

An Improved Lower Bound for n -Brinkhuis k -Triples

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Abstract

Let s_n be the number of words in the ternary alphabet $\Sigma = \{0, 1, 2\}$ such that no subword (or factor) is a square (a word concatenated with itself, e.g., 11, 1212, or 102102). From computational evidence, s_n grows exponentially at a rate of about 1.317277^n . While known upper bounds are already relatively close to the conjectured rate, effective lower bounds are much more difficult to obtain. In this paper, we construct a 54-Brinkhuis 952-triple, which leads to an improved lower bound on the number of n -letter ternary squarefree words: $952^{n/53} \approx 1.1381531^n$.

1 Introduction

A word w of length n is a string of n symbols from an alphabet Σ . A word w is said to be *squarefree* if it does not contain an adjacent repetition of a *subword* (or *factor*), i.e., w cannot be written as $axxb$ for nonempty subwords a , x , and b . In the field of combinatorics on words, the literature on pattern-avoiding words is vast and there has always been much

progress in the study of powerfree words such as the *binary cubefree* and *ternary squarefree* words (see [10, 16]).

It is easy to see that there are only six nonempty binary squarefree words: $\{0, 1, 01, 10, 101, 010\}$. Using the Prouhet-Thue-Morse sequence (see [13]) the number of ternary squarefree words was proven to be infinite.

We denote by s_n the exponentially growing number [14] of ternary squarefree words of length n [1, 13]. We denote by $\mathcal{A}(n)$ the set of ternary squarefree words of length n .

In Section 2, we define basic properties of n -Brinkhuis k -triples. In Section 3, we define how we searched for the 54-Brinkhuis 952-triple. In Section 4, we produce the newly discovered 54-Brinkhuis 952-triple. In Section 5, we describe how to get the code that found the specialized Brinkhuis triple of Section 4 and how to run it.

2 n -Brinkhuis k -triples

In this section, we define a n -Brinkhuis k -triple, prove a theorem about the lower bound on the growth rate s , and provide a history of estimates for the lower bound.

Definition 1. An n -Brinkhuis k -triple is a set $\mathcal{B} = \{\mathcal{B}^0, \mathcal{B}^1, \mathcal{B}^2\}$ of three sets of words $\mathcal{B}^i = \{w_j^i \mid 1 \leq j \leq k\}$. The w_j^i are squarefree words of length n such that for all squarefree words $i_1 i_2 i_3$ with $i_1, i_2, i_3 \in \{0, 1, 2\}$ has the property that the word $w_{j_1}^{i_1} w_{j_2}^{i_2} w_{j_3}^{i_3}$ of length $3n$ with $j_1, j_2, j_3 \in \{1, 2, \dots, k\}$ is also squarefree.

An example of an 18-Brinkhuis 2-triple [9] is given by,

$$\mathcal{B} = \left\{ \begin{array}{l} \mathcal{B}^0 = \{210201202120102012, \quad 210201021202102012\}, \\ \mathcal{B}^1 = \{021012010201210120, \quad 021012102010210120\}, \\ \mathcal{B}^2 = \{102120121012021201, \quad 102120210121021201\} \end{array} \right\}.$$

The lower bound on the growth rate is given in the following theorem [5]:

Theorem 2. *The existence of a special n -Brinkhuis k -triple implies that the lower bound on the growth rate of the ternary squarefree words is*

$$k^{1/(n-1)} \leq s = \lim_{m \rightarrow \infty} (s_m)^{1/m}.$$

Proof. We define the a set of uniformly growing morphisms by

$$\rho : \begin{cases} 0 \rightarrow w_{j_0}^0, \\ 1 \rightarrow w_{j_1}^1, \\ 2 \rightarrow w_{j_2}^2, \end{cases}$$

where $1 \leq j_0, j_1, j_2 \leq k$. As proven in [4, 7, 12], the ρ are squarefree morphisms mapping each squarefree word of length m to k^m squarefree words of length nm . Thus, existence of an n -Brinkhuis k -triple indicates that

$$s_{mn}/s_m \geq k^m, \quad \forall m, n \geq 1.$$

Since $s = \lim_{m \rightarrow \infty} (s_m)^{1/m}$,

$$s^{n-1} = \lim_{m \rightarrow \infty} \left(\frac{s_{mn}}{s_m} \right)^{1/m} \geq k,$$

which yields the lower bound of $s \geq k^{1/(n-1)}$. □

A history of estimates for the lower bound for s is given in Table 1. As is obvious from Theorem 2, the discovery of a n -Brinkhuis k -triple for a new pair (n, k) potentially gives us a new a lower bound for s .

n	k	Lower bound	Year	Authors
25	2	$2^{n/24} \approx 1.0293022^n$	1983	Brinkhuis [5]
22	2	$2^{n/21} \approx 1.0335578^n$	1983	Brandenburg [4]
18	2	$2^{n/17} \approx 1.0416160^n$	1998	Ekhad and Zeilberger [9]
41	65	$65^{n/40} \approx 1.1099996^n$	2001	Grimm [11]
43	110	$110^{n/42} \approx 1.1184191^n$	2003	Sun [17]
54	952	$952^{n/53} \approx 1.1381531^n$	2016	Sollami, Douglas, and Liebmann

Table 1: Lower bounds.

