{F}}_v^{(0)}$ denote the set of all $\gamma \in \mathbb{N}^n$ such that $\gamma_i = 0$ if and only if $v_i = 1$. For each $1 \leq k \leq n$, let $\overline{\mathcal{F}}_v^{(k)}$ be the set of vectors $\gamma \in \mathbb{N}^n$ such that

(1) $\gamma_i = 0$ if and only if $v_i = 1$,
(2) each number $0, 1, 2, \ldots, k$ appears in γ.

We also allow certain entries to be barred. Specifically, $\gamma_j \in \overline{\mathcal{F}}_v^{(k)}$ may be barred if

(1) $0 < \gamma_j \leq k$,
(2) γ_j is unique in γ, and
(3) for each $i < j$ we have $\gamma_i < \gamma_j$, i.e. γ_j is a left-to-right maximum in γ.

Note that $\overline{\mathcal{F}}_v^{(n)} = \overline{\mathcal{F}}_v$, and is therefore finite. For convenience, we set

$$\mathrm{wt}_{\gamma,\pi} = \mathrm{wt}_{\gamma,\pi}(x;q,t) = q^{\mathrm{area}(\gamma)+\mathrm{bar}(\gamma)} t^{\mathrm{dinv}(\gamma,\pi)} (1-q)^{-\mathrm{bar}(\gamma)} x^\pi. \tag{33}$$

where the dinv statistic is the one we defined for barred Fubini words. Our goal is to show that

$$\sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k-1)} \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} = \sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k)} \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} \tag{34}$$

for each $1 \leq k \leq n$. Then we can chain together these identities for $k = 1, 2, \ldots, n$ to obtain the desired result.

First, we remove the intersection $\overline{\mathcal{F}}_v^{(k-1)} \cap \overline{\mathcal{F}}_v^{(k)}$ from both summands in (34) to obtain the equivalent statement

$$\sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k-1)} \setminus \overline{\mathcal{F}}_v^{(k)} \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} = \sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k)} \setminus \overline{\mathcal{F}}_v^{(k)} \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi}\,. \tag{35}$$

Now we wish to describe the γ that appear in the left- and right-hand summands of (35). $\gamma \in \overline{\mathcal{F}}_v^{(k-1)}$ is not in $\overline{\mathcal{F}}_v^{(k)}$ if and only if it does not contain a k; similarly, $\gamma \in \overline{\mathcal{F}}_v^{(k)}$ is not in $\overline{\mathcal{F}}_v^{(k-1)}$ if and only if it contains a single k and that k is barred. This allows us to rewrite (35) as

$$\sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k-1)} \\ k \notin \gamma \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} = \sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k)} \\ \overline{k} \in \gamma \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} \,. \tag{36}$$

Specifically, for each subset $S \subseteq \{1, 2, \ldots, n\}$ we will show that

$$\sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k-1)} \\ k \notin \gamma \\ \gamma_i < k \Leftrightarrow i \in S \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} = \sum_{\substack{\gamma \in \overline{\mathcal{F}}_v^{(k)} \\ \overline{k} \in \gamma \\ \gamma_i < k \Leftrightarrow i \in S \\ \pi \in \mathbb{P}^n}} \mathrm{wt}_{\gamma,\pi} \,. \tag{37}$$

Then summing over all S will conclude the proof.

We consider the left-hand side of (37). Note that there cannot be any dinv between entries i and j if $\gamma_i < k$ and $\gamma_j > k$. In this sense, the entries i with $\gamma_i < k$ are independent of the columns j with $\gamma_j > k$. This allows us to write the left-hand side of (37) as a product

$$q^{n-|S|} L_{0^{n-|S|}} F_{v,S} \tag{38}$$

where $F_{v,S}$ is a certain symmetric function that accounts for all contribution to the weights coming from columns $i \in S$. The factor of q appears because each of the entries $j \notin S$ has an empty box in the diagram that is not counted by either of the other factors. Now we can use Lemma 2.1 to rewrite this product as

$$\frac{q^{n-|S|}}{1-q} L_{10^{n-|S|-1}} F_{v,S}. \tag{39}$$

Let m be the minimal index not in S. Our last goal is to show that the product in (39) is equal to the right-hand side of (37).

