Quantum Information Processing manuscript No.
(will be inserted by the editor)

Generalized Ramsey numbers through adiabatic quantum optimization

Mani Ranjbar, William G. Macready,
Lane Clark, Frank Gaitan

June 6, 2016

Abstract Ramsey theory is an active research area in combinatorics whose central theme is the emergence of order in large disordered structures, with Ramsey numbers marking the threshold at which this order first appears. For generalized Ramsey numbers $r(G,H)$, the emergent order is characterized by graphs G and H. In this paper we: (i) present a quantum algorithm for computing generalized Ramsey numbers by reformulating the computation as a combinatorial optimization problem which is solved using adiabatic quantum optimization; and (ii) determine the Ramsey numbers $r(\mathscr{T}_m, \mathscr{T}_n)$ for trees of order $m,n = 6,7,8$, most of which were previously unknown.

Keywords Adiabatic quantum algorithms, generalized Ramsey numbers, tree Ramsey numbers, Ramsey theory, combinatorial optimization

PACS 03.67.Ac, 02.10.Ox, 89.75.Hc

Mathematics Subject Classification (2000) 05C55, 05D10, 68Q12, 81P68, 90C27

1 Introduction

To get a taste of the type of problem considered in Ramsey theory, consider an arbitrary gathering of N people. One might wonder whether there is a group of m people at the party who are all mutual acquaintances, or a group of n people who are all mutual strangers. Using Ramsey theory [1,2] it can be shown that once the party size

Frank Gaitan
Laboratory for Physical Sciences, 8050 Greenmead Dr, College Park, MD 20740
Tel.: 301-935-6531
Fax: 301-935-6723
E-mail: fgaitan@lps.umd.edu

Mani Ranjbar, William G. Macready
D-Wave Systems, Inc., 3033 Beta Avenue, Burnaby, British Columbia V5G 4M9, Canada

Lane Clark
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901-4401

N reaches a threshold size $r(m,n)$, *every* party with $N \geq r(m,n)$ people *must* contain either m mutual acquaintances or n mutual strangers. The unforced and guaranteed emergence of order (viz. a cluster of m mutual friends or n mutual strangers) upon reaching the threshold size is an essential characteristic of problems in Ramsey theory. The threshold $r(m,n)$ is an example of a two-color Ramsey number.

It proves fruitful to represent the N-person party problem by an N-vertex graph. Each party-goer is identified with a vertex, and a red (blue) edge is drawn between a pair of vertices when the corresponding people are acquaintances (strangers). Since any two people attending will either know each other or not, every pair of vertices is joined by a red or blue edge. The party graph is thus the complete graph K_N (all vertex pairs joined by an edge) with edges colored red or blue. Notice that the group of m mutual acquaintances (strangers) corresponds to a red K_m (blue K_n) subgraph of K_N. The Ramsey theory result for the party problem becomes a theorem in graph theory [3]: if the order N of the complete graph K_N satisfies $N \geq r(m,n)$, then every red/blue coloring of the edges of K_N contains either a red K_m or a blue K_n subgraph.

The classical two-color Ramsey numbers $r(m,n)$ are extremely difficult to calculate, with only 9 values currently known [4]. It was once hoped that by considering proper subgraphs $G \subset K_m$ and $H \subset K_n$, generalized Ramsey numbers $r(G,H)$ might prove easier to calculate and inspire new techiques that would also work for $r(m,n) \equiv r(K_m,K_n)$. Although these hopes have not been borne out to date, the study of generalized Ramsey numbers is now an active, well established part of Ramsey theory. Formally, for given graphs G and H, the generalized Ramsey number $r(G,H)$ is defined to be the smallest positive integer p for which every red/blue edge-coloring of the complete graph K_p contains either a red G or a blue H subgraph [1,3]. Generalized Ramsey numbers can also be defined for families of graphs $\mathscr{G}$ and $\mathscr{H}$. Such families typically partition into graph isomorphism (GI) classes $\{\mathscr{G}^i \subset \mathscr{G}\}$ and $\{\mathscr{H}^j \subset \mathscr{H}\}$, and associated with each pair of classes is a generalized Ramsey number $r(\mathscr{G}^i, \mathscr{H}^j)$. We write $r(\mathscr{G}, \mathscr{H})$ for the set of all such Ramsey numbers. Early tabulations of generalized Ramsey numbers with G and H of order at most 5 appear in Refs. [5,6,7,8], while Ref. [4] presents the current state-of-the-art.

In this paper we present a quantum algorithm for computing generalized Ramsey numbers. We reformulate the computation as a combinatorial optimization problem which is solved using adiabatic quantum optimization; and determine the Ramsey numbers $r(\mathscr{T}_m, \mathscr{T}_n)$ for trees of order $m,n = 6,7,8$, most of which were previously unknown. The quantum algorithm presented here generalizes an earlier adiabatic quantum algorithm for classical Ramsey numbers $r(m,n)$ [9] which was used to *experimentally* determine a number of small Ramsey numbers [10].

The structure of this paper is as follows. In Section 2 we summarize the basic concepts from graph theory that will be needed in the remainder of the paper. Section 3 then shows how the computation of $r(G,H)$ can be transformed into a combinatorial optimization problem whose solution is found using adiabatic quantum optimization [11]. Calculation of the generalized Ramsey numbers $r(\mathscr{T}_m, \mathscr{T}_n)$ for trees of order $m,n = 6,7,8$ appears in Section 4. In the interests of clarity, this section focuses on the simplest case with $m,n = 6$; the remaining tree Ramsey numbers appear in Appendix B. The paper closes with a summary of our results in Section 5, and for the reader's convenience, we collect previously known results for tree Ramsey numbers in Appendix A.

2 Preliminaries

We begin by reviewing those ideas from graph theory [3] that will be central to our discussion. In the following all sets will be finite. We denote the cardinality of the set X by $|X|$, and the set of all 2-subsets of X by $X^{(2)}$.

A graph G is specified by a non-empty set of vertices V_G and a set of edges $E_G \subseteq V_G^{(2)}$. The order (size) of G is denoted $|V_G|$ ($|E_G|$). A graph G' is a subgraph of a graph G iff $V_{G'} \subseteq V_G$ and $E_{G'} \subseteq E_G$. We denote by K_n, P_n, and $K_{1,n-1}$ the complete graph, path, and star of order n, respectively. Lastly, we denote by $\mathscr{L}_n$ the set of $2^{\binom{n}{2}}$ distinct vertex-labelled graphs with fixed n-vertex set, and by $\mathscr{U}_n$ the set of vertex-unlabelled graphs of order n.

Two graphs G_1 and G_2 are isomorphic ($G_1 \cong G_2$) iff there there exists a bijection $f : V_{G_1} \to V_{G_2}$ such that $\{u,v\} \in E_{G_1}$ iff $\{f(u), f(v)\} \in E_{G_2}$. The bijection f is called an isomorphism of G_1 and G_2. We write $G_1 \sqsubseteq G_2$ iff there exists a subgraph $G' \subseteq G_2$ such that $G_1 \cong G'$.

A red-blue coloring of the edges of a graph G is a map $c : E_G \to \{\text{red}, \text{blue}\}$. Given a graph G and an edge-coloring c of G, the red subgraph $G^r(c)$ has vertex set V_G and edge set $\{e \in E_G : c(e) = \text{red}\}$. Similarly, the blue subgraph $G^b(c)$ has vertex set V_G and edge set $\{e \in E_G : c(e) = \text{blue}\}$. Finally, we define the *arrow* relation between graphs F, G, and H. We write $F \to (G,H)$ iff, for all edge-colorings $c : E_F \to \{\text{red}, \text{blue}\}$, either $G \sqsubseteq F^r(c)$ or $H \sqsubseteq F^b(c)$.

With these definitions in place, we are now in a position to define the generalized Ramsey numbers.

Definition 1 Given graphs G and H, the generalized Ramsey number $r(G,H)$ is:

$$r(G,H) = \min\{n \in \mathbb{P} : K_n \to (G,H)\}.$$

A red-blue edge-colored graph F is said to be (G,H)-critical iff: (i) F has order $r(G,H)-1$, and (ii) $G \not\sqsubseteq F^r(c)$ and $H \not\sqsubseteq F^b(c)$.

We collect literature results pertaining to generalized Ramsey numbers for certain families of trees in Appendix A. These results allow us to determine which of the tree Ramsey numbers calculated in Section 4 and Appendix B are new, and provide checks for the rest.