3 Searching for n -Brinkhuis k -triples

In this section we describe how we searched for n -Brinkhuis k -triples.

We can pare down the search by systematically determining the prefixes and suffixes of the words in a special n -Brinkhuis k -triple. Grimm [11] proved that only two classes of special n -Brinkhuis k -triples must be searched, namely

$$\mathcal{A}_1(n) = \{w \in \mathcal{A}(n) \mid w = 012021\{012\} * 120210\} \subseteq \mathcal{A}(n)$$

and

$$\mathcal{A}_2(n) = \{w \in \mathcal{A}(n) \mid w = 012102\{012\} * 201210\} \subseteq \mathcal{A}(n),$$

where recall that $\mathcal{A}(n)$ is the set of ternary squarefree words of length n .

Let \bar{w} be the reversal of symbols in w . We use this notation to be consistent with earlier papers in this field, e.g., [11, 17] (it is sometimes denoted by w^R by other authors). For example, if

$$w = 0122, \bar{w} = 2210$$

and an example palindrome is

$$w = 2112 = \bar{w}.$$

We denote the number of potential words, palindromes, and nonpalindromes for each set $\mathcal{A}_i(n)$, $i \in \{1, 2\}$, by

$$\begin{aligned} a_i(n) &= |\mathcal{A}_i(n)|, \\ a_{ip}(n) &= |\{w \in \mathcal{A}_i(n) \mid w = \overline{w}\}|, \\ a_{in}(n) &= |\{w \in \mathcal{A}_i(n) \mid w \neq \overline{w}\}|. \end{aligned}$$

Clearly, there are no palindromic squarefree words of even length. Thus, $a_{1p}(2n) = a_{2p}(2n) = 0$, $a_{1n}(2n) = a_1(2n)/2$, and $a_{2n}(2n) = a_2(2n)/2$ [11]. If a word in $\mathcal{A}_1(n)$ or $\mathcal{A}_2(n)$ is a member of a special n -Brinkhuis k -triple, then it must at least generate a Brinkhuis triple by itself, which motivates the following definition:

Definition 3. A word w is *admissible* if $\{w, \tau(w), \tau^2(w)\}$ is a special n -Brinkhuis k -triple, where τ is the permutation

$$\tau : \begin{cases} 0 & \rightarrow & 1, \\ 1 & \rightarrow & 2, \\ 2 & \rightarrow & 0. \end{cases} \quad (1)$$

As before, we denote the number of admissible words, palindromes, and nonpalindromes for each set $\mathcal{A}_i(n)$, $i \in \{1, 2\}$, by

$$\begin{aligned} b_i(n) &= |\{w \in \mathcal{A}_i(n) \text{ and } w \text{ is admissible}\}|, \\ b_{ip}(n) &= |\{w \in \mathcal{A}_i(n) \mid w = \overline{w} \text{ and } w \text{ is admissible}\}|, \\ b_{in}(n) &= |\{w \in \mathcal{A}_i(n) \mid w \neq \overline{w} \text{ and } w \text{ is admissible}\}|. \end{aligned}$$

The strategy we used to find a special n -Brinkhuis k -triple begins by enumerating the set of all admissible words of length n . From this enumeration we determine the largest subset in which any three words w_1, w_2, w_3 , form a special n -Brinkhuis triple.

The method we used to find a special n -Brinkhuis k -triple is summarized below in three steps:

Step 1. Find all admissible words in $\mathcal{A}_1(n)$ and $\mathcal{A}_2(n)$.

Step 2. Find all triples of admissible words that generate a special n -Brinkhuis k -triple.

Step 3. Find the largest set of admissible words such that all three-elemental subsets are contained in our list of admissible triples.

Steps 1 and 2 are essentially precomputations which involve checking the squarefreeness of words. A naive algorithm for detecting squares has time complexity of order $O(n^3)$ for words of length n and a fixed length alphabet. This algorithm was first improved to of order $O(n \log n)$ [7] and then further improved to of order $O(n)$ [1].

Experimentally it seems that \mathcal{A}_1 is more likely to provide maximum sized n -Brinkhuis triples for large n than generators from the set \mathcal{A}_2 and so we have focused our search to this specific class [11]. It is also simpler to find n -Brinkhuis k -triples in $\mathcal{A}_1(n)$ where n is even since a maximum number of generators does not necessarily give the largest Brinkhuis triple (i.e., unless we know that none of the words are palindromes, as they are for even n).