We note that, by the definition of dinv for barred words, there are no dinv pairs (i, j) with $i \in S$ and $j \notin S$, i.e. $\gamma_i < k$ and $\gamma_j \geq k$ for γ that appear in the sum on the right-hand side of (37). We also note that $L_{10^{n-|S|-1}}$ accounts for the contribution from columns $j \notin S$ except that it does not account for the bar on γ_m. This bar contributes a factor of $q/(1-q)$. Now there are $q^{n-|S|-1}$ columns with an extra box; these are the columns $j \notin S$ and $j \neq m$. The same polynomial $F_{v,S}$ accounts for the contributions of columns $i \in S$. Multiplying these together, we obtain (39).

Finally, we must address the case $v = 0^n$. In this case, we immediately use $L_{0^n} = (1-q)^{-1} L_{10^{n-1}}$ and then proceed as above. This is why Fubini words associated with 0^n have an "extra" zero at the beginning. This also slightly adjusts the weight of the summands, explaining the $\chi(v = 0^n)$ in the statement of the theorem. □

As in Section 2, we give a formula for computing $f_v(q, a, t)$ directly. Given a barred Fubini word γ, we define

$$\mathrm{dinv}_i(\gamma) = \#\{j < i : \gamma_j = \gamma_i\} + \#\{j > i : \gamma_j = \gamma_i + 1, \gamma_j \text{ is not barred}\}. \tag{40}$$

t^2			
t	qt	q^2t	
1	q	q^2	q^3

FIGURE 4. This is the Ferrers diagram of the partition $\mu = (4, 3, 1)$. In each cell we have written the monomial $q^i t^j$ that corresponds to the cell, yielding $B_\mu = \{1, q, q^2, q^3, t, qt, q^2t, t^2\}$.

Corollary 3.1.

$$f_v(q, a, t) = \sum_{\gamma \in \overline{\mathcal{F}}_v} q^{\operatorname{area}(\gamma)+\operatorname{bar}(\gamma)} (1-q)^{-\operatorname{bar}(\gamma)-\chi(v=0^n)} \prod_{i=1}^{n} \left(a + t^{\operatorname{dinv}_i(\gamma)}\right) \tag{41}$$

4. CONJECTURES

So far, we have used the inner product $\langle L_v, e_{n-d}h_d \rangle$ to compute $f_v(q, a, t)$; one might wonder if there is any value in studying the full symmetric function L_v. In this section, we conjecture that the link symmetric function L_v is closely related to the combinatorics of Macdonald polynomials, hinting at a stronger connection between Macdonald polynomials and link homology. Following [EH16], we must first define a "normalized" version of the link symmetric function L_v.

Definition 4.1.

$$\widetilde{L}_v = \widetilde{L}_v(x; q, t) = (1-q)^{n-|v|} L_v(x; q, t). \tag{42}$$

We could also define $\widetilde{L}_v$ in terms of diagrams; each box that contains a number contributes an additional factor of $1-q$. Theorem 3.1 implies that $\widetilde{L}_v$ has coefficients in $\mathbb{Z}[q, t]$, whereas the coefficients of L_v are elements of $\mathbb{Z}[[q, t]]$. We conjecture that the normalized link symmetric function $\widetilde{L}_v$ is closely connected to the Macdonald eigenoperators ∇ and Δ.

The modified Macdonald polynomials $\widetilde{H}_\mu$ form a basis for the ring of symmetric functions with coefficients in $\mathbb{Q}(q, t)$. They can be defined via triangularity relations of combinatorially [HHL05, Hag08]. Given a partition μ, let B_μ be the alphabet of monomials $q^i t^j$ where (i, j) ranges over the coordinates of the cells in the Ferrers diagram of μ. We compute an example in Figure 4.