3 Quantum algorithm for generalized Ramsey numbers

In this Section we present an adiabatic quantum algorithm for computing generalized Ramsey numbers. We first show (Section 3.1) how a computation of the generalized Ramsey number $r(G,H)$ can be transformed into a combinatorial optimization problem (COP) which is then solved (Section 3.2) using adiabatic quantum optimization.

The resulting algorithm generalizes an earlier adiabatic quantum algorithm for classical two-color Ramsey numbers [9] which has been used to *experimentally* determine a number of small Ramsey numbers [10].

3.1 Generalized Ramsey numbers through combinatorial optimization

Red-blue edge-colorings of K_N are an essential ingredient in the definition of $r(G,H)$. Each such coloring can be represented by a Boolean string

$$e = (e_{1,2}, \ldots, e_{i,j}, \ldots e_{N-1,N})$$

of length $\binom{N}{2}$, where $e_{i,j} = 1$ (0) if the edge $\{i,j\}$ (with $i<j$) is colored red (blue). For a given coloring e of K_N, let $K_N^r(e)$ and $K_N^b(e)$ denote, respectively, its red and blue subgraphs.

Let e be a coloring of K_N. The following procedure counts the number of red subgraphs of $K_N^r(e)$ that are isomorphic to G. To begin, choose $|V_G|$ vertices from the N vertices of K_N, and denote this choice by $S_\alpha = \{v_1, \ldots v_{|V_G|}\}$. Next, let $K_\alpha^r(e)$ be the subgraph of $K_N^r(e)$ with vertex set S_α and edge set $E_\alpha^r(e) = \{\,\{i,j\} \,|\, (i,j \in S_\alpha) \bigwedge (i < j) \bigwedge (e_{i,j} = 1)\,\}$. We show below that the following Boolean function evaluates to 1 (True) if $G \cong K_\alpha^r(e)$ is True, and to 0 (False) otherwise:

$$f[G \cong K_\alpha^r(e)] = \bigvee_{\pi \in Sym(V_G)} \; \bigwedge_{\{i,j\} \in E_G} e_{\pi(i),\pi(j)}. \tag{1}$$

Here V_G (E_G) is the vertex (edge) set of G; and $Sym(V_G)$ is the symmetric group on V_G. Notice that if $G \cong K_\alpha^r(e)$ is True, there exists a permutation π that transforms $V_G \to S_\alpha$ and preserves adjacency so that $e_{\pi(i),\pi(j)} = 1$ iff $\{i,j\} \in E_G$. Thus the conjunction over E_G evaluates to 1 for this permutation, and so the disjunction evaluates to 1. On the other hand, if $G \cong K_\alpha^r(e)$ is False, no permutation π exists which preserves adjacency, and so for each permutation, at least one $e_{\pi(i),\pi(j)} = 0$ for $\{i,j\} \in E_G$. The conjunction thus evaluates to 0 for all permutations π, and the disjunction then evaluates to 0. Summing $f[G \cong K_\alpha^r(e)]$ over all vertex choices S_α gives the number of red subgraphs of $K_\alpha^r(e)$ that are isomorphic to G. Denoting this sum as $\mathscr{O}_N(e;G)$, we have

$$\mathscr{O}_N(e;G) = \sum_{S_\alpha} f[G \cong K_\alpha^r(e)]. \tag{2}$$

In a similar manner, the number of blue subgraphs of $K_N^b(e)$ isomorphic to H is

$$\mathscr{O}_N(e;H) = \sum_{S_\beta} f[H \cong K_\alpha^b(e)], \tag{3}$$

where: (i) $S_\beta = \{v_1, \ldots, v_{|V_H|}\}$ is a choice of $|V_H|$ vertices from the N vertices of K_N; (ii) $K_\beta^b(e)$ is the subgraph of $K_N^b(e)$ with vertex set S_β and edge set $E_\beta^b(e) = \{\,\{i,j\} \,|\, (i,j \in S_\beta) \bigwedge (i<j) \bigwedge (e_{i,j} = 0)\,\}$; and

$$f[H \cong K_\beta^b(e)] = \bigvee_{\pi \in Sym(V_G)} \; \bigwedge_{\{i,j\} \in E_G} \overline{e}_{\pi(i),\pi(j)}, \tag{4}$$

where $\overline{e}_{\pi(i),\pi(j)} = 1 - e_{\pi(i),\pi(j)}$.

We now define an objective function $\mathscr{O}_N(e;G,H)$ which assigns to each coloring e (of K_N) the total number of red and blue subgraphs it contains that are, respectively, isomorphic to G and H:

$$\mathscr{O}_N(e;G,H) = \mathscr{O}_N(e;G) + \mathscr{O}_N(e;H). \tag{5}$$

From Ramsey theory we know that if $N < r(G,H)$, then there is a coloring e_* for which $G \not\sqsubseteq K_N^r(e_*)$ and $H \not\sqsubseteq K_N^b(e_*)$. For this coloring the objective function vanishes and so

$$\min_e [\mathscr{O}_N(e;G,H)] = 0, \qquad (N < r(G,H)). \tag{6}$$

On the other hand, if $N \geq r(G,H)$, we know that $K_N \to (G,H)$ and so

$$\min_e [\mathscr{O}_N(e;G,H)] > 0, \qquad (N \geq r(G,H)). \tag{7}$$

The above discussion suggests the following COP for $r(G,H)$. Given graphs G and H, and a positive integer N, find a coloring e_* of K_N that minimizes the objective function $\mathscr{O}_N(e;G,H)$. As we have just seen, if $N < r(G,H)$, the minimum value of the objective function will be 0, while if $N \geq r(G,H)$, the minimum value will be positive. This motivates the following classical optimization algorithm for finding $r(G,H)$ which will guide our construction of the quantum algorithm in Section 3.2.

1. *Choose N to be a strict lower bound for $r(G,H)$.* In principle, the probabilistic method [12] can always be used to produce such a lower bound, though in some cases, such lower bounds may already be available in the literature (e. g., see Ref. [4]).
2. *Solve the COP for the minimum value of $\mathscr{O}_N(e;G,H)$.* Since $N < r(G,H)$ we know that $\min_e [\mathscr{O}_N(e;G,H)] = 0$.
3. *Increment $N \to N+1$ and determine the minimum value of $\mathscr{O}_N(e;G,H)$ for this new N.* If the minimum is zero, continue incrementing $N \to N+1$ and finding the minimum value of the new objective function until $\min [\mathscr{O}_N(e;G,H)]$ first becomes positive. When this first occurs, the algorithm returns the current value of N as the Ramsey number $r(G,H)$ since this is the smallest N for which $K_N \to (G,H)$.

We next show how this classical optimization algorithm can be promoted to an adiabatic quantum optimization for $r(G,H)$.

3.2 Adiabatic quantum algorithm for $r(G,H)$

Here we show how the classical optimization algorithm for $r(G,H)$ (Section 3.1) can be converted into an quantum algorithm.

The adiabatic quantum optimization (AQO) algorithm [11] exploits the adiabatic dynamics of a quantum system to solve COPs. The AQO algorithm uses the objective function for the COP to define a problem Hamiltonian H_P whose ground-state subspace encodes all optimal solutions. The algorithm evolves the state of an L-qubit register from the ground-state of an initial Hamiltonian H_i to the ground-state of H_P with probability approaching 1 in the adiabatic limit. An appropriate measurement at the end of the adiabatic evolution yields a solution of the COP almost certainly. The time-dependent Hamiltonian $H(t)$ for AQO is

$$H(t) = A(t/T)H_i + B(t/T)H_P, \tag{8}$$

where T is the algorithm runtime, adiabatic dynamics corresponds to $T \to \infty$, and $A(t/T)$ $[B(t/T)]$ is a positive monotonically decreasing [increasing] function with $A(1) = 0$ $[B(0) = 0]$.