It is in the Step 3 that our main difficulty becomes apparent. The way we found n -Brinkhuis k -triples involved solving a purely combinatorial problem that is an instance of the NP-complete maximum clique problem for hypergraphs [6]. A maximal hyperclique in a hypergraph on n vertices with hyperedges of cardinality at most \aleph can be found using a branching algorithm in $\mathcal{O}(2^{\kappa n})$ time for some $\kappa < 1$, depending only on \aleph [2].

4 A 54-Brinkhuis 952-triple

Theorem 4. *A special 54-Brinkhuis 952-triple exists, and thus shows*

$$s \geq 952^{1/53} \approx 1.1381531 > 110^{1/42} \approx 1.1184191.$$

Proof. The proof is by a computational construction of a special Brinkhuis triple. \mathcal{B}^0 is explicitly listed below. \mathcal{B}^1 is constructed by applying the τ permutation 1 on \mathcal{B}^0 . \mathcal{B}^2 is constructed by applying the τ permutation on \mathcal{B}^1 . All three sets are available as plain ASCII files on the journal web site and [8].

Practical algorithms have been developed to solve the maximum clique problem (see [3] for a comprehensive survey on methods of finding maximum cliques). These methods were adapted to solve the corresponding problem for hypergraphs. We used the *Random Hyperclique Search algorithm* (RHCS) to perform our computer searches for maximum hypercliques [15].

Formally, the first 476 elements of \mathcal{B}^0 are given below. The remaining 476 elements are reversals of the first 476 elements.

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□

5 Code availability

The 54-Brinkhuis 952-triple can be verified using the code and script to run it found at [8]. The code is general and can be used to find special Brinkhuis triples for general values of n . While the code is fast for small values of n , e.g., $n \leq 35$, it will take a *very* long time for $n = 54$. In addition, the output files will take Terabytes of disk space.

A sample build and run of the code for $n = 35$ is the following:

```
% script log
Script started on Sat Apr 16 19:20:23 2016
% make runit N=35
clang -w -O3 brinkhuis.c -o brinkhuis
clang -w -O3 brinkhuis2t1.c -o brinkhuis2t1
clang -w -O3 brinkhuis2t2.c -o brinkhuis2t2
./doit 35
Success [ 1]: 012021020102120102012021020120102120210
Success [ 2]: 01202102010212010201210120102120210
```

Success [3]: 01202102010212012101202120102120210
 Success [4]: 01202102010212021012021201020120210 (palindromic)
 Success [5]: 01202102012101201020121020102120210
 Success [6]: 01202102012101202120102012102120210
 Success [7]: 01202102012101202120121020102120210
 Success [8]: 01202102012102120210121020102120210
 Success [9]: 01202120102012101201021012102120210
 Success [10]: 01202120102012101202102012102120210
 Success [11]: 01202120102012102010210120102120210
 Success [12]: 01202120102012102010212012102120210
 Success [13]: 01202120102012102120121020102120210 (palindromic)
 Success [14]: 01202120102101210201210120102120210 (palindromic)
 Success [15]: 01202120102120210120212012102120210
 Success [16]: 01202120102120210201021012102120210
 Success [17]: 01202120121020102120102012102120210 (palindromic)
 Success [18]: 01210201021012010212021012010201210
 Success [19]: 01210201021012010212021012021201210
 Success [20]: 01210201021012021020120212010201210
 Success [21]: 01210201021012021201020121021201210
 Success [22]: 01210201021012021201210212010201210
 Success [23]: 01210201021012102120102012021201210
 Success [24]: 01210201021012102120121012010201210 (palindromic)
 Success [25]: 01210201021012102120210201021201210
 Success [26]: 01210201021201020120210121021201210
 Success [27]: 01210201021201020120210201021201210
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 Success [29]: 01210201021201210212021012021201210
 Success [30]: 01210201021202101201020121021201210
 Success [31]: 01210201021202101201021012021201210
 Success [32]: 01210201021202102010210121021201210
 Success [33]: 01210201021202102012101201021201210
 Success [34]: 01210212010201202101210201021201210
 Success [35]: 01210212010201210120102012021201210
 Success [36]: 01210212010201210120210121021201210
 Success [37]: 01210212012101202120102012021201210
 Success [38]: 01210212012102010212021012021201210
 Success [39]: 01210212012102012021020121021201210 (palindromic)
 Success [40]: 01210212021020102120102012021201210 (palindromic)
 Done: a1=109, a1p= 9, a1n= 50; b1= 30, b1p= 4, b1n= 13
 a2=142, a2p= 6, a2n= 68; b2= 43, b2p= 3, b2n= 20
 Generated a1 and a2 files.
 17 admissible words of length 35 read in

admissible triples:
328 admissible triples found
Generated t1.
23 admissible words of length 35 read in
admissible triples:
483 admissible triples found
Generated t2.
% exit
exit

Script done on Sat Apr 16 19:20:39 2016

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