Given a symmetric function F and a set of monomials $A = \{a_1, a_2, \ldots, a_n\}$, we let $F[A]$ be the result of setting $x_i = a_i$ for $1 \leq i \leq n$ and $x_i = 0$ for $i > n$. Then we define two operators on symmetric functions by setting, for $\mu \vdash n$,

$$\Delta_F \widetilde{H}_\mu = F[B_\mu] \widetilde{H}_\mu \tag{43}$$

$$\nabla \widetilde{H}_\mu = \Delta_{e_n} \widetilde{H}_\mu \tag{44}$$

and expanding linearly. Note that, for $\mu \vdash n$, $e_n[B_\mu]$ is simply the product of the n monomials in B_μ; we will sometime write T_μ for the product $e_n[B_\mu]$.

Conjecture 4.1.

$$\nabla p_{1^n} = \widetilde{L}_{0^n} \tag{45}$$

$$\Delta_{e_{n-1}} p_{1^n} = \sum_{\substack{v \in \{0,1\}^n \\ |v|=1}} \widetilde{L}_v \tag{46}$$

In fact, both conjectures follow from the conjecture that

$$\widetilde{L}_{v0} = \nabla p_1 \nabla^{-1} \widetilde{L}_v. \tag{47}$$

We should mention that Eugene Gorsky first noticed that the identity

$$\sum_{a=0}^{d} \langle \nabla p_{1^n}, e_{n-d} h_d \rangle \, a^d = (1-q)^n f_{0^n}(q,a,t) \tag{48}$$

seemed to hold and communicated this observation to the author via Jim Haglund. Gorsky's conjectured identity is a special case of Conjecture 4.1. It is also interesting to note that the operator in (47) appears in the setting of the Rational Shuffle Conjecture as $-\mathbf{Q}_{1,1}$ [BGLX15].

Proof. We prove that (47) implies (45) and (46). The fact that (47) implies (45) is clear. For the second implication, consider $v \in \{0,1\}^n$ with $|v| = 1$. Say k is the unique position such that $v_k = 1$. By (45), $\widetilde{L}_{0^{k-1}} = \nabla p_{1^{k-1}}$. By definition, $\widetilde{L}_{0^{k-1}1}$ considers γ such that $\gamma_i = 0$ if and only if $i = k$. It follows that π_k cannot be involved in any dinv pairs, and that γ_k contributes no new area. Therefore

$$\widetilde{L}_{0^{k-1}1} = p_1 \nabla p_{1^{k-1}}. \tag{49}$$

Using (45) again, we get

$$\widetilde{L}_{0^{k-1}10^{n-k}} = \nabla p_{1^{n-k}} \nabla^{-1} p_1 \nabla p_{1^{k-1}}. \tag{50}$$

We define the *Macdonald Pieri coefficients* $d_{\mu,\nu}$ by

$$p_1 \widetilde{H}_\nu = \sum_{\mu \leftarrow \nu} d_{\mu,\nu} \widetilde{H}_\mu. \tag{51}$$

where the sum is over partitions μ obtained by adding a single cell to ν. Given a standard tableau τ, let $\mu^{(i)}$ be the partition obtained by taking the cells containing $1, 2, \ldots, i$ in τ. Then each $\mu^{(i+1)}$ is obtained by adding a single cell to $\mu^{(i)}$. Let d_τ denote the product of the Macdonald Pieri coefficients

$$d_\tau = d_{\mu^{(1)},\emptyset} d_{\mu^{(2)},\mu^{(1)}} \ldots d_{\mu^{(n)},\mu^{(n-1)}}. \tag{52}$$