The point of departure for converting the classical optimization algorithm for $r(G,H)$ into an adiabatic quantum algorithm is the set of binary edge-coloring strings e introduced in Section 3.1 for graphs of order N. Each of the $L = \binom{N}{2}$ bits in e is promoted to a qubit so that the adiabatic quantum algorithm uses L qubits. The 2^L strings e are used to label the 2^L computational basis states (CBS) that span the L-qubit Hilbert space: $e \to |e\rangle = |e_0 \cdots e_{L-1}\rangle$, with $e_i = 0,1$ for $i = 0,\ldots,L-1$. The problem Hamiltonian H_P is defined to be diagonal in the computational basis $|e\rangle$, with eigenvalue $\lambda(e) = \mathscr{O}_N(e;G,H)$, where $\mathscr{O}_N(e;G,H)$ is the objective function for the classical optimization algorithm for $r(G,H)$:

$$H_P|e\rangle = \mathscr{O}_N(e;G,H)|e\rangle. \tag{9}$$

Notice that the smallest eigenvalue (viz. ground-state energy) of H_P will be zero iff there exists a coloring e_* with no red subgraph isomorphic to G or blue subgraph isomorphic to H. The initial Hamiltonian H_i is chosen to be

$$H_i = -\sum_{k=0}^{L-1} \sigma_x^k, \tag{10}$$

where σ_x^k acts like a NOT operator on the k^{th} qubit,

$$\sigma_x^k|e_0 \cdots e_k \cdots e_{L-1}\rangle = |e_0 \cdots (e_k \oplus 1) \cdots e_{L-1}\rangle,$$

where $\oplus$ indicates binary addition. The ground-state of H_i is easily shown to be the uniform superposition of L-qubit CBS.

As with the classical optimization algorithm for $r(G,H)$, the adiabatic quantum algorithm begins by setting the graph order N equal to a strict lower bound for $r(G,H)$, obtained using the probabilistic method, or a lower bound from the literature. The AQO algorithm is run on $L_N = \binom{N}{2}$ qubits, and at the end of the adiabatic evolution, the qubits are measured in the computational basis. The result is a binary string e_* of length L_N. In the *adiabatic limit* ($T \to \infty$), the string e_* will be an optimal string, almost certainly, with $\mathscr{O}_N(e;G,H) = 0$ since $N < r(G,H)$. The value of N is now incremented $N \to N+1$, the AQO algorithm is re-run on L_{N+1} qubits, and the qubits measured in the computational basis at the end of adiabatic evolution. This process is repeated until the objective function value for the measured string is first positive. When this first occurs, in the adiabatic limit, the current N value will be equal to $r(G,H)$, almost certainly. Note that any real application of AQO will only be approximately adiabatic. Thus the probability that the measured string e_* will be an optimal string is $1-\varepsilon$. In this case, the algorithm must be run $k \sim \mathscr{O}(\ln[1-\delta]/\ln\varepsilon)$ times so that, with probability $\delta > 1-\varepsilon$, at least one of the measurement outcomes will be an optimal string. We can make δ arbitrarily close to 1 by choosing k sufficiently large. This then gives an adiabatic quantum algorithm for computing generalized Ramsey numbers.

4 Numerical results for $r(\mathscr{T}_m, \mathscr{T}_n)$

In this Section we numerically determine the generalized Ramsey numbers $r(\mathscr{T}_m, \mathscr{T}_n)$ associated with trees of order $m,n = 6,7,8$. These Ramsey numbers are of interest as many are unknown, and only determined to within loose lower and upper bounds [4]. Ideally, these Ramsey numbers would be found by simulating the quantum dynamics of the AQO algorithm presented in Section 3.2. However, the exponential growth of Hilbert space dimension with number of qubits makes simulation of quantum systems with more than 20 qubits impracticable [11, 13, 14]. From Ref. [4], the Ramsey numbers for 6-vertex trees satisfy $7 \leq r(\mathscr{T}_6, \mathscr{T}_6) \leq 25$. Thus simulating the AQO algorithm at the lower bound is already impractical as this requires $\binom{7}{2} = 21$ qubits. The situation is even worse for the other tree Ramsey numbers listed above. However, the classical optimization algorithm of Section 3.1 does allow us to determine these tree Ramsey numbers, and as we shall see below, most of the tree Ramsey numbers found are new.

In Section 4.1 we discuss the methodology and complexity of our numerical computation. In the interests of clarity we limit our presentation of numerical results in Section 4.2 to $r(\mathscr{T}_6, \mathscr{T}_6)$; the remaining tree Ramsey numbers are presented in Appendix B. Specifically, the Ramsey numbers $r(\mathscr{T}_7, \mathscr{T}_n)$ with $n = 6,7$ appear in Section B.1; and $r(\mathscr{T}_8, \mathscr{T}_n)$ with $6 \leq n \leq 8$ appear in Section B.2. Section 4.2 and Appendix B.1 also present, for each input pair of GI classes, the number of non-isomorphic critical graphs, and at the Ramsey threshold $N = r(\mathscr{T}_m^i, \mathscr{T}_n^j)$, the number of non-isomorphic optimal graphs and their associated minimum objective function values.

4.1 Sources of complexity

The difficulty of calculating Ramsey numbers was noted in the Introduction, and the above optimization algorithm does not evade this difficulty. Here we describe three sources of exponential complexity which the COP contains, and discuss how they impact the numerical work presented in Section 4.2 and Appendix B.

For a given coloring e of K_N, the algorithm examines all choices of m-sets S_α and n-sets S_β. There are $\binom{N}{m}$ and $\binom{N}{n}$ such choices which, respectively, scale exponentially with m and n. As $m,n \leq 8$ in the numerical work presented in this paper, this source of intractability proved managable.

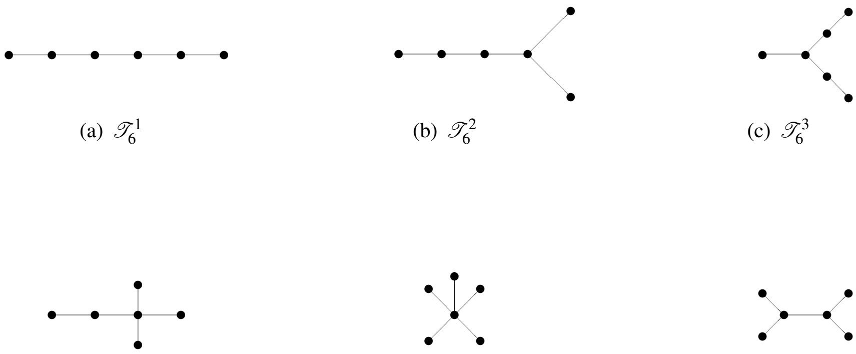
The second source of intractability arises from the need to consider all possible two-colorings of K_N. There are $2^{\binom{N}{2}}$ such colorings which is super-exponential in N. Note, however, that (two-)colorings of K_N that are isomorphic to a given coloring e

Table 1 The number of unlabelled (u_N) and labelled (l_N) colorings of K_N [15].

N	u_N	$l_N = 2^{\binom{N}{2}}$
1	1	1
2	2	2
3	4	8
4	11	64
5	34	1024
6	156	32768
7	1044	2097152
8	12346	268435456
9	274668	68719476736
10	12005168	35184372088832
11	1018997864	36028797018963968
12	165091172592	73786976294838206464
13	50502031367952	302231454903657293676544
14	29054155657235488	2475880078570760549798248448

contain a red G or a blue H iff e does. Thus, when calculating $r(G,H)$, we only need to consider vertex-unlabelled colorings of K_N. Since there are far fewer unlabelled colorings of K_N than labelled colorings (see Table 1), it was possible to exhaustively examine all unlabelled colorings of K_N for $N \leq 11$. For a given N, the graph isomorphism algorithm NAUTY [16] was used to generate the unlabelled colorings of K_N. To go to larger N (viz. $N \geq 12$), it was necessary to give up on exhaustive examination of colorings to find the objective function minimum, and instead work with the heuristic algorithm Tabu search [17]. If, for a given N, Tabu search returned a coloring e_* with $\mathcal{O}_N(e_*;G,H) = 0$, then we know that e_* does not contain a red G or a blue H, and so $r(G,H) > N$. However, if the smallest objective value returned by Tabu search is positive, we cannot rule out that Tabu search missed a coloring with vanishing objective. In this case, absent further information, the most that can be concluded is that $r(G,H) \geq N$. We return to this point in Appendix B.2.

The final source of exponential complexity arises when computing the look-up tables for $f[G \cong K^r_\alpha(e)]$ and $f[H \cong K^b_\beta(e)]$. As discussed above, each choice S_α (S_β) of m (n) vertices gives rise to a subgraph $K^r_\alpha(e)$ ($K^b_\beta(e)$) which must be examined to see if it is isomorphic to G (H). A separate look-up table was used to store the values of $f[G \cong K^r_\alpha(e)]$ and $f[H \cong K^b_\beta(e)]$. Naively, each table requires an entry for each of the $2^{\binom{m}{2}}$ and $2^{\binom{n}{2}}$ possible $K^r_\alpha(e)$ and $K^b_\beta(e)$, respectively. When m and/or n equals 8, this becomes unmanagable. The solution is again to store only unlabelled graphs in the look-up tables for $f[G \cong K^r_\alpha(e)]$ and $f[H \cong K^b_\beta(e)]$. NAUTY was used to find all unlabelled graphs $\mathcal{G}$ of order 8, and for each $\mathcal{G}$, Eq. (1) and/or Eq. (4) was used to evaluate $f[G \cong \mathcal{G}]$ and/or $f[H \cong \mathcal{G}]$, depending upon whether m and/or n was equal to 8. Then, for a given coloring e of K_N, to determine whether $G \cong K^r_\alpha(e)$ or $H \cong K^b_\beta(e)$, NAUTY was used to find the unlabelled graph isomorphic to $K^r_\alpha(e)$ ($K^b_\beta(e)$), and the look-up table value for the unlabelled graph used to determine whether $G \cong K^r_\alpha(e)$ ($H \cong K^b_\beta(e)$).



(a) $\mathscr{T}_6^1$ (b) $\mathscr{T}_6^2$ (c) $\mathscr{T}_6^3$

(d) $\mathscr{T}_6^4$ (e) $\mathscr{T}_6^5$ (f) $\mathscr{T}_6^6$

Fig. 1 Graph isomorphism classes $\mathscr{T}_6^j$ $(j = 1, \ldots 6)$ for trees of order 6.

By combining all of the above mitigation procedures, we were able to handle complete graphs K_N with $N \leq 14$, and graphs G and H corresponding to trees with $6 \leq |V_G|,\ |V_H| \leq 8$.

4.2 Tree Ramsey numbers $r(\mathscr{T}_6, \mathscr{T}_6)$

Trees of order 6 partition into six GI classes [18] which we denote by $\{\mathscr{T}_6^j : j = 1, \ldots 6\}$, and show as unlabelled graphs in Figure 1. All GI classes except $\mathscr{T}_6^3$ correspond to known unlabelled graphs (Section 2 and Appendix A):

$$\begin{array}{lll} \mathscr{T}_6^1 = P_6 = S_{1,1}^{(4)}; & \mathscr{T}_6^4 = S_{1,3}^{(2)}; & \mathscr{T}_6^6 = S_{2,2}^{(2)}. \\ \mathscr{T}_6^2 = S_{1,2}^{(3)}; & \mathscr{T}_6^5 = K_{1,5}; & \end{array} \tag{11}$$

Using the numerical procedure described in Section 4.1 we determined the tree Ramsey numbers $r(\mathscr{T}_6^i, \mathscr{T}_6^j)$ for $i, j = 1, \ldots, 6$ which we displayed in Table 2 Only the upper triangular table entries are shown as the lower triangular entries follow from $r(\mathscr{T}_6^j, \mathscr{T}_6^i) = r(\mathscr{T}_6^i, \mathscr{T}_6^j)$. A superscript "x" on a table entry indicates that Theorem A.x in Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 24 tree Ramsey numbers (to the best of our knowledge) are new.

As explained in Section 4.1, for $m, n = 6$, and for graphs of order $7 \leq N \leq 10$, our numerical procedure can exhaustively search over all non-isomorphic graphs, and so can find the number of non-isomorphic optimal graphs and their associated objective function value. For graphs with order $N = r(\mathscr{T}_6^i, \mathscr{T}_6^j) - 1$, the optimal graphs

Table 2 Numerical results for tree Ramsey numbers $r(\mathscr{T}_6^i, \mathscr{T}_6^j)$ with $1 \leq i, j \leq 6$. Table rows (columns) are labelled by i (j). Only the upper triangular table entries are shown as the lower triangular entries follow from $r(\mathscr{T}_6^j, \mathscr{T}_6^i) = r(\mathscr{T}_6^i, \mathscr{T}_6^j)$. A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 24 tree Ramsey numbers (to the best of our knowledge) are new.

$\mathbf{r}(\mathscr{T}_6^{\mathbf{i}}, \mathscr{T}_6^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6
1	$8^{2,6}$	8	8	8	9^4	8
2		7	8	7	9^4	8
3			8	8	9^4	8
4				7^7	9^4	8
5					10^3	9
6						8^7

Table 3 Numerical results for the number of non-isomorphic critical graphs $N_c(\mathscr{T}_6^i, \mathscr{T}_6^j)$ with $1 \leq i, j \leq 6$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_6^i, \mathscr{T}_6^j) - 1$. Only the upper triangular table entries are shown as the lower triangular entries follow from symmetry under interchange of colors: $N_c(\mathscr{T}_6^j, \mathscr{T}_6^i) = N_c(\mathscr{T}_6^i, \mathscr{T}_6^j)$.

$\mathbf{N_c}(\mathscr{T}_6^{\mathbf{i}}, \mathscr{T}_6^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6
1	4	2	4	2	1	3
2		8	2	9	1	1
3			4	2	1	3
4				14	6	1
5					16	1
6						2

are $(\mathscr{T}_6^i, \mathscr{T}_6^j)$-critical graphs (Section 2). Table 3 lists the number of non-isomorphic critical graphs for each pairing of GI classes $(\mathscr{T}_6^i, \mathscr{T}_6^j)$. These graphs are all found to have vanishing objective function value, which is expected, since critical graphs have order $N < r(\mathscr{T}_6^i, \mathscr{T}_6^j)$.

At the Ramsey threshold, $N = r(\mathscr{T}_6^i, \mathscr{T}_6^j)$, optimal graphs first acquire a non-vanishing objective function value (see Section 3.1). In Tables 4 and 5 we list, respectively, the number of non-isomorphic optimal graphs for each pairing of GI classes $(\mathscr{T}_6^i, \mathscr{T}_6^j)$ and the corresponding minimum objective function value.

As noted earlier, in the interests of clarity, we present the remainder of our tree Ramsey number results in Appendix B.

5 Summary

In this paper we presented an adiabatic quantum algorithm for computing generalized Ramsey numbers. We showed how such a computation can be reformulated as a combinatorial optimization problem whose solution is found using adiabatic quan-

Table 4 Numerical results for the number of non-isomorphic optimal graphs $N_{opt}(\mathscr{T}_6^i, \mathscr{T}_6^j)$ with $1 \leq i, j \leq 6$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_6^i, \mathscr{T}_6^j)$. Only the upper triangular table entries are shown as the lower triangular entries follow from symmetry under interchange of colors: $N_{opt}(\mathscr{T}_6^j, \mathscr{T}_6^i) = N_{opt}(\mathscr{T}_6^i, \mathscr{T}_6^j)$.

$\mathbf{N_{opt}}(\mathscr{T}_\mathbf{6}^\mathbf{i}, \mathscr{T}_\mathbf{6}^\mathbf{j})$ $i \setminus j$	1	2	3	4	5	6
1	4	2	4	2	1	3
2		8	2	5	1	1
3			4	2	1	3
4				2	1	1
5					8	1
6						2

Table 5 Numerical results for the minimum objective function value $\mathscr{O}_N(e_*; \mathscr{T}_6^i, \mathscr{T}_6^j)$ at the Ramsey threshold with $1 \leq i, j \leq 6$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_6^i, \mathscr{T}_6^j)$. Only the upper triangular table entries are shown as the lower triangular entries follow from symmetry of the objective function under interchange of colors.

$\mathscr{O}_\mathbf{N}(\mathbf{e}_*; \mathscr{T}_\mathbf{6}^\mathbf{i}, \mathscr{T}_\mathbf{6}^\mathbf{j})$ $i \setminus j$	1	2	3	4	5	6
1	1	1	1	1	4	1
2		1	1	1	4	1
3			1	1	4	1
4				1	4	1
5					5	4
6						1

tum optimization. We determined all generalized Ramsey numbers for trees of order 6-8, resulting in 1600 tree Ramsey numbers, of which (to the best of our knowledge) 1479 are new. All results are consistent with a conjectured upper bound on tree Ramsey numbers [24].

Acknowledgements F. G. thanks T. Howell III for continued support.

Conflicts of Interest The authors declare that they have no conflict of interest.

A Tree Ramsey numbers - literature survey

In this Appendix we quote five theorems from the literature pertaining to generalized Ramsey numbers for certain families of trees. These theorems: (i) allow us to identify which of our tree Ramsey numbers are new; and (ii) provide checks on the remainder of our results. Section 2 provides a brief review of the graph theory concepts needed in this paper.