Now we can express the right-hand side of (50) as

$$\nabla p_{1^{n-k}} \nabla^{-1} p_1 \nabla \sum_{\nu \vdash k-1} \sum_{\tau \in \mathrm{SYT}(\nu)} d_\tau \widetilde{H}_\nu \tag{53}$$

$$= \nabla p_{1^{n-k}} \nabla^{-1} p_1 \sum_{\nu \vdash k-1} \sum_{\tau \in \mathrm{SYT}(\nu)} d_\tau T_\nu \widetilde{H}_\nu \tag{54}$$

$$= \nabla p_{1^{n-k}} \sum_{\lambda \vdash k} \sum_{\tau \in \mathrm{SYT}(\lambda)} d_\tau B_\lambda(\tau, n)^{-1} \widetilde{H}_\lambda \tag{55}$$

where by $B_\lambda(\tau, n)$ we mean the monomial $q^i t^j$ associated to the cell containing n in τ. Completing the computation, we get

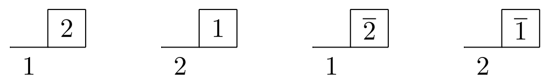
$$\sum_{\mu \vdash n} \widetilde{H}_\mu \sum_{\tau \in \mathrm{SYT}(\mu)} d_\tau \prod_{i \neq k} B_\mu(\tau, i). \tag{56}$$

Summing over all k, we obtain $\Delta_{e_{n-1}} p_{1^n}$. □

As an example of our conjecture, we can use Sage to compute

$$\langle \nabla p_{1,1}, p_{1,1} \rangle = 1 + q + t - qt. \tag{57}$$

This expression should equal $\left\langle \widetilde{L}_{00}, p_{1,1} \right\rangle$ by Conjecture 4.1. To compute this inner product using Theorem 3.1, we consider the barred Fubini words 01 and $0\overline{1}$, each of which can receive labels $\pi = 12$ or 21. The corresponding diagrams are



where we have moved the bars from γ_i to the corresponding π_i. The weights of these diagrams coming from Theorem 3.1 are

$$\frac{t}{1-q} \qquad \frac{1}{1-q} \qquad \frac{q}{(1-q)^2} \qquad \frac{q}{(1-q)^2} \tag{58}$$

respectively. After multiplying by the normalizing factor $(1-q)^2$ to go from L_{00} to $\widetilde{L}_{00}$, we sum the resulting weights to get

$$(1-q)t + 1 - q + q + q = 1 + q + t - qt \tag{59}$$

as desired.

After reading an earlier version of this paper, François Bergeron contacted the author with the following additional conjectures.

Conjecture 4.2 (Bergeron, 2016)**.**

$$L_{v0} = L_{1v} + qL_{0v} \tag{60}$$

$$L_{0^n} = \sum_{v \in \{0,1\}^k} q^{n-|v|} L_{v0^{n-k}} \tag{61}$$

$$t\,(L_{u011v} - L_{u101v}) = L_{u101v} - L_{u110v} \tag{62}$$

$$\widetilde{L}_{0^a 1^b 0^c} = \nabla p_{1^c} \nabla^{-1} \widetilde{H}_{1^b} \nabla p_{1^a} \tag{63}$$

$$L_{1^a 0 1^b} = \frac{t^a - 1}{t^{a+b} - 1} \left[\nabla p_1 \nabla^{-1}, \widetilde{H}_{1^{a+b}} \right] + \widetilde{H}_{1^{a+b}} p_1 \tag{64}$$

where the bracket represents the Lie bracket and operators are applied to 1 if nothing is explicitly specified. Bergeron also observed that $L_v(x; q, 1+t)$ is e-positive. (For more context on this last statement, see Section 4 of [Ber16]*.)*

It is clear that (60) implies (61). We do not know of any other relations between these conjectures. We close with two more open questions.

(1) Is there a Macdonald eigenoperator expression for $\widetilde{L}_v$ for other v? Perhaps we can use ideas from the Rational Shuffle Conjecture [BGLX15], recently proved by Mellit [Mel16].

(2) Can we generalize our conjecture for ∇p_{1^n} to "interpolate" between our conjecture and the Shuffle Theorem [CM15], or maybe the Square Paths Theorem [Ser16]?

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