Theorem 1 **(Gerecsér and Gyárfás [19])** *For paths P_m and P_n with $2 \leq m \leq n$,*

$$r(P_m, P_n) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

Theorem 2 **(Harary [20])** *For stars $K_{1,m-1}$ and $K_{1,n-1}$ with $m,n \geq 2$ and diameter 2,*

$$r(K_{1,m-1}, K_{1,n-1}) = \begin{cases} m+n-3, & \text{odd } m,n \\ m+n-2, & \text{otherwise.} \end{cases}$$

Theorem 3 **(Cockayne [21])** *If T_m is a tree of order m containing a vertex of degree one adjacent to a vertex of degree two, then $r(T_m, K_{1,n-1}) = m+n-3$ ($n \geq 2$), provided one of the following holds:*

(1) $n-1 \equiv 0,2 \mod (m-1)$ *;*
(2) $n-1 \not\equiv 1 \mod (m-1)$ *and* $n-1 \geq (m-3)^2$*;*
(3) $n-1 \not\equiv 1 \mod (m-1)$ *and* $n-1 \equiv 1 \mod (m-2)$*;*
(4) $n-1 \equiv m-2 \mod (m-1)$ *and* $n-1 > m-2$.

The following definition proves convenient.

Definition 2 Suppose $a,b \geq 1$ and $k \geq 2$. Then $S^{(k)}_{a,b}$ is the graph obtained from the disjoint stars $K_{1,a}$ and $K_{1,b}$ by joining their centers with a path of order k (viz. length $k-1$).

Thus the order of $S^{(k)}_{a,b}$ is $a+b+k$ and the path $P_n = S^{(n-2)}_{1,1}$ for $n \geq 4$.

Theorem 4 **(Burr and Erdös [22])** *Consider the graph $S^{(4)}_{a,b}$ with $a \geq b \geq 1$ and diameter 5. Then*

$$r(S^{(4)}_{a,b}, S^{(4)}_{a,b}) = \max\{2a+3, a+2b+5\}.$$

Theorem 5 **(Grossman, Harary, and Klawe [23])** *Consider the graph $S^{(2)}_{a,b}$ with $a \geq b \geq 1$ and diameter 3. Then*

$$r(S^{(2)}_{a,b}, S^{(2)}_{a,b}) \geq \begin{cases} \max\{2a+1, a+2b+2\}, & a \text{ odd, and } b = 1,2 \\ \max\{2a+2, a+2b+2\}, & \text{otherwise.} \end{cases}$$

Furthermore,

$$r(S^{(2)}_{a,b}, S^{(2)}_{a,b}) = \begin{cases} \max\{2a+1, a+2b+2\}, & a \text{ odd, and } b = 1,2 \\ \max\{2a+2, a+2b+2\}, & a \text{ even, or } b \geq 3 \text{ provided } a \leq \sqrt{2}b \text{ or } a \geq 3b. \end{cases}$$

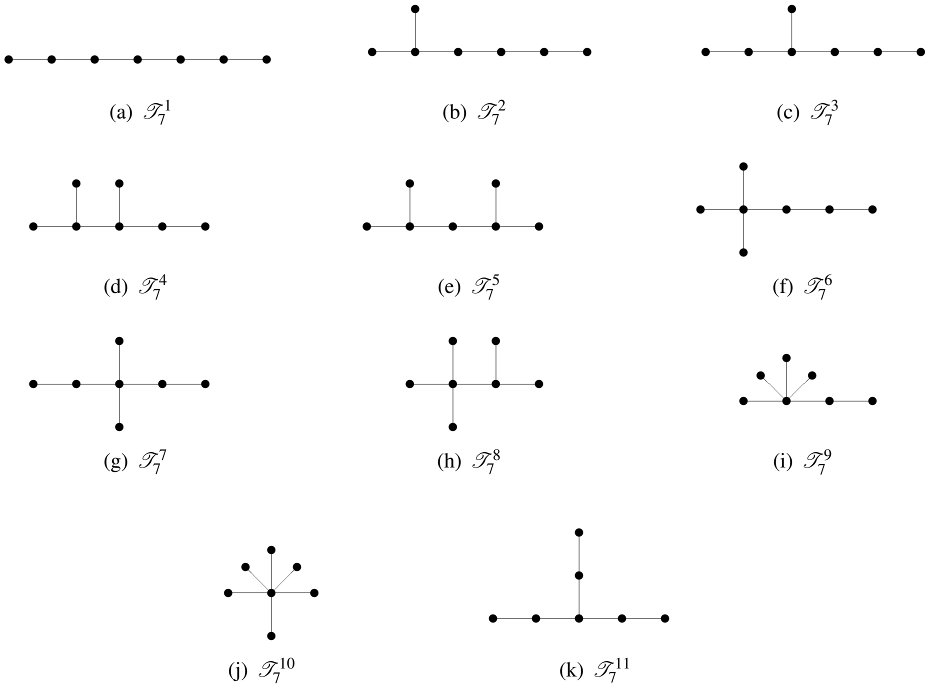
B Remaining tree Ramsey numbers

In this Appendix we present our remaining tree Ramsey number results. Appendix B.1 contains our results for $r(\mathscr{T}_7, \mathscr{T}_n)$ with $n = 6,7$; and Appendix B.2 contains $r(\mathscr{T}_8, \mathscr{T}_n)$ for $n = 6,7,8$. Appendix B.1 also present, for each input pair of GI classes, the number of non-isomorphic critical graphs, and at the Ramsey threshold $N = r(\mathscr{T}^i_m, \mathscr{T}^j_n)$, the number of non-isomorphic optimal graphs and their associated minimum objective function values.

B.1 Tree Ramsey numbers $r(\mathscr{T}_7, \mathscr{T}_n)$ for $n = 6,7$

Trees of order 7 partition into eleven GI classes [18] which we denote by $\{\mathscr{T}^j_7 : j = 1,\ldots 11\}$, and show as unlabelled graphs in Figure 2. Seven of these GI classes correspond to known (unlabelled) graphs (Section 2 and Appendix A):

$$\begin{array}{lll} \mathscr{T}^1_7 = P_7 = S^{(5)}_{1,1}; & \mathscr{T}^6_7 = S^{(3)}_{3,1}; & \mathscr{T}^{10}_7 = K_{1,6}. \\ \mathscr{T}^2_7 = S^{(4)}_{2,1}; & \mathscr{T}^8_7 = S^{(2)}_{3,2}; & \\ \mathscr{T}^5_7 = S^{(3)}_{2,2}; & \mathscr{T}^9_7 = S^{(2)}_{4,1}; & \end{array} \qquad (12)$$



(a) $\mathscr{T}_7^1$ (b) $\mathscr{T}_7^2$ (c) $\mathscr{T}_7^3$ (d) $\mathscr{T}_7^4$ (e) $\mathscr{T}_7^5$ (f) $\mathscr{T}_7^6$ (g) $\mathscr{T}_7^7$ (h) $\mathscr{T}_7^8$ (i) $\mathscr{T}_7^9$ (j) $\mathscr{T}_7^{10}$ (k) $\mathscr{T}_7^{11}$

Fig. 2 Graph isomorphism classes $\mathscr{T}_7^j$ $(j = 1, \ldots 11)$ for trees of order 7.

B.1.1 $r(\mathscr{T}_7^i, \mathscr{T}_6^j)$

Using our numerical procedure (Section 4.1) we determined the tree Ramsey numbers $r(\mathscr{T}_7^i, \mathscr{T}_6^j)$ for $1 \leq i \leq 11$ and $1 \leq j \leq 6$ which are displayed in Table 6. A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 64 tree Ramsey numbers (to the best of our knowledge) are new.

As explained in Section 4.1, for trees of order 6 and 7, and for graphs of order $8 \leq N \leq 11$, our numerical procedure can exhaustively search over all non-isomorphic graphs, and so can find the number of non-isomorphic optimal graphs and their associated objective function value. For graphs with order $N = r(\mathscr{T}_6^i, \mathscr{T}_6^j) - 1$, the optimal graphs are $(\mathscr{T}_7^i, \mathscr{T}_6^j)$-critical graphs (Section 2). Table 7 lists the number of non-isomorphic critical graphs for each pairing of GI classes $(\mathscr{T}_7^i, \mathscr{T}_6^j)$. These graphs are all found to have vanishing objective function value, which is expected, since critical graphs have order $N < r(\mathscr{T}_7^i, \mathscr{T}_6^j)$.

At the Ramsey threshold, $N = r(\mathscr{T}_7^i, \mathscr{T}_6^j)$, optimal graphs first acquire a non-vanishing objective function value (see Section 3.1). In Tables 8 and 9 we list, respectively, the number of non-isomorphic optimal graphs for each pairing of GI classes $(\mathscr{T}_7^i, \mathscr{T}_6^j)$ and the corresponding minimum objective function value.

B.1.2 $r(\mathscr{T}_7^i, \mathscr{T}_7^j)$

Using the numerical procedure described in Section 4.1, we determined the tree Ramsey numbers $r(\mathscr{T}_7^i, \mathscr{T}_7^j)$ for $1 \leq i, j \leq 11$ which are displayed in Table 10. Only the upper triangular table entries are shown as the

Table 6 Numerical results for tree Ramsey numbers $r(\mathscr{T}_7^i, \mathscr{T}_6^j)$ with $1 \leq i \leq 11$ and $1 \leq j \leq 6$. Table rows (columns) are labelled by i (j). A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 64 tree Ramsey numbers (to the best of our knowledge) are new.

$\mathbf{r}(\mathscr{T}_7^{\mathbf{i}}, \mathscr{T}_6^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6
1	9^2	8	9	8	9	9
2	9	8	9	8	9	9
3	9	8	9	8	9	9
4	9	8	9	8	9	9
5	9	9	9	9	9	9
6	9	9	9	9	9	9
7	9	8	9	8	9	9
8	9	8	9	8	9	9
9	9	9	9	9	10	9
10	11	11	11	11	11^3	11
11	9	8	9	8	9	9

Table 7 Numerical results for the number of non-isomorphic $(\mathscr{T}_7^i, \mathscr{T}_6^j)$-critical graphs $N_c(\mathscr{T}_7^i, \mathscr{T}_6^j)$ with $1 \leq i \leq 11$ and $1 \leq j \leq 6$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_7^i, \mathscr{T}_6^j) - 1$.

$\mathbf{N_c}(\mathscr{T}_7^{\mathbf{i}}, \mathscr{T}_6^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6
1	2	8	2	5	1	1
2	2	6	2	3	1	1
3	2	6	2	3	1	1
4	2	6	2	3	1	1
5	3	1	3	1	6	2
6	3	1	3	1	10	2
7	2	6	2	3	7	1
8	2	5	2	5	8	1
9	3	1	3	6	16	2
10	1	1	1	1	60	1
11	2	9	2	7	3	1

lower triangular entries follow from $r(\mathscr{T}_7^j, \mathscr{T}_7^i) = r(\mathscr{T}_7^i, \mathscr{T}_7^j)$. A superscript "x" on a table entry indicates that Theorem A.x in Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 102 tree Ramsey numbers (to the best of our knowledge) are new.

As explained in Section 4.1, for $m, n = 7$, and for graphs of order $9 \leq N \leq 11$, our numerical procedure can exhaustively search over all non-isomorphic graphs, and so can find the number of non-isomorphic optimal graphs and their associated objective function value. For graphs with order $N = r(\mathscr{T}_7^i, \mathscr{T}_7^j) - 1$, the optimal graphs are $(\mathscr{T}_7^i, \mathscr{T}_7^j)$-critical graphs (Section 2). Table 11 lists the number of non-isomorphic critical graphs for each pairing of GI classes $(\mathscr{T}_7^i, \mathscr{T}_7^j)$. These graphs are all found to have vanishing objective function value, which is expected, since critical graphs have order $N < r(\mathscr{T}_7^i, \mathscr{T}_7^j)$.

At the Ramsey threshold, $N = r(\mathscr{T}_7^i, \mathscr{T}_7^j)$, optimal graphs first acquire a non-vanishing objective function value (see Section 3.1). In Tables 12 and 13 we list, respectively, the number of non-isomorphic

Table 8 Numerical results for the number of non-isomorphic optimal graphs $N_{opt}(\mathscr{T}_7^i, \mathscr{T}_6^j)$ with $1 \leq i \leq 11$ and $1 \leq j \leq 6$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_7^i, \mathscr{T}_6^j)$.

$\mathbf{N_{opt}(\mathscr{T}_7^i, \mathscr{T}_6^j)}$ $i \setminus j$	1	2	3	4	5	6
1	2	8	2	5	1	1
2	2	6	2	3	1	1
3	2	6	2	3	1	1
4	2	6	2	3	1	1
5	2	1	2	1	1	1
6	2	1	2	1	1	1
7	2	6	2	3	1	1
8	2	5	2	2	1	1
9	2	1	2	1	1	1
10	1	1	1	1	1	1
11	2	8	2	5	1	1

Table 9 Numerical results for the minimum objective function value $\mathscr{O}_N(e_*; \mathscr{T}_7^i, \mathscr{T}_6^j)$ at the Ramsey threshold with $1 \leq i \leq 11$ and $1 \leq j \leq 6$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_7^i, \mathscr{T}_6^j)$.

$\mathbf{\mathscr{O}_N(e_*; \mathscr{T}_7^i, \mathscr{T}_6^j)}$ $i \setminus j$	1	2	3	4	5	6
1	1	1	1	1	4	1
2	1	1	1	1	4	1
3	1	1	1	1	4	1
4	1	1	1	1	4	1
5	1	6	1	6	4	1
6	1	6	1	6	4	1
7	1	1	1	1	4	1
8	1	1	1	1	4	1
9	1	6	1	6	5	1
10	6	6	6	6	6	6
11	1	1	1	1	4	1

optimal graphs for each pairing of GI classes $(\mathscr{T}_7^i, \mathscr{T}_7^j)$ and the corresponding minimum objective function value.

B.2 $r(\mathscr{T}_8, \mathscr{T}_n)$ for $n = 6, 7, 8$

Trees of order 8 partition into twenty-three GI classes [18] which we denote by $\{\mathscr{T}_8^j : j = 1, \ldots 23\}$, and show as unlabelled graphs in Figure 3. Ten of these GI classes correspond to known (unlabelled) graphs (Section 2 and Appendix A):

$$\begin{array}{llll} \mathscr{T}_8^1 = K_{1,7}; & \mathscr{T}_8^4 = S_{4,2}^{(2)}; & \mathscr{T}_8^9 = S_{3,1}^{(4)}; & \mathscr{T}_8^{23} = P_8 = S_{1,1}^{(6)}. \\ \mathscr{T}_8^2 = S_{5,1}^{(2)}; & \mathscr{T}_8^5 = S_{3,2}^{(3)}; & \mathscr{T}_8^{12} = S_{2,2}^{(4)}; & \\ \mathscr{T}_8^3 = S_{4,1}^{(3)}; & \mathscr{T}_8^6 = S_{3,3}^{(2)}; & \mathscr{T}_8^{14} = S_{2,1}^{(5)}; & \end{array} \tag{13}$$

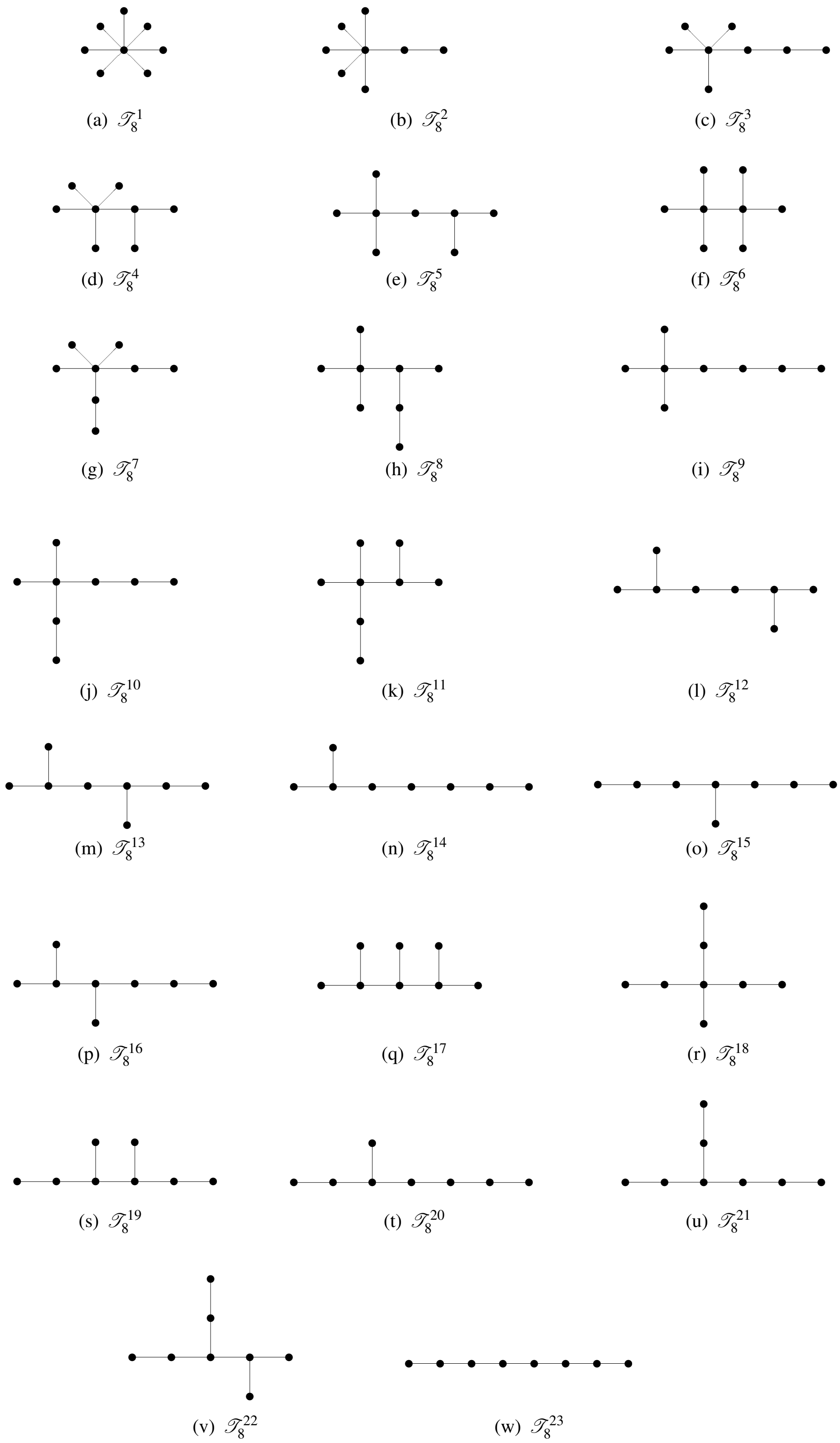


Fig. 3 Graph isomorphism classes $\mathcal{T}_8^j$ $(j = 1, \ldots 23)$ for trees of order 8.

Table 10 Numerical results for tree Ramsey numbers $r(\mathscr{T}_7^i, \mathscr{T}_7^j)$ with $1 \leq i, j \leq 11$. Table rows (columns) are labelled by i (j). Only the upper triangular table entries are shown as the lower triangular entries follow from $r(\mathscr{T}_7^j, \mathscr{T}_7^i) = r(\mathscr{T}_7^i, \mathscr{T}_7^j)$. A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 102 tree Ramsey numbers (to the best of our knowledge) are new.

$\mathbf{r}(\mathscr{T}_7^{\mathbf{i}}, \mathscr{T}_7^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6	7	8	9	10	11
1	9^{2}	9	9	9	9	9	9	9	9	11^{4}	9
2		9^{6}	9	9	9	9	9	9	9	11^{4}	9
3			9	9	9	9	9	9	9	11^{4}	9
4				9	9	9	9	9	9	11^{4}	9
5					9	9	9	9	9	11	9
6						9	9	9	9	11^{4}	9
7							9	9	9	11^{4}	9
8								9^{7}	9	11	9
9									10^{7}	11^{4}	9
10										11^{3}	11
11											9

Table 11 Numerical results for the number of non-isomorphic critical graphs $N_c(\mathscr{T}_7^i, \mathscr{T}_7^j)$ with $1 \leq i, j \leq 11$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_7^i, \mathscr{T}_7^j) - 1$. Only the upper triangular table entries are shown as the lower triangular entries follow from symmetry under interchange of colors: $N_c(\mathscr{T}_7^j, \mathscr{T}_7^i) = N_c(\mathscr{T}_7^i, \mathscr{T}_7^j)$.

$\mathbf{N_c}(\mathscr{T}_7^{\mathbf{i}}, \mathscr{T}_7^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6	7	8	9	10	11
1	8	6	6	6	5	5	6	5	5	1	8
2		4	4	4	3	3	4	3	3	1	6
3			4	4	3	3	4	3	3	1	6
4				4	3	3	4	3	3	1	6
5					2	2	4	2	6	1	6
6						2	4	2	10	1	6
7							4	3	9	1	6
8								2	9	1	5
9									16	60	7
10										9638	1
11											8

B.2.1 $r(\mathscr{T}_8^i, \mathscr{T}_6^j)$

The numerical procedure described in Section 4.1 was used to determine the tree Ramsey numbers $r(\mathscr{T}_8^i, \mathscr{T}_6^j)$ for $1 \leq i \leq 23$ and $1 \leq j \leq 6$, and the results are displayed in Table 14. A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. Recall from Section 4.1 that for graphs with order $N \geq 12$, exhaustive search over non-isomorphic graphs was not feasible. For such graphs we used the heuristic algorithm Tabu search to look for minima of the tree Ramsey number objective function. As noted there, Tabu search only yields a lower bound for a tree Ramsey number. However, because of Theorem A.3, we know that for $r(\mathscr{T}_8^1, \mathscr{T}_6^5)$, the

Table 12 Numerical results for the number of non-isomorphic optimal graphs $N_{opt}(\mathscr{T}_7^i, \mathscr{T}_7^j)$ with $1 \leq i, j \leq 11$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_7^i, \mathscr{T}_7^j)$. Only the upper triangular table entries are shown as the lower triangular entries follow from symmetry under interchange of colors: $N_{opt}(\mathscr{T}_7^j, \mathscr{T}_7^i) = N_{opt}(\mathscr{T}_7^i, \mathscr{T}_7^j)$.

$\mathbf{N_{opt}}(\mathscr{T}_7^{\mathbf{i}}, \mathscr{T}_7^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6	7	8	9	10	11
1	8	6	6	6	4	4	6	5	4	1	8
2		4	4	4	2	2	4	3	2	1	6
3			4	4	2	2	4	3	2	1	6
4				4	2	2	4	3	2	1	6
5					2	2	2	1	6	1	4
6						2	2	1	6	1	4
7							4	3	2	1	6
8								2	1	1	5
9									22	1	4
10										7502	1
11											8

Table 13 Numerical results for the minimum objective function value $\mathscr{O}_N(e_*; \mathscr{T}_7^i, \mathscr{T}_7^j)$ at the Ramsey threshold with $1 \leq i, j \leq 11$. Table rows (columns) are labelled by i (j), and for table entry (i, j), the graph order is $r(\mathscr{T}_7^i, \mathscr{T}_7^j)$. Only the upper triangular table entries are shown as the lower triangular entries follow from symmetry of the objective function under interchange of colors.

$\mathscr{O}_{\mathbf{N}}(\mathbf{e}_*; \mathscr{T}_7^{\mathbf{i}}, \mathscr{T}_7^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	5	1
2		1	1	1	1	1	1	1	1	5	1
3			1	1	1	1	1	1	1	5	1
4				1	1	1	1	1	1	5	1
5					6	6	1	1	6	5	1
6						6	1	1	6	5	1
7							1	1	1	5	1
8								1	1	5	1
9									20	5	1
10										1	5
11											1

value 12 is in fact the *actual* value of this Ramsey number. Thus no lower bounds appear in Table 14—all values are actual Ramsey number values. Of the 138 tree Ramsey numbers appearing in Table 14, (to the best of our knowledge) 132 are new.

B.2.2 $r(\mathscr{T}_8^i, \mathscr{T}_7^j)$

The numerical results for the tree Ramsey numbers $r(\mathscr{T}_8^i, \mathscr{T}_7^j)$ for $1 \leq i \leq 23$ and $1 \leq j \leq 11$ are shown in Table 15. A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. Recall from Section 4.1 that for graphs with order $N \geq 12$, exhaustive search over non-isomorphic graphs was not feasible. For such graphs we used the heuristic algorithm Tabu search to look for minima of the tree Ramsey number objective function. As noted there, Tabu search only yields a lower bound for a tree Ramsey number. Numbers in Table 15 marked

Table 14 Numerical results for tree Ramsey numbers $r(\mathscr{T}_8^i, \mathscr{T}_6^j)$ with $1 \leq i \leq 23$ and $1 \leq j \leq 6$. Table rows (columns) are labelled by i (j). A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. The remaining 132 tree Ramsey numbers (to the best of our knowledge) are new.

$\mathbf{r}(\mathscr{T}_8^{\mathbf{i}}, \mathscr{T}_6^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6
1	11^4	11^4	11^4	11^4	12^3	11
2	11	11	11	11	11	11
3	11	11	11	11	11	11
4	10	9	10	9	10	10
5	11	11	11	11	11	11
6	10	9	10	9	10	10
7	10	9	10	9	10	10
8	10	9	10	9	9	10
9	10	9	10	9	9	10
10	10	9	10	9	9	10
11	10	9	10	9	9	10
12	10	9	10	9	9	10
13	10	9	10	9	9	10
14	10	9	10	9	9	10
15	10	9	10	9	9	10
16	10	9	10	9	9	10
17	10	9	10	9	9	10
18	10	9	10	9	9	10
19	10	9	10	9	9	10
20	10	9	10	9	9	10
21	10	9	10	9	9	10
22	10	9	10	9	9	10
23	10^2	9	10	9	9	10

with an asterisk correspond to lower bounds on the associated tree Ramsey number. Of the 253 numbers appearing in this Table, (to the best of our knowledge) 241 are new tree Ramsey numbers, 10 are lower bounds, and 2 were previously known.

Before moving on we note that it has been conjectured [24] that $r(\mathscr{T}_m, \mathscr{T}_n) \leq n + m - 2$ for all trees $\mathscr{T}_m$ and $\mathscr{T}_n$. For the trees in Table 15 the conjecture gives the upper bound $r(\mathscr{T}_8, \mathscr{T}_7) \leq 13$. The conjecture is seen to be consistent with all entries in Table 15. Furthermore, if the conjecture is true, then all lower bounds in row 1 become exact values since we would have $13 \leq r(\mathscr{T}_8^1, \mathscr{T}_7^j) \leq 13$.

B.2.3 $r(\mathscr{T}_8^i, \mathscr{T}_8^j)$

The numerical results for the tree Ramsey numbers $r(\mathscr{T}_8^i, \mathscr{T}_8^j)$ for $1 \leq i, j \leq 23$ are shown in Table 16. Only the upper triangular table entries are shown as the lower triangular entries follow from $r(\mathscr{T}_8^j, \mathscr{T}_8^i) = r(\mathscr{T}_8^i, \mathscr{T}_8^j)$. A superscript "x" on a table entry indicates that Theorem A.x in Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. Recall from Section 4.1 that for graphs with order $N \geq 12$, exhaustive search over non-isomorphic graphs was not feasible. For such graphs we used the heuristic algorithm Tabu search to look for minima of the tree Ramsey number objective function. As noted there, Tabu search only yields a lower bound for a tree Ramsey number. Numbers in Table 16 marked with an asterisk correspond to lower bounds on the associated tree Ramsey number. Of the 529 numbers appearing in this Table, (to the best of our knowledge) 479 are new tree Ramsey numbers, 10 are

Table 15 Numerical results for tree Ramsey numbers $r(\mathscr{T}_8^i, \mathscr{T}_7^j)$ with $1 \leq i \leq 23$ and $1 \leq j \leq 11$. Table rows (columns) are labelled by i (j). A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. Numbers marked by an asterisk indicate that these numbers are lower bounds on the corresponding tree Ramsey number (see text). The remaining 241 tree Ramsey numbers (to the best of our knowledge) are new.

$\mathbf{r}(\mathscr{T}_\mathbf{8}^\mathbf{i}, \mathscr{T}_\mathbf{7}^\mathbf{j})$ $i \setminus j$	1	2	3	4	5	6	7	8	9	10	11
1	13^*	13^*	13^*	13^*	13^*	13^*	13^*	13^*	13^*	13^3	13^*
2	11	11	11	11	11	11	11	11	11	11	11
3	11	11	11	11	11	11	11	11	11	11	11
4	10	10	10	10	9	9	10	10	10	11	10
5	11	11	11	11	11	11	11	11	11	11	11
6	10	10	10	10	10	10	10	10	10	11	10
7	10	10	10	10	9	9	10	10	10	11	10
8	10	10	10	10	9	9	10	10	9	11	10
9	10	10	10	10	9	9	10	10	9	11	10
10	10	10	10	10	9	9	10	10	9	11	10
11	10	10	10	10	10	10	10	10	10	11	10
12	10	10	10	10	10	10	10	10	10	11	10
13	10	10	10	10	9	9	10	10	9	11	10
14	10	10	10	10	9	9	10	10	9	11	10
15	10	10	10	10	9	9	10	10	9	11	10
16	10	10	10	10	10	10	10	10	10	11	10
17	10	10	10	10	9	9	10	10	9	11	10
18	10	10	10	10	10	10	10	10	10	11	10
19	10	10	10	10	10	10	10	10	10	11	10
20	10	10	10	10	10	10	10	10	10	11	10
21	10	10	10	10	10	10	10	10	10	11	10
22	10	10	10	10	9	9	10	10	9	11	10
23	10^2	10	10	10	10	10	10	10	10	11	10

lower bounds, and 40 were previously known. The conjectured upper bound [24], $r(\mathscr{T}_m, \mathscr{T}_n) \leq n+m-2$, evaluates to $r(\mathscr{T}_8, \mathscr{T}_8) \leq 14$ and is seen to be consistent with all entries in Table 16.

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Table 16 Numerical results for tree Ramsey numbers $r(\mathscr{T}_8^i, \mathscr{T}_8^j)$ with $1 \leq i, j \leq 23$. Table rows (columns) are labelled by i (j). Only the upper triangular table entries are shown as the lower triangular entries follow from $r(\mathscr{T}_8^j, \mathscr{T}_8^i) = r(\mathscr{T}_8^i, \mathscr{T}_8^j)$. A superscript "x" on a table entry indicates that Theorem A.x of Appendix A applies, and so these tree Ramsey numbers were known prior to this work. The reader can verify that our numerical results are in agreement with the theorems of Appendix A. Numbers marked by an asterisk indicate that these numbers are lower bounds on the corresponding tree Ramsey number (see text). Of the 529 numbers appearing in this Table, (to the best of our knowledge) 479 are new tree Ramsey numbers, 10 are lower bounds, and 40 were previously known.

$\mathbf{r}(\mathscr{T}_8^{\mathbf{i}}, \mathscr{T}_8^{\mathbf{j}})$ $i \setminus j$	1	2	3	4	5	6	7	8	9	10	11	12
1	14^3	13^4	13^4	13^*	13^*	13^*	13^4	13^4	13^4	13^4	13^4	13^*
2		11^7	11	11	11	11	11	11	11	11	11	11
3			11	11	11	11	11	11	11	11	11	11
4				10	11	11	10	10	10	10	11	11
5					10	11	11	11	11	11	11	11
6						11^7	11	11	11	11	11	11
7							10	10	10	10	11	11
8								10	10	10	11	11
9									10^6	10	11	11
10										10	11	11
11											11	11
12												11^6

$\mathbf{r}(\mathscr{T}_8^{\mathbf{i}}, \mathscr{T}_8^{\mathbf{j}})$ $i \setminus j$	13	14	15	16	17	18	19	20	21	22	23
1	13^4	13^4	13^4	13^4	13^*	13^4	13^4	13^4	13^4	13^4	13^4
2	11	11	11	11	11	11	11	11	11	11	11
3	11	11	11	11	11	11	11	11	11	11	11
4	10	10	10	11	10	11	11	11	11	10	11
5	11	11	11	11	11	11	11	11	11	11	11
6	11	11	11	11	11	11	11	11	11	11	11
7	10	10	10	11	10	11	11	11	11	10	11
8	10	10	10	11	10	11	11	11	11	10	11
9	10	10	10	11	10	11	11	11	11	10	11
10	10	10	10	11	10	11	11	11	11	10	11
11	11	11	11	11	11	11	11	11	11	11	11
12	11	11	11	11	11	11	11	11	11	11	11
13	10	10	10	11	10	11	11	11	11	10	11
14		10	10	11	10	11	11	11	11	10	11
15			10	11	10	11	11	11	11	10	11
16				11	11	11	11	11	11	11	11
17					10	11	11	11	11	10	11
18						11	11	11	11	11	11
19							11	11	11	11	11
20								11	11	11	11
21									11	11	11
22										10	11
23											11^2